The Structure Theorem and Its Spinoffs

work in progress

joint with

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Groups of local characteristic p

Let G be finite group and p a prime.

Definition 1 *G has* **characteristic** *p if*

$$C_G(O_p(G)) \leq O_p(G).$$

*H is a p-***local subgroup** *of G if $H = N_G(P)$ for some non-trivial p-subgroup of G.*

G has **local characteristic** *p if all p-local subgroups of G have characteristic p.*

*G is a $\mathcal{K}_p$-***group** *if the simple sections of the p-locals subgroups of G are known finite simple groups.*

Goals:

1. Understand the finite groups of local characteristic p.

2. Classify $\mathcal{K}_p$-groups of local characteristic p whose p-local structure is not too small.

The Structure Theorem

Definition 2 *Let H be group, $\mathbb{F}$ a field and V an $\mathbb{F}H$-module.*

(a) H acts **nilpotently** *on V if there exists an ascending series*

$$0 = V_0 \leq V_1 \leq V_2 \ldots, V_{n-1} \leq V_n$$

of $\mathbb{F}H$-submodules of V such that H centralizes each of the factor V_{i+1}/V_i.

*(b) V is H-***reduced** *if $[V, N] = 0$ whenever $N \trianglelefteq H$ and N acts nilpotently on V.*

(c) If H is finite, then the largest elementary abelian normal H-reduced p-subgroup of H is denoted by Y_H.

Definition 3 *Let A and B be subgroups of G. The relation $\ll$ on the subgroups of G is defined by*

$$A \ll B :\Longleftrightarrow A \subseteq C_G(Y_A)B \text{ and } Y_A \leq Y_B.$$

Furthermore, we define

$$A^\dagger := C_G(Y_A)A$$

$$\mathcal{S}^\dagger = \{L \leq G \mid L = L^\dagger\} = \{L \leq G \mid C_G(Y_L) \leq L\}$$

Lemma 4 *(a) For all $L \leq G$, $A \ll A^\dagger$ and $A^\dagger \in \mathcal{S}^\dagger$.*

(b) $\ll$ is reflexive and transitive.

(c) Restricted to $\mathcal{S}^\dagger$, $\ll$ is a partial ordering.

Definition 5 *$\mathcal{S}^\dagger(S) = \{L \in \mathcal{S}^\dagger \mid S \leq L\}$ and $\mathcal{F}(S)$ is the set of maximal elements of $\ll$ in $\mathcal{S}^\dagger(S)$.*

Definition 6 *Let Q be a p-subgroup of a finite group G. We say that Q is* **large subgroup** *of G provided that $C_G(Q) \leq Q$ and*

$$Q \trianglelefteq N_G(A)$$

for all $1 \neq A \leq Z(Q)$.

Theorem 7 (Structure Theorem)

Let p be a prime, G be a finite $\mathcal{K}_p$-group of local characteristic p. Suppose that Q is a large p-subgroup of G and $Q \leq S \in \mathrm{Syl}_p(G)$. Let $M \in \mathcal{F}(S)$ with $Q \ntrianglelefteq M$. Put $M^\circ = \langle Q^M \rangle$, $\overline{M} = M/C_M(Y_M)$ and $I = [Y_M, M^\circ]$.

Suppose that $Y_M \leq O_p(N_G(Q))$. Then one the following holds.

1. *$\overline{M^\circ} \cong SL_n(q), Sp_{2n}(q)$ or $Sp_4(2)'$ and I is the corresponding natural module.*

2. *There exists a normal subgroup K of $\overline{M}$ such that*

   (a) *$K = K_1 \times \cdots \times K_r$, $K_i \cong Sl_2(q)$ and*

   $$Y_M = V_1 \times \cdots \times V_r$$

   where $V_i := [Y_M, K_i]$ is a natural K_i-module.

   (b) *Q permutes the K_i's transitively.*

Suppose that $Y_M \not\leq O_p(N_G(Q))$. Then one of the following holds:

(a) There exists a normal subgroup K of $\overline{M}$ such that $K = K_1 \circ K_2$ with $K_i \cong SL_{m_i}(q)$, $Y_M \cong V_1 \otimes V_2$ where V_i is a natural module for K_i and $\overline{M^\circ}$ is one of K_1, K_2 or $K_1 \circ K_2$.

