Norddeutsches Gruppentheorie-Kolloquium

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A Pushing Up Theorem

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work in progress

History

Aschbacher's local CGT Theorem

Baumann/Niles Pushing Up $SL_2(q)$

Gaubermann-Niles: A pair of characteristic subgroups for pushing up in finite groups.

Timmesfeld: A pushing up result and some consequences for the embedding of 2-constrained subgroups.

Timmesfeld: Simultaneous pushing up.

p-reduced normal subgroups

H has **characteristic** p if $C_H(O_p(H)) \leq O_p(H)$.

An elementary abelian normal p-subgroups V of H is called p-**reduced** if $O_p(H/C_H(V)) = 1$.

Lemma: Let H be a finite group of characteristic p and $T \in \mathrm{Syl}_p(H)$. Then

(a) There exists a unique **maximal** p-**reduced normal subgroup** Y_L of L.

(b) Let $T \leq R \leq H$ and X a p-reduced normal subgroup of R. Then $\langle X^H \rangle$ is a p-reduced normal subgroup of H. In particular, $Y_R \leq Y_H$.

(c) Let $T_H = C_T(Y_H)$ and $H_T = N_H(T_H)$. Then $H = H_T C_H(Y_H)$, $T_H = O_p(H_T)$ and $Y_H = \Omega_1 Z(T_H)$.

(d) $Y_T = \Omega_1 \mathrm{Z}(T)$, $Z_H := \langle \Omega_1 \mathrm{Z}(T)^H \rangle$ is p-reduced for H and $\Omega_1 \mathrm{Z}(T) \leq Z_H \leq Y_H$.

(e) Let V be p-reduced normal subgroup of H and K a subnormal subgroup of H. Then $[V, O^p(K)]$ is a p-reduced normal subgroup of K.

Point-Stabilizers

Let $T \in \mathsf{Syl}_p(H)$. Then

$$P_H(T) := O^{p'}(C_H(\Omega_1 \mathsf{Z}(T)))$$

is the **point-stabilizer** of H with respect to T.

Lemma: Let H be a finite group of characteristic p, $T \in \mathsf{Syl}_p(H)$ and L a subnormal subgroup of H. Then

(a) $C_L(\Omega_1 \mathsf{Z}(T)) = C_L(\Omega_1 \mathsf{Z}(T \cap L))$

(b) $P_L(T \cap L) = O^{p'}(P_H(T) \cap L)$

(c) $C_L(Y_L) = C_L(Y_H)$

(d) Suppose $L = \langle L_1, L_2 \rangle$ for some subnormal subgroups L_1, L_2 of H. Then

(da) $P_L(T \cap L) = \langle P_{L_1}(T \cap L_1), P_{L_2}(T \cap L_2) \rangle$.

(db) For $i = 1, 2$ let P_i be a point stabilizer of L_i. Then $\langle P_1, P_2 \rangle$ contains a point stabilizer of L.

Notation

For a p-group R we let $\mathcal{PU}_1(R)$ be the class of all finite $\mathcal{CK}$- groups L containing R such

i) L is of characteristic p,

ii) $R = O_p(N_L(R))$

iii) $N_L(R)$ contains a point stabilizer of L.

Let R be a group and Σ a set of groups containing R. Then

$$O_R(\Sigma) = \langle N \leq R \mid N \trianglelefteq L \forall L \in \Sigma \rangle$$

So $O_R(\Sigma)$ is the largest subgroup of R which is normal in all the $L \in \Sigma$.

For example if $R \in \mathrm{Syl}_p(G)$ and $R \leq L \leq G$ for all $L \in \Sigma$ then

$$O_R(\Sigma) = O_p(\langle \Sigma \rangle)$$

Goal

Given a finite p-group R and a subset Σ of $\mathcal{PU}_1(R)$. Suppose that $O_R(\Sigma) = 1$.

For each $L \in \Sigma$, determine the structure of $\langle \mathsf{B}(R)^L \rangle$.

Here $\mathsf{B}(R)$ is the **Baumann subgroup** of R defined as follows:

$\mathcal{A}(R)$ is the set of elementary abelian subgroups of maximal order in R.

$\mathsf{J}(S) = \langle \mathcal{A}(R) \rangle$ is the **Thompson subgroup** of R

$\mathsf{B}(R) = C_R(\Omega_1 \mathsf{Z}(\mathsf{J}(R)))$.

Example

Let G be a finite group of **local characteristic** p, that is all p-local subgroups of G have characteristic p.

Let $S \in \mathrm{Syl}_p(G)$, $1 \neq x \in \Omega_1 \mathrm{Z}(S)$ and $C = C_G(x)$. Then one of the following holds:

1. C is contained in a unique maximal p-local subgroup of G.

2. Put $R = O_p(C)$ and let Σ be the set of maximal p-locals subgroups of G containing C. Then

$$\Sigma \subseteq \mathcal{P}U_1(R) \text{ and } O_R(\Sigma) = 1$$

Complication

Let I be a finite set and for $i \in I$ let R_i be a finite p group and

$$\Sigma_i \subseteq \mathcal{PU}_1(R_i) \text{ with } O_{R_i}(\Sigma_i) = 1$$

Put

$$R = \bigtimes_{i \in I} R_i$$

and

$$\Sigma = \{ \bigtimes_{i \in I} L_i \mid L_i \in \Sigma_i \, \forall i \in I \}$$

Then

$$\Sigma \subseteq \mathcal{P}U_1(R) \text{ and } O_R(\Sigma) = 1$$

FF-modules

Theorem: Let H be a finite group, V a faithful, irreducible $\mathbb{F}_pH$-module, L a point stabilizer for H on V and $1 \neq A \leq O_p(L)$. Suppose that

i) $|V/C_V(A)| \leq |A/C_A(V)|$.

ii) $F^*(H)$ is quasi-simple and $H = \langle A^H \rangle$.

