Eigenvalues of the Orr-Sommerfeld Equation in an Unbounded Domain

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1. Introduction

The main purpose of this paper is to prove that in the space \((L^2(0, \infty))^4\) the generalized Orr-Sommerfeld equation [10] has only finitely many eigenvalues when the mean flow exponentially approaches a constant. This surprising fact was discovered by numerical studies of eigenvalues for Blasius mean flow [4, 9]. It has been proven [10] that when the mean flow approaches a constant slowly enough, the generalized Orr-Sommerfeld equation can have infinitely many eigenvalues. There is also a nontrivial condition [10] (involving the Reynolds number) which implies that the Orr-Sommerfeld equation has no eigenvalues. The proof given here is based on Lemma 2, which can be considered a generalization of some standard results [e.g. 3, 11].

Given are several properties of, and bounds for eigenvalues which can be used to estimate the critical Reynolds number and to help in the numerical search for eigenvalues.

An expectation [12] that eigenvalues should not be imbeded in the continuous spectrum [10] is also proven. These facts may suggest a way [3, 10] to obtain a spectral resolution. One can show, however, that there can exist (finitely many) spectral singularities not corresponding to eigenvalues, i.e. \(-1\) can be an eigenvalue of \(RQ_0(\sqrt{1-R^2})\) (see Section 3) even if \(z\) is not an eigenvalue of the generalized Orr-Sommerfeld equation. In such a case it is still possible to define a spectral resolution in a suitable subspace [3, 10]. If the Reynolds number is sufficiently small, then the corresponding operator is spectral. This can easily be seen from [3, 7, 10]. Since these spectral results are rather far from what one would want [1, 2, 12] and since the proofs are very cluttered, details will not be presented.

In Section 2 the main theorem is given. The idea of the proof is worked out in Section 3 and the proof of the main theorem is presented in Section 4.

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2. The Main Theorem

The generalized Orr-Sommerfeld equation is given by [6, 10]

\[
\frac{\partial u_1}{\partial t} - \frac{1}{R} \frac{\partial^2 u_1}{\partial y^2} + h(y) u_1 + i\beta_1 (y) u_2 + i\alpha p = 0, \tag{1}
\]

\[
\frac{\partial u_2}{\partial t} - \frac{1}{R} \frac{\partial^2 u_2}{\partial y^2} + h(y) u_2 + \frac{\partial p}{\partial y} = 0, \tag{2}
\]

\[
\frac{\partial u_3}{\partial t} - \frac{1}{R} \frac{\partial^2 u_3}{\partial y^2} + h(y) u_3 + i\beta_2 (y) u_2 + i\alpha p = 0, \tag{3}
\]

\[
ixu_1 + i\beta u_3 + \frac{\partial u_2}{\partial y} = 0, \tag{4}
\]

\[
ixu_k(0, t) = 0 \quad \text{for} \quad j = 1, 2, 3 \tag{5}
\]

where \( h(y) = (\alpha^2 + \beta^2)/R + i\alpha v_1(y) + i\beta v_3(y) \) and the primes denote derivatives \( u = u(y, t) = (u_1, u_2, u_3) \) and \( p = p(y, t) \) denote the velocity and pressure of the fluid at a point \( y \geq 0 \) and time \( t \geq 0 \) respectively; \( R \) as usual is the Reynolds number. \( v_1 \) and \( v_3 \) are the \( x \) and \( z \) components of the mean flow while \( \alpha \) and \( \beta \) are the wave numbers in the \( x \) and \( z \) directions of the mean flow.

Throughout \( \mathcal{H}(\mathcal{H}) \) denotes the Hilbert space \( L^2(0, \infty) \) (\( j \)-fold product of \( L^2(0, \infty) \)). The set of all complex-valued functions which are absolutely continuous on \([0, a]\) for every \( a > 0 \) is denoted by \( \mathcal{A} \).