(b) $(\overline{M^\circ}, p, I)$ is as given in the following table:

$\overline{M^\circ}$	p	I
$\mathrm{SL}_n(q)$	p	natural
$\mathrm{SL}_n(q)$	p	$\bigwedge^2$(natural)
$\mathrm{SL}_n(q)$	p	S^2(natural)
$\mathrm{SL}_n(q^2)$	p	natural $\otimes$ natural^q
$3\,\mathrm{Alt}(6), 3\,\mathrm{Sym}(6)$,	2	2^6
$\Gamma\mathrm{SL}_2(4), \Gamma\,\mathrm{GL}_2(4)$	2	natural
$\mathrm{Sp}_{2n}(q)$	2	natural
$\Omega_n^{\pm}(q)$	p	natural
$\mathrm{O}_4^{+}(2)$	2	natural
$\Omega_{10}^{\pm}(q)$	2	half-spin
$\mathrm{E}_6(q)$	p	q^{27}
Mat_{11}	3	3^5
$2\,\mathrm{Mat}_{12}$	3	3^6
Mat_{22}	2	2^{10}
Mat_{24}	2	2^{11}

$2F$-**stability**

Definition 8 *Let A be an elementary abelian p-group and V a finite dimensional $GF(p)A$-module. Then A is*

(a) **quadratic** *on V if $[V, A, A] = 0$,*

(b) **cubic** *on V if $[V, A, A, A] = 0$,*

(c) **nearly quadratic** *on V if A is cubic and*

$$[V, A] + C_V(A) = [v, A] + C_V(A)$$

for every $v \in V \setminus [V, A] + C_V(A)$,

(d) an **offender** *on V if $|V/C_V(A)| \leq |A/C_A(V)|$,*

*(e) a $2F$-***offender** *if $|V/C_V(A)| \leq |A/C_A(V)|^2$,*

(f) **non-trivial** *on V if $[V, A] \neq 0$.*

*Let A be an elementary abelian p-subgroup A of G. Then A is F-***stable** *in G if none of the elementary abelian p-subgroups of $N_G(A)/C_G(A)$ are non-trivial offenders on A.*

*Similarly, A is $2F$-***stable** *in G if none of the elementary abelian p-subgroups of $N_G(A)/C_G(A)$ are non-trivial nearly quadratic $2F$-offenders on A.*

Let H be a finite group, p a prime and V an elementary abelian p-subgroup of H. Suppose that

(i) H is of characteristic p.

(ii) $V \not\leq \mathrm{O}_p(H)$.

(iii) V is weakly closed in H.

Choose $V \leq L \leq H$ minimal with $V \not\leq \mathrm{O}_p(L)$.

Put $A := \langle (V \cap \mathrm{O}_p(L))^L \rangle$. Then $[V, A] \neq 1$ and

A is a nearly quadratic $2F$-offender on V

Definition 9 *Let S be Sylow p-subgroup of G.*

$$B(S) := C_S(\Omega_1 \mathrm{Z}(J(S)))$$

$$C^*(G,S) := \langle C_G(\Omega_1 \mathrm{Z}(S))), N_G(C) \mid 1 \neq C \text{ char } B(S) \rangle$$

Definition 10 *Let G be a finite group and $H \leq G$.*

(a) *H is called a* **parabolic subgroup** *of G if H contains a Sylow p-subgroup of G.*

(b) *G has* **parabolic characteristic** *p if all p-local, parabolic subgroups of G have characteristic p.*

Theorem 11 *Let G be a finite group of parabolic characteristic p and $S \in \mathrm{Syl}_p(G)$. Suppose $M \in \mathcal{F}(S)$ such that Y_M is $2F$-stable. Then*

(a) *$C^*(G,S) \leq M$.*

(b) *$C^*(H,T) \leq H \cap M < H$ for all $H \leq G$ with $B(S) \leq H$ and $H \not\leq N$, where $B(S) \leq T \in \mathrm{Syl}_p(H)$.*

(c) *If $N \in \mathcal{F}(S)$ with $N \neq M$, then Y_N is not F-stable.*

Corollary 12 *Let G be a finite group of parabolic characteristic p and $S \in \mathrm{Syl}_p(G)$. If S is contained in at least two maximal p-local subgroups of G, then there exists $M \in \mathcal{F}(S)$ such that Y_M is not $2F$-stable.*

The Fitting Submodule

Let $\mathbb{F}$ be a field, H a finite group and V a finite dimensional $\mathbb{F}H$-module.