Then $H \cong SL_n(q)$, $Sp_{2n}(q)$, $G_2(q)$ or Sym(n), where $p = 2$ in the last two cases and $n \equiv 2, 3$ (mod 4) in the last case. Moreover, V is the corresponding natural module.

The Baumann Argument

Lemma: Let L be a finite group, R a p-subgroup of L, $V := \Omega_1 \mathrm{Z}(O_p(L))$, $K := \langle \mathrm{B}(R)^L \rangle$, $\tilde{V} = V/C_V(O^p(K))$, and suppose that each of the following holds:

i) $O_p(L) \leq R$ and $L = \langle \mathrm{J}(R)^L \rangle N_L(\mathrm{J}(R))$.

ii) $C_K(\tilde{V})$ is p-closed.

iii) $|\tilde{V}/C_{\tilde{V}}(A)| \geq |A/C_A(\tilde{V})|$ for all elementary abelian subgroups A of R.

iv) If U is an $L/O_p(L)$ module with $\tilde{V} \leq U$ and $U = C_U(\mathrm{B}(R))\tilde{V}$, then $U = C_U(O^p(K))\tilde{V}$.

Then $O_p(K) \leq \mathrm{B}(R)$.

The Reduction Theorem

Let $\mathcal{PU}_3(R)$ be the class of all finite $\mathcal{CK}$-groups L such that

i) L is of characteristic p.

ii) $R \leq L$ and $L = \langle R^L \rangle$

iii) $L/O_p(L) \cong SL_n(q), Sp_{2n}(q)$ or $G_2(q)$, where q is a power of p and $p = 2$ in the last case.

iv) $Y_L/C_{Y_L}(L)$ is the corresponding natural module.

v) $O_p(L) \leq R$ and $N_L(R)$ contains a point-stabilizer of L.

vi) If $L/C_L(Y_L) \ncong G_2(q)$ then $R = O_p(N_L(R))$

Let $\mathcal{PU}_4(R)$ be the class of all finite groups L containg R such that L is of characteristic p and

$$L = \langle N_L(R), H \mid R \leq H \leq L, H \in \mathcal{PU}_3(R) \rangle.$$

Theorem: Let R be a p-group. Then

$$\mathcal{PU}_1(R) \subseteq \mathcal{PU}_4(\mathsf{B}(R)).$$

The First Pushing Up Theorem

Theorem: Let T be a p-group with $T = \mathrm{B}(T)$ and Σ a subset of $\mathcal{PU}_3(T)$ with $O_R(\Sigma) = 1$.

Then there exists $H \in \Sigma$ such $O^p(H)$ has one of the following structures (where q is a power of p)

$q^n SL_n(q)'$;

$q^{2n} Sp_{2n}(q)', p$ odd;

$q^{1+2n} Sp_{2n}(q)', p = 2$;

$2^6 G_2(2)', p = 2$;

$q^{1+6+8} Sp_6(q), p = 2$;

$2^{1+4+6} L_4(2), p = 2$; or

$q^{1+2+2} SL_2(q)', p = 3$.

On the Proof

Lemma: Let $L, H \in \Sigma$.

(a) If $L/O_p(L) \not\cong Sp_{2n}(q), n \geq 2$ and $Y_L \not\leq Y_H$, then $[Y_L, R] \leq Z(H)$.

(b) If $L/O_p(L) \cong Sp_{2n}(q), n \geq 2$ and $[Y_L, R] \not\leq Y_H$, then $[Y_L, R, R] \leq Z(H)$.

(c) If $L/O_p(L) \cong Sp_{2n}(q), n \geq 2$ and N is a normal p-subgroup of H with $[Y_L, R, N] = 1$, then $[Y_L, N] \leq Z(H)$

Proposition Let $L, H \in \Sigma$. Then one of the following holds

1. $O_p(H) \cap O_p(L)$ is normal in H and L.

2. $\langle {Y_H}^L \rangle$ is not abelian.

3. $\langle {Y_L}^H \rangle$ is not abelian.

Proposition: There exists H, L in Σ such that $\langle {Y_H}^L \rangle$ is not abelian.

Proof: Otherwise $\bigcap_{H \in \Sigma} O_p(H)$ is normal in each $L \in \Sigma$.

Also $\Omega_1 Z(R) \leq \bigcap_{H \in \Sigma} O_p(H)$ and

$$\bigcap_{H \in \Sigma} O_p(H) \neq 1.$$

A guess

Let R be a finite p-group, $\Sigma \subseteq \mathcal{PU}_1(R)$ with $O_R(\Sigma) = 1$ and $L \in \Sigma$. Then we guess that there exist subgroups $L_i, i \in I$, of L such that

$$\langle \mathsf{B}(R)^L \rangle = \bigcurlywedge_{i \in I} L_i$$

and for $i \in I$, $O^p(L_i)$ has one of the following structures:(where q is a power of p)

$q^n SL_n(q)'$;

$q^{2n} Sp_{2n}(q)', p$ odd;

$q^{1+2n} Sp_{2n}(q)', p = 2$;

$2^6 G_2(2)', p = 2$;

$q^{1+6+8} Sp_6(q), p = 2$;

$2^{1+4+6} L_4(2), p = 2$;

$q^{1+2+2} SL_2(q)', p = 3$.

$2^{n-1}\,\mathrm{Alt}(n)$, $p = 2$, $n \equiv 2, 3 \pmod 4$

$2^{1+2+1\cdot 2m+2\cdot 2k} SL_2(2)'$, $p = 2$.

maybe a case similar to the last one with $SL_2(2)$ replaced by $Sp_{2n}(2)$.