Nearly Quadratic Modules

Work in progress

joint with

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Definitions

Let A be an abelian group, $\mathbb{F}$ a field and V an $\mathbb{F}A$-module.

The module V is a

quadratic $\mathbb{F}A$-module if $[V, A, A] = 0$,

cubic $\mathbb{F}A$-module if $[V, A, A, A] = 0$,

nearly quadratic $\mathbb{F}A$-module if V is a cubic $\mathbb{F}A$-module and

$$[V, A] + C_V(A) = [\mathbb{F}v, A] + C_V(A)$$

for every $v \in V \setminus [V, A] + C_V(A)$.

In the corresponding cases we also say that A is quadratic, cubic and nearly quadratic on V.

$\mathbf{Q}_V(A)$ is the largest quadratic $\mathbb{F}A$-submodule of V.

Motivation

Let G be a finite group, p a prime and V an elementary abelian p-subgroup of G. Suppose that

(i) $\mathsf{C}_G(\mathsf{O}_p(G)) \leq \mathsf{O}_p(G)$

(ii) $V \nleq \mathsf{O}_p(G)$.

(iii) V is weakly closed in G.

Choose $V \leq L \leq G$ minimal with $V \nleq \mathsf{O}_p(L)$.

Put $A := \langle (V \cap \mathsf{O}_p(L))^L \rangle$. Then $[V, A] \neq 1$ and

A is nearly quadratic on V

Lemma 1 *Let V be a nearly quadratic $\mathbb{F}A$-module and W be an $\mathbb{F}A$-submodule of V. Then*

(a) Either

$$W \leq [V, A] + C_V(A)$$

or

$$[V, A] \leq [W, A] + C_V(A).$$

(b) One of the following holds:

1. A acts quadratically on V, $Q_V(A) = V$ and

$$[V, A] + C_V(A) = C_V(A).$$

2. A does not act quadratically and

$$Q_V(A) = [V, A] + C_V(A).$$

(c) Both W and V/W are nearly quadratic $\mathbb{F}A$-modules.

(d) A acts quadratically on W or on V/W.

Lemma 2 *Let V be a nearly quadratic, but not quadratic $\mathbb{F}A$-module. Let X and Y be $\mathbb{F}A$-submodules of V such that*

$$V = X \oplus Y$$

Then A centralizes X or Y.

Lemma 3 *Let V be a faithful, nearly quadratic, but not quadratic $\mathbb{F}A$-module, and let Δ be a system of imprimitivity for A on V. Suppose that A acts non-trivially on Δ.*

Then exists a unique A-orbit $W^A \subseteq \Delta$ with $[W, A] \neq 0$. Moreover $C_A(W)$=1, $B := N_A(W)$ acts quadratically on V, and one of the following holds:

1. $\operatorname{char} \mathbb{F} = 2$, $|W^A| = 4$, $\dim_{\mathbb{F}} W = 1$, $B = 1$ *and* $A \cong C_2 \times C_2$.

2. $\operatorname{char} \mathbb{F} = 3$, $|W^A| = 3$, $\dim_{\mathbb{F}} W = 1$, $B = 1$, *and* $A \cong C_3$.

3. $\operatorname{char} \mathbb{F} = 2$, $|W^A| = 2$, *and*

$$[W, B] = C_W(B)$$

is an $\mathbb{F}$-hyperplane of W.

Lemma 4 *Let* $\mathbb{K}$ *be a field,* A *a group and* V_1 *and* V_2 *be nilpotent* $\mathbb{K}A$*-modules and*

$$V = V_1 \otimes_{\mathbb{K}} V_2.$$

Suppose that there exists a subfield $\mathbb{F} \leq \mathbb{K}$ *such that* V *is a nearly quadratic, but not quadratic* $\mathbb{F}A$*-module. Then for* $j = 1, 2$

$$[V_j, A] = C_{V_j}(A)$$

is a $\mathbb{K}$*-hyperplane of* V_j*.*

Lemma 5 *Let V be faithful, nearly quadratic,not quadratic $\mathbb{F}A$-module. Suppose that there exists a field $\mathbb{K}$ with $\mathbb{F} \leq \mathbb{K}$ such that V is a semi-linear, but not linear, $\mathbb{K}A$-module. Put $A_{\mathbb{K}} = C_A(\mathbb{K})$. Then $p := \operatorname{char} \mathbb{K} \in \{2, 3\}$, A is a elementary abelian p-group, $\dim_{\mathbb{F}} \mathbb{K} = |A/A_{\mathbb{K}}|$ and one of the following holds:*

1. $A \cong C_3$, $A_{\mathbb{K}} = 1$ *and* $\dim_{\mathbb{K}} V = 1$.

2. $A \cong C_2 \times C_2$, $A_{\mathbb{K}} = 1$ *and* $\dim_{\mathbb{K}} V = 1$.

