Friedrichs Extension and Galerkin Approximations for weakly nonlinear evolution equations

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Suppose that

(**S**1) $\mathcal{H}$ is a complex Hilbert space

(**S**2) $\phi_n \in \mathcal{H}$ for $n \geq 1$, $\mathcal{V}_n = \text{span}\{\phi_1, \ldots, \phi_n\}$ and $\cup_{n=1}^{\infty} \mathcal{V}_n$ is dense in $\mathcal{H}$

(**S**3) S is a linear operator in $\mathcal{H}$ with domain $\mathcal{D}(S) = \cup_{n=1}^{\infty} \mathcal{V}_n$ and such that there exist $r \in \mathbb{R}$ and $M \in (0, \infty)$ such that

$$|\text{Im}(Sx, x)| \leq M \text{Re}(Sx - rx, x) \quad \text{for all} \quad x \in \mathcal{D}(S).$$

Schechter [5] has shown that S has **Friedrichs extension**, which will be denoted by S_F. Integration of his proof with the finite element method gives the following Theorem A which is applicable to linear elliptic problems. For details see [2,3].

Theorem A *If* $f \in \mathcal{H}$ *and* $\lambda \in \mathbb{C}$ *with* $\text{Re}\lambda < r$*, then for each* $n \geq 1$ *there exists a unique* $x_n \in \mathcal{V}_n$ *such that*

$$(Sx_n - \lambda x_n, \phi_k) = (f, \phi_k) \quad \textit{for} \quad k = 1, \ldots, n,$$

moreover,

$$\lim_{n \to \infty} \|x_n - (S_F - \lambda)^{-1} f\| = 0.$$

The above x_n are the usual Galerkin approximations of x which satisfies $S_F x - \lambda x = f$. For examples see [1].

Suppose that the nonlinearity G satisfies

($\mathbf{S}4$) $T \in (0,\infty)$ and $G : [0,T] \times \mathcal{H} \to \mathcal{H}$ is continous and for some $L < \infty$ we have

$$\|G(t,x) - G(t,y)\| \le L\|x-y\| \quad \text{for} \quad x,y \in \mathcal{H},\ t \in [0,T].$$

For much weaker, but much more technical, conditions on G see [2].

Theorem B [3] *Choose any $u_0 \in \mathcal{H}$. For each $n \ge 1$ there exists a unique $u_n \in C^1([0,T],\mathcal{V}_n)$ such that $(u_n(0),\phi_k) = (u_0,\phi_k)$ and*

$$(u_n'(t),\phi_k) + (Su_n(t),\phi_k) = (G(t,u_n(t)),\phi_k) \quad \textit{for} \quad t \in [0,T],\ k = 1,\dots,n.$$

Moreover, there exists $u \in C([0,T],\mathcal{H})$ such that

$$\lim_{n\to\infty} \sup_{0\le t\le T} \|u(t) - u_n(t)\| = 0,$$

furthermore, this u is the unique element of $C([0,T],\mathcal{H})$ which satisfies

$$u(t) = e^{-S_F t}u_0 + \int_0^t e^{-S_F(t-s)}G(s,u(s))ds \quad \textit{for} \quad t \in [0,T].$$

The above u_n are the usual Galerkin approximations of u which is the mild solution of the parabolic type equation

$$u'(t) + S_F u(t) = G(t,u(t)) \quad \text{for} \quad t \in [0,T], \qquad u(0) = u_0.$$

To make the initial value problem for the wave type problems

$$v''(t) + S_F v(t) = G(t, v(t)) \quad \text{for} \quad t \in [0, T], \qquad v(0) = x_0, v'(0) = y_0 \tag{1}$$

well posed we assume that there exists $c < \infty$ such that

(**S**5) $|(Sx, y) - (x, Sy)| \le c(\mathrm{Re}(Sx, x) + (1 + r)\|x\|^2 + \|y\|^2)$ for all $x, y \in \mathcal{D}(S)$.

Theorem C [4] *Choose any $x_0, y_0 \in \mathcal{H}$. For each $n \ge 1$ there exists a unique $v_n \in C^2([0, T], \mathcal{V}_n)$ such that $(v_n(0), \phi_k) = (x_0, \phi_k)$, $(v_n'(0), \phi_k) = (y_0, \phi_k)$ and*

$$(v_n''(t), \phi_k) + (S v_n(t), \phi_k) = (G(t, v_n(t)), \phi_k) \quad \textit{for} \quad t \in [0, T],\ k = 1, \ldots, n.$$

Moreover, there exists $v \in C([0, T], \mathcal{H})$ such that

$$\lim_{n\to\infty} \sup_{0 \le t \le T} \|v_n(t) - v(t)\| = 0,$$

furthermore, this v is a mild solution of (1).

References

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