Definition 13

(a) $\mathrm{rad}_V(H)$ *is the intersection of the maximal* $\mathbb{F}H$-*submodules of* V

(b) Let W *be an* $\mathbb{F}H$ *submodule of* V *and* $N \trianglelefteq H$*. Then* W *is* N-**quasisimple** *if* W *is* H*-reduced,* $W/\mathrm{rad}_W(H)$ *is simple for* $\mathbb{F}H$*,* $W = [W, N]$ *and* N *acts nilpotently on* $\mathrm{rad}_W(H)$*.*

(c) $\mathrm{S}_V(H)$ *is the sum of all simple* $\mathbb{F}H$*-submodules of* V*.*

(d) $\mathrm{E}_H(V) := \mathrm{C}_{\mathrm{F}^*(H)}(\mathrm{S}_V(H))$*.*

(e) W *is a* **component** *of* V *if either* W *is a simple* $\mathbb{F}H$*-submodule with* $[W, \mathrm{F}^*(H)] \neq 0$ *or* W *is an* $\mathrm{E}_H(V)$*-quasisimple* $\mathbb{F}H$*-submodule.*

(f) The **Fitting submodule** $\mathrm{F}_V(H)$ *of* V *is the sum of all components of* V*.*

(g) $\mathrm{R}_V(H) := \sum \mathrm{rad}_W(H)$*, where the sum runs over all components* W *of* V

Theorem 14 *(a) The Fitting submodule* $\mathsf{F}_V(H)$ *is* H*-reduced.*

(b) $\mathsf{R}_V(H)$ *is a semisimple* $\mathbb{F}\,\mathsf{F}^*(H)$*-module.*

(c) $\mathsf{R}_V(H) = \mathsf{rad}_{\mathsf{F}_V(H)}(H)$*.*

(d) $\mathsf{F}_V(H)/\,\mathsf{R}_V(H)$ *is a semisimple* $\mathbb{F}H$*-module*

Theorem 15 *Let* V *be faithful and* H*-reduced. Then also* $\mathsf{F}_V(H)$ *and* $\mathsf{F}_V(H)/\,\mathsf{R}_V(H)$ *are faithful and* H*-reduced.*

Definition 16 *Let A be a subgroup of G such that $A/C_A(V)$ is an elementary abelian p-group. A is a* **best offender** *of G on V if $|B|\cdot|C_V(B)| \leq |A|\cdot|C_V(A)|$ for every $B \leq A$.*

Definition 17 *The normal subgroup of G generated by the best offenders of G on V is denoted by $J_G(V)$.*

*A $\mathsf{J}_G(V)$-***component** *is non-trivial subgroup K of $\mathsf{J}_G(V)$ minimal with respect to $K = [K, \mathsf{J}_G(V)]$.*

Theorem 18 (The Other $\mathcal{P}(G, V)$-Theorem.)

Suppose that V is a faithful finite dimensional, reduced $\mathbb{F}_pG$-module. Then

$$[E, K] = 1 \text{ and } [V, E, K] = 0$$

for any two distinct $\mathsf{J}_G(V)$-components E and K.

Definition 19 *A finite group is a $\mathcal{CK}$-group if all its compositions factors are known finite simple groups.*

Theorem 20 (FF-Module Theorem, Guralnick-Malle) *Let M be a finite $\mathcal{CK}$ group with $\mathsf{F}^*(M)$ quasisimple and V a faithful simple $\mathbb{F}_p M$-module. Suppose that $M = \mathsf{J}_M(V)$.*

Then (M, p, V) is one of the following:

M	p	V
$\mathrm{SL}_n(q)$	p	natural
$\mathrm{Sp}_{2n}(q)$	p	natural
$\mathrm{SU}_n(q)$	p	natural
$\Omega_n^\epsilon(q)$	p	natural
$\mathrm{O}_{2n}^\epsilon(q)$	2	natural
$\mathrm{G}_2(q)$	2	q^6
$\mathrm{SL}_n(q)$	p	$\bigwedge^2$(natural)
$\mathrm{Spin}_7(q)$	p	Spin
$\mathrm{Spin}_{10}^+(q)$	p	Spin
3. Alt(6)	2	2^6
Alt(7)	2	2^4
$\mathrm{Sym}(n)$	2	natural
$\mathrm{Alt}(n)$	2	natural

Theorem 21 (J-Module Theorem) *Let M be a finite $\mathcal{CK}$-group, V a faithful, reduced $\mathbb{F}_pM$-module. Put $J = \mathsf{J}_V(M)$ and let $\mathcal{J} = \mathcal{J}_V(M)$ be the set of J_V-components of V. Put $W = [V, \mathcal{J}]C_V(\mathcal{J})/C_V(\mathcal{J})$ and let $K \in \mathcal{J}$.*

(a) K *is either quasisimple or* $p = 2$ *or* 3 *and* $K \cong SL_2(p)'$.