A map \((u, p)\) from the interval \((0, \infty)\) into \( \mathcal{H}^4 \) is said to be a solution of equations (1–5) if for each \( t \in (0, \infty) \) the following conditions are satisfied [10]:

(i) \( u_j, p, \frac{\partial u_j}{\partial y} \in \mathcal{H} \cap \mathcal{A} \), \( \frac{\partial^2 u_j}{\partial y^2} \in \mathcal{H} \) for \( j = 1, 2, 3 \) and \( u = (u_1, u_2, u_3) \)

(ii) \( u \) is continuously differentiable in \( t \)

(iii) \( \frac{\partial u_2}{\partial t} \in \mathcal{A} \) and \( \frac{\partial^2 u_2}{\partial t \partial y} = \frac{\partial^2 u_2}{\partial y \partial t} \)

(iv) \( u, p \) satisfy equations (1–5)

(v) \( \lim_{t \to 0^+} u(t) \) exists.

\( \mathcal{S}_0 \) is the set of all such maps. \((u, p) \in \mathcal{S}_0 \) is an eigenvector if \( u(t) = e^{-\sigma t} u_0 \) and \( p(t) = e^{-\sigma t} p_0 \) for some \( z \in \mathbb{C} \), \( u_0 \in \mathcal{H}^3 \setminus \{0\} \) and \( p_0 \in \mathcal{H} \). The set of all such \( z \) is denoted by \( \sigma_{0z} \).

The main theorem may now be stated.

Theorem 1. Suppose:

(i) \( v_1, v_3 \in \mathcal{A} \),

(ii) the limits \( \lim_{y \to \infty} v_1(y) = \overline{v}_1 \) and \( \lim_{y \to \infty} v_3(y) = \overline{v}_3 \) exist and are finite,

(iii) \( v_1 - \overline{v}_1, v_3 - \overline{v}_3, v'_1, v'_3 \in \mathcal{H} \),

(iv) \( \alpha^2 + \beta^2 \in \mathbb{C} \setminus (-\infty, 0] \), \( R > 0 \).
Let $\lambda = \sqrt{\alpha^2 + \beta^2}$, $\text{Re}(\lambda) > 0$, $\mu = \lambda^2/R + i\alpha \tilde{v}_1 + i\beta \tilde{v}_3$ and $g_1 = \alpha(\tilde{v}_1 - v_1) + \beta(\tilde{v}_3 - v_3)$. Then

a) If $g_1, g_1' \in L^1(0, \infty)$, then for every $z \in \sigma_0$,

$$|z - \mu| \leq R \left( \|g_1\|_1 + \frac{2 |\lambda| + 1}{(\text{Re}(\lambda))^2} \|g_1'\|_1 \right) \left( \|g_1\|_1 + \left( \frac{2 |\lambda|}{(\text{Re}(\lambda))^2} + 1 \right) \|g_1'\|_1 \right).$$

b) If for some $\varepsilon > 0 \int_0^\infty e^{\varepsilon x} |g_1'(x)| \, dx < \infty$, then $\sigma_0$ is finite.

c) If $g_1$ and $\lambda$ are real valued, then for every $z \in \sigma_0$,

$$\text{Re}(z - \mu) > -\frac{1}{\lambda} \inf_{\rho \in [2, \infty)} \left( \frac{\lambda}{2} \right)^{1/\rho} \|g_1'\|_p.$$  

d) If $g_1', g_1'' \in \mathcal{H}$, $\lambda > 0$ and if $g_1(x) > 0$, $2\lambda^2g_1(x) + g_1''(x) \geq 0$ for all $x \in (0, \infty)$, then $\text{Im}(z - \mu) < 0$ for every $z \in \sigma_0$.

Remark: If $v_3 = 0$ and $v_1$ is the usual Blasius mean flow, then the assumptions in parts a and b are satisfied. If, in addition, $\alpha > 0$ and $\beta \in \mathbb{R}$ then the assumptions in parts c and d are also satisfied.

3. Preliminaries

In this section the stage is set for the proof of parts a and b of the Main Theorem. The main idea is represented in the following lemmas. The notation used is standard [8]; for $a \in \mathbb{R}$ let $\varphi(a) = \{z \in \mathbb{C} \mid \text{Re}(z) > a\}$.

Lemma 1. Suppose:

(i) $T, A_1, \ldots, A_n, B_1, \ldots, B_n$ are operators on a Banach space $X$, $R \in (0, \infty)$. Set $S = \frac{1}{R} T + B_1 A_1 + \ldots + B_n A_n$.