3. $|A/A_{\mathbb{K}}| = 2$ *and* $[V, A_{\mathbb{K}}] = C_V(A_{\mathbb{K}})$ *is a* $\mathbb{K}$*-hyperplane of* V.

Theorem 6 *Let $\mathbb{F}$ be field, H a group and V be a faithful semisimple $\mathbb{F}H$-module. Let $\mathcal{Q}$ be the set of nearly quadratic, but not quadratic subgroups of H. Suppose that $H = \langle \mathcal{Q} \rangle$. Then there exists a partition $(\mathcal{Q}_i)_{i \in I}$ of $\mathcal{Q}$ such that*

(a) $H = \bigoplus_{i \in I} H_i$, *where* $H_i = \langle \mathcal{Q}_i \rangle$.

(b) $V = C_V(H) \oplus \bigoplus_{i \in I} [V, H_i]$.

(c) For each $i \in I$, $[V, H_i]$ *is a simple* $\mathbb{F}H_i$*-module.*

Theorem 7 * *Let H be a finite group, and V a faithful simple $\mathbb{F}_p H$-module. Suppose that H is generated by elementary abelian, nearly quadratic, but not quadratic subgroups of H.*

Let W a simple $\mathbb{F}_p\,\mathsf{F}^*(H)$-submodule of V and

$$\mathbb{K} = \mathsf{End}_{\mathsf{F}^*(H)}(W).$$

Then $H, V, W, \mathbb{K}$ and $H/C_H(\mathbb{K})$ are as in the following table:

H	V	W	$\mathbb{K}$	$H/C_H(\mathbb{K})$
$(C_2 \wr \mathsf{Sym}(n))'$	$\mathbb{F}_3^n$	$\mathbb{F}_3$	$\mathbb{F}_3$	–
$\mathsf{SL}_n(\mathbb{F}_2) \wr C_2$	$\mathbb{F}_2^n \oplus \mathbb{F}_2^n$	$\mathbb{F}_2^n$	$\mathbb{F}_2$	–
$\mathsf{SL}_2(\mathbb{F}_2) \times \mathsf{SL}_2(\mathbb{F}_2)$	$\mathbb{F}_2^2 \otimes \mathbb{F}_2^2$	$\mathbb{F}_4$	$\mathbb{F}_4$	–
$\mathsf{Frob}(39)$	$\mathbb{F}_{27}$	V	$\mathbb{F}_{27}$	C_3
$\Gamma\mathsf{GL}_n(\mathbb{F}_4)$	$\mathbb{F}_4^n$	V	$\mathbb{F}_4$	C_2
$\Gamma\mathsf{SL}_n(\mathbb{F}_4)$	$\mathbb{F}_4^n$	V	$\mathbb{F}_4$	C_2
$3\cdot\mathsf{Sym}(6)$	$\mathbb{F}_4^3$	V	$\mathbb{F}_4$	C_2
$\mathsf{SL}_n(\mathbb{K}) \circ \mathsf{SL}_m(\mathbb{K})$	$\mathbb{K}^n \otimes \mathbb{K}^m$	V	*any*	1
$(C_2 \wr \mathsf{Sym}(4))'$	$\mathbb{F}_3^4$	V	$\mathbb{F}_3$	1
$\mathsf{F}^*(H)$ *quasisimple*	?	V	?	1

* In the cases where $|H/C_H(\mathbb{K})| = 2$ we currently use the FF-module Theorem (and so a $\mathcal{K}$-group assumption) to identify H

Some examples for the last case (under stronger assumptions, including 2F and $\mathcal{K}$-group, these are all the examples.)

H	V	$\mathbb{K}$
$\mathrm{SL}_n(\mathbb{K})$	$\mathbb{K}^n$	any
$\mathrm{Sp}_{2n}(\mathbb{K})$	$\mathbb{K}^{2n}$	any
$\Omega_n^\epsilon(\mathbb{K})$	$\mathbb{K}^n$	any
$\mathrm{SL}_n(\mathbb{K})$	$\bigwedge^2 \mathbb{K}^n$	any
$\mathrm{SL}_n(\mathbb{K})$	$\mathrm{S}^2 \mathbb{K}^n$	char $\mathbb{K}$ odd
$\mathrm{Spin}_{10}^+(\mathbb{K})$	$\mathbb{K}^{16}$, spin	any
$\mathrm{E}_6(\mathbb{K})$	$\mathbb{K}^{27}$	any
Mat_{22},	$\mathbb{F}_2^{10}$	$\mathbb{F}_2$
Mat_{24}	$\mathbb{F}_2^{11}$	$\mathbb{F}_2$
Mat_{11}	$\mathbb{F}_3^5$	$\mathbb{F}_3$
$2\cdot \mathrm{Mat}_{12}$	$\mathbb{F}_3^6$	$\mathbb{F}_3$