(b) $[V, K, L] = 0$ *for all* $K \neq L \in \mathcal{J}$.

(c) $W = \bigoplus_{K \in \mathcal{J}} [W, K]$.

(d) $J^p J' = 0^p(J) = \mathsf{F}^*(J) = \bigtimes \mathcal{J}$.

(e) W *is a semisimple* $\mathbb{F}_pJ$*-module.*

(f) Let $J_K = J/C_J([W,K])$. Then $K \cong O^p(J_K)$ and one of the following holds:

1. $[W,K]$ is a simple K-module and $(J_K,[W,K])$ fullfills the assumptions and so also the conclusion of Theorem 20.

2. J_K and $[W,K]$ are as follows (where N denotes a natural module and N^* its dual):

J_K	$[W,K]$	conditions
$\mathsf{SL}_n(q)$	$N^r \oplus N^{*s}$	$\sqrt{r}+\sqrt{s} \le \sqrt{n}$
$\mathsf{Sp}_{2n}(q)$	N^r	$r \le n$
$\mathsf{SU}_n(q)$	N^r	$r \le \frac{n}{4}$
$\Omega_n^\epsilon(q)$	N^r	$r \le \frac{n-2}{4}$
$\mathsf{O}_{2n}^\epsilon(q)$	N^r	$p=2, r \le \frac{2n-2}{4}$

Nearly Quadratic Modules

Lemma 22 *Let V be a nearly quadratic, but not quadratic $\mathbb{F}A$-module. Let X and Y be $\mathbb{F}A$-submodules of V such that*

$$V = X \oplus Y$$

Then A centralizes X or Y.

Theorem 23 *Let $\mathbb{F}$ be field, H a group and V be a faithful semisimple $\mathbb{F}H$-module. Let $\mathcal{Q}$ be the set of nearly quadratic, but not quadratic subgroups of H. Suppose that $H = \langle \mathcal{Q} \rangle$. Then there exists a partition $(\mathcal{Q}_i)_{i \in I}$ of $\mathcal{Q}$ such that*

(a) $H = \bigoplus_{i \in I} H_i$, *where* $H_i = \langle \mathcal{Q}_i \rangle$.

(b) $V = C_V(H) \oplus \bigoplus_{i \in I} [V, H_i]$.

(c) *For each $i \in I$, $[V, H_i]$ is a simple $\mathbb{F}H_i$-module.*

Theorem 24 *Let H be a finite group, and V a faithful simple $\mathbb{F}_pH$-module. Suppose that H is generated by nearly quadratic, but not quadratic subgroups of H.*

Let W a Wedderburn-component for $\mathbb{F}_p\,\mathsf{F}^*(H)$ in V and $\mathbb{K} := \mathsf{Z}(\mathsf{End}_{\mathsf{F}^*(H)}(W))$. Then W is a simple $\mathbb{F}_p\mathbb{F}^*(H)$-module and one of the following holds for $H, V, W, \mathbb{K}$ and (if $V = W$) $H/C_H(\mathbb{K})$

H	V	W	$\mathbb{K}$	$H/C_H(\mathbb{K})$	
$(C_2 \wr \mathsf{Sym}(m))'$	$\mathbb{F}_3^m$	$\mathbb{F}_3$	$\mathbb{F}_3$	$-$	$m \geq 3, m \neq 4$
$\mathsf{SL}_n(\mathbb{F}_2) \wr \mathsf{Sym}(m)$	$(\mathbb{F}_2^n)^m$	$\mathbb{F}_2^n$	$\mathbb{F}_2$	$-$	$m \geq 2, n \geq 3$
$\mathsf{Wr}(\mathsf{SL}_2(\mathbb{F}_2), m)$	$(\mathbb{F}_2^n)^m$	$\mathbb{F}_2^n$	$\mathbb{F}_4$	$-$	$m \geq 2$
$\mathsf{Frob}(39)$	$\mathbb{F}_{27}$	V	$\mathbb{F}_{27}$	C_3	
$\Gamma\,\mathsf{GL}_n(\mathbb{F}_4)$	$\mathbb{F}_4^n$	V	$\mathbb{F}_4$	C_2	$n \geq 2$
$\Gamma\mathsf{SL}_n(\mathbb{F}_4)$	$\mathbb{F}_4^n$	V	$\mathbb