(ii) There exists a family of operators $K(z)$ on $X$ for $z \in \overline{\varphi}(0)$ such that $K(z)(T + z^2)f = f$ for all $f \in \mathcal{D}(T)$ and all $z \in \overline{\varphi}(0)$.

(iii) There exists a family of operators $C_i(z)$ on $X$ for $z \in \overline{\varphi}(0) \setminus \{0\}, i = 1, \ldots, n$ such that $C_i(z) \supset A_i K(z)$ and Range $(B_i) \subset \mathcal{D}(C_i(z))$ for all $z \in \overline{\varphi}(0) \setminus \{0\}$ and all $i, j \in \{1, \ldots, n\}$.

(iv) There exists a family of operators $Q_{ij}(z)$ on $X$ for $z \in \overline{\varphi}(0) \setminus \{0\}, i, j \in \{1, \ldots, n\}$ such that $Q_{ij}(z) \supset C_i(z) B_j$ and $\|Q_{ij}(z)\| \leq q_{ij}/|z| < \infty$ for all $z \in \overline{\varphi}(0) \setminus \{0\}$ and all $i, j \in \{1, \ldots, n\}$.

Then for every $z \in \sigma_0(S)$

$$|z| \leq R \sum_{ij} q_{ij}^2.$$
Lemma 2. If assumptions (i) through (iv) of Lemma 1 are satisfied and if there is an \( \varepsilon > 0 \) such that \( Q_{\phi}(\tilde{z}) \) are holomorphic families of compact operators on \( X \) for \( z \in \mathcal{V}(\varepsilon) \) and \( i, j \in \{1, \ldots, n\} \), then \( \sigma_{\phi}(S) \) is a finite set.

Proof. Suppose that \( z \in \sigma_{\phi}(S) \setminus \{0\} \). Let \( f \neq 0 \) be such that
\[
\sum_{i=1}^{n} B_{i}(A_{i}f) = \left( z - \frac{1}{R} T \right) f.
\]
Then
\[
RK \left( \sqrt{-Rz} \right) \sum_{i=1}^{n} B_{i}(A_{i}f) = -f, \quad \Re \left( \sqrt{-Rz} \right) \geq 0,
\]
\[RA_{j}K \left( \sqrt{-Rz} \right) \sum_{i=1}^{n} B_{i}(A_{i}f) = -A_{j}f, \text{for } j = 1, \ldots, n,
\]
\[
RC_{j} \left( \sqrt{-Rz} \right) \sum_{i=1}^{n} B_{i}(A_{i}f) = -A_{j}f,
\]
\[
R \sum_{i=1}^{n} Q_{ji} \left( \sqrt{-Rz} \right) (A_{i}f) = -A_{j}f.
\]
Let \( x = (A_{1}f, \ldots, A_{n}f) \in X^{n} \) and let \( Q_{\phi}(\tilde{z}) = \{Q_{\phi}(\xi)_{ij}\} \). Clearly, \( x \neq 0 \) and
\[
1 \leq \| RQ_{\phi} \left( \sqrt{-Rz} \right) \|^{2} \leq \frac{R}{|z|} \sum_{ij} q_{ij}^{2},
\]
which proves Lemma 1. Lemma 2 is now obvious [8].

Now several operators on \( \mathcal{H} \) will be introduced. For \( z \in \mathcal{V}(0) \) and \( g \in \mathcal{H} \), define \( F_{z}, G_{z} \in \mathcal{B}(\mathcal{H}) \) by
\[
(F_{z}g)(x) = \int_{0}^{x} e^{z(s-x)} g(s) \, ds
\]
and
\[
(G_{z}g)(x) = \int_{x}^{\infty} e^{z(s-x)} g(s) \, ds.
\]
The operator \( T \) is defined by \( Tf = -f'' \) for \( f \in \mathcal{D}(T) = \{f \mid f, f' \in \mathcal{H} \cap \mathcal{H}' \}, f'' \in \mathcal{H}, f(0) = 0 \).

For \( z \in \mathcal{C} \) and \( x, y \in [0, \infty) \) define
\[
k(z, x, y) = \int_{0}^{\min(x, y)} e^{z(t-x-y)} \, ds.
\]
Observe that \( |k(z, x, y)| \leq \frac{1}{|z|} \) for \( z \in \mathcal{V}(0) \setminus \{0\} \). If \( \varepsilon \in (0, \infty) \), \( \delta \in [0, \varepsilon) \) and \( z \in \mathcal{V}(\varepsilon) \), then
\[
|k(z, x, y)| \leq \frac{1}{\varepsilon - \delta} e^{\varepsilon(x+y)}.
\]
If $\xi \in \mathbb{C} \setminus \{0\}$, $\varepsilon \in (0, \infty)$, $\delta \in [0, \varepsilon)$ and $z \in \mathcal{V}(\frac{\xi}{\varepsilon} - \delta)$, then
\[
\left| \frac{k(z + \xi, x, y) - k(z, x, y)}{\varepsilon} - \frac{\partial k(z, x, y)}{\partial z} \right| \leq |\xi| \left( \frac{3}{\varepsilon - \delta} \right)^3 e^{\varepsilon(x+y)}.
\] (7)

Define the family of operators $K(z)$ for $z \in \overline{\mathcal{V}}(0)$ by $\mathcal{D}(K(z)) = \{ f \in \mathcal{H} \}$ for all $x \in [0, \infty)$ \( \lim_{z \to \infty} \int_0^x k(z, x, y)f(y) \, dy = g(x) \), and $g \in \mathcal{D}(T)$,
\[
(K(z)f)(x) = \lim_{z \to \infty} \int_0^x k(z, x, y)f(y) \, dy \quad \text{for } f \in \mathcal{D}(K(z)).
\]

Integration by parts gives $K(z)(T + z^2)f = f$ for every $f \in \mathcal{D}(T)$ and every $z \in \overline{\mathcal{V}}(0)$. Note that if $z \in \mathbb{C}$ and $e^{-z(t)}f(\cdot) \in L^1(0, \infty)$, then $k(z, x, \cdot)f(\cdot) \in L^1(0, \infty)$ for all $x \in [0, \infty)$.

Suppose that $h_1, h_2 \in \mathcal{H}$ and that $\lambda, \lambda_1 \in \mathcal{V}(0)$. In $\mathcal{H}$ define operators $A$ and $B$ in the following way:

Case I: \( A = h_1, \ B = h_2, \)

Case II: \( A = h_1, \ B = G_{\lambda_1}h_2, \)

Case III: \( A = h_1F_{\lambda_1}, \ B = h_2, \)

Case IV: \( A = h_1F_{\lambda_1}, \ B = G_{\lambda_1}h_2. \)

$G_{\lambda_1}h_2$ is a product of operators $G_{\lambda_1}$ and the multiplication operator $h_2$.

**Case I.** Define the family $C(z)$ for $z \in \overline{\mathcal{V}}(0) \setminus \{0\}$ by $\mathcal{D}(C(z)) = \{ f \in \mathcal{H} \}$ for all $x \in [0, \infty)$ \( \lim_{z \to \infty} \int_0^x k(z, x, y)f(y) \, dy = g(x) \), and $g \in \mathcal{H}$,
\[
(C(z)f)(x) = h_1(x) \lim_{z \to \infty} \int_0^x k(z, x, y)f(y) \, dy \quad \text{for } f \in \mathcal{D}(C(z)).
\]

Clearly, $C(z) \supset AK(z)$ for all $z \in \overline{\mathcal{V}}(0) \setminus \{0\}$.

For $z \in \overline{\mathcal{V}}(0) \setminus \{0\}$ define the family $Q(z)$ by
\[
(Q(z)f)(x) = h_1(x) \int_0^\infty k(z, x, y) h_2(y)f(y) \, dy, \quad f \in \mathcal{H}.
\]

Clearly, $\|Q(z)\| \leq \|h_1\|_2 \|h_2\|_2 / |z|$, Range $(B) \subset \mathcal{D}(C(z))$ and $C(z)B \subset Q(z)$ for all $z \in \overline{\mathcal{V}}(0) \setminus \{0\}$.

**Case II.** Define the family $C(z)$ as in Case I. For $z \in \overline{\mathcal{V}}(0) \setminus \{0\}$ define the family $Q(z)$ by
\[
(Q(z)f)(x) = h_1(x) \int_0^\infty k(z, x, y) \left( \int_y^x e^{\lambda_1(y-s)} h_2(s)f(s) \, ds \right) \, dy
\]
\[
= h_1(x) \int_0^x \left( \int_0^y k(z, x, s) e^{\lambda_1(s-y)} \, ds \right) h_2(y)f(y) \, dy, \quad f \in \mathcal{H}.
\]
Thus \( \|Q(z)\| \leq \frac{\|h_1\|_2 \|h_2\|_2}{|z| \text{ Re } (\lambda_1)} \), Range \((B) \subset \mathcal{D}(C(z))\) and \(C(z) B \subset Q(z)\) for all \(z \in \mathcal{V}(0) \setminus \{0\}\).

**Case III.** Now, define the family \(C(z)\) for \(z \in \mathcal{V}(0) \setminus \{0\}\) by \(\mathcal{D}(C(z)) = \{f \in \mathcal{H} | f(x) = \lim_{s \to \infty} \int_0^x k(z, x, y) f(y) dy = g(x)\text{ and if } h(x) = \int_0^x e^{\lambda(t-x)} g(s) ds \text{ then } hh_1 \in \mathcal{H}\} \),

\[
(C(z)f)(x) = h_1(x) \int_0^x e^{\lambda(t-x)} \left( \lim_{t \to \infty} \int_0^t k(z, t, y) f(y) dy \right) dt, \quad f \in \mathcal{D}(C(z)).
\]

Clearly, \(C(z) \supset AK(z)\) for all \(z \in \mathcal{V}(0) \setminus \{0\}\).

For \(z \in \mathcal{V}(0) \setminus \{0\}\) and \(f \in \mathcal{H}\) let

\[
(Q(z)f)(x) = \int_0^x h_1(x) e^{\lambda(t-x)} \left( \int_0^\infty k(z, x, y) h_2(y) f(y) dy \right) ds
\]

\[
\quad = \int_0^\infty h_1(x) \left( \int_0^x k(z, s, y) e^{\lambda(t-x)} ds \right) h_2(y) f(y) dy.
\]

Again \(\|Q(z)\| \leq \frac{\|h_1\|_2 \|h_2\|_2}{|z| \text{ Re } (\lambda)}\), Range \((B) \subset \mathcal{D}(C(z))\) and \(C(z) B \subset Q(z)\) for all \(z \in \mathcal{V}(0) \setminus \{0\}\).

**Case IV.** Let the family \(C(z)\) be as in Case III. For \(z \in \mathcal{V}(0) \setminus \{0\}\) and \(f \in \mathcal{H}\) define

\[
(Q(z)f)(x) = h_1(x) \int_0^x e^{\lambda(t-x)} \left( \int_y^\infty e^{\lambda(y-s)} h_2(s) f(s) ds \right) dy \right) dt
\]

\[
\quad = \int_0^\infty h_1(x) \left( \int_0^x dt \int_0^y ds k(z, t, s) e^{\lambda(t-x)+\lambda(s-x)} \right) h_2(y) f(y) dy.
\]

Thus \(\|Q(z)\| \leq \frac{\|h_1\|_2 \|h_2\|_2}{|z| \text{ Re } (\lambda) \text{ Re } (\lambda_1)}\), Range \((B) \subset \mathcal{D}(C(z))\) and \(C(z) B \subset Q(z)\) for all \(z \in \mathcal{V}(0) \setminus \{0\}\).

If, in addition, there is an \(\epsilon > 0\) such that \(\int_0^\infty |h_1(x) e^{\lambda x}|^2 dx < \infty\), then inequalities (6) and (7) imply that in all of the above cases \(Q(z)\) can be extended to a holomorphic family of compact operators for \(z \in \mathcal{V}(-\epsilon)\).

### 3. Proof of the Main Theorem

**Parts a and b.** It has been shown [10] that

\[
\sigma_{0a} - \mu \subset \sigma_p(D_{11}) \cup \sigma_p(D_{22})
\]
The Orr-Sommerfeld Equation

\[ D_{11} = \frac{1}{R} T - i g_1, \]
\[ D_{22} = \frac{1}{R} T - i g_1 - 2i\lambda G_i g_1 F_\lambda + i g_1 F_2. \]

Therefore, it is enough to prove the following theorem.

**Theorem 2.** Suppose that \( \phi_1, \phi_2, \phi_3 \in L^2(0, \infty) \cap L^1(0, \infty), \ R \in (0, \infty) \) and \( \lambda, \lambda_1 \in \mathcal{Y}(0). \) Set \( S = \frac{1}{R} T + \phi_1 + G_i \phi_2 F_\lambda + \phi_3 F_2. \) Then

a) For every \( z \in \sigma_p(S) \)

\[ |z| \leq R \left( \|\phi_1\|_1 + \frac{\|\phi_2\|_1}{(\text{Re} \ (\lambda))^2} + \frac{\|\phi_3\|_1}{(\text{Re} \ (\lambda))^2} \right) \left( \|\phi_1\|_1 + \frac{\|\phi_2\|_1}{(\text{Re} \ (\lambda))^2} + \|\phi_3\|_1 \right). \]

b) If, in addition, there is an \( \epsilon > 0 \) such that \( \int_0^\infty |\phi_i(x)| e^{\epsilon x} \ dx < \infty, \ i = 1, 2, 3 \) then \( \sigma_p(S) \) is finite.

**Proof.** Define

\[ A_1 = |\phi_1|^{\frac{1}{2}}, \quad B_1 = \text{sgn} (\phi_1) |\phi_1|^{\frac{1}{2}}, \]
\[ A_2 = |\phi_2|^{\frac{1}{2}} F_\lambda, \quad B_2 = G_i \text{sgn} (\phi_2) |\phi_2|^{\frac{1}{2}}, \]
\[ A_3 = |\phi_3|^{\frac{1}{2}} F_2, \quad B_3 = \text{sgn} (\phi_3) |\phi_3|^{\frac{1}{2}}, \]

where \( \text{sgn} (\phi) (x) = \phi(x) / |\phi(x)| \) if \( \phi(x) \neq 0 \) and 1 otherwise. \( B_2 \) is considered as a product of operators. Hence

\[ S = \frac{1}{R} T + B_1 A_1 + B_2 A_2 + B_3 A_3, \]

Define the families \( C(z), Q_0(z) \) as in the above cases. An application of Lemma 1 and Lemma 2 completes the proof.

**Parts c and d.** Suppose that \( \lambda > 0 \) and that \( g_1 \) is a real valued function. If \( z \in \sigma_p(D_{11}) \), then

\[ \frac{1}{R} Tf - ig_1 f = zf, \quad f \in \mathcal{D}(T) \setminus \{0\}, \]

so that

\[ \|f\|_2^2 \text{Im} (z) = -(g_1 f, f), \]
\[ \|f\|_2^2 \text{Re} (z) = \frac{1}{R} \|f'\|_2^2. \]

If \( z \in \sigma_p(D_{22}) \), then

\[ \frac{1}{R} Tf - ig_1 f - 2i\lambda G_i g_1 F_\lambda f + i g_1 F_2 f = zf, \quad f \in \mathcal{D}(T) \setminus \{0\}. \]
Hence

\[ \|f\|_2^2 \Re (z) = \frac{1}{R} \|f'\|_2^2 - \Im (g_1 F_2 f, f) \]

\[ > - \|f\|_2^2 \frac{1}{\lambda} \inf_{\rho \in [2, \infty]} \left( \frac{1}{2} \right)^{\frac{1}{p}} \|g_1'\|_p \]

which proves part c. This bound is somewhat weaker than those obtained in the bounded domain [5]; however, it does not require that \( g_1 \in L^\infty(0, \infty) \). Assuming, in addition, that \( g_1 \in \mathcal{A}^\infty \) and \( g_1' \in \mathcal{H} \) gives

\[ - \|f\|_2^2 \Im (z) = (g_1, f, f) + 2\lambda (g_1 F_2 f, F_2 f) - \Re (g_1' F_2 f, f) \]

\[ = (g_1, f, f) + 2\lambda (g_1 F_2 f, F_2 f) - \left( \left( \lambda g_1' - \frac{1}{2} g_1'' \right) F_2 f, F_2 f \right) \]

\[ = (g_1 (F_2 f)', (F_2 f)'), \left( \left( \lambda^2 g_1 + \frac{1}{2} g_1'' \right) F_2 f, F_2 f \right) \]

which proves part d. Note that this equality can also give bounds on \( \Im (z) \), which are similar to those in [5].

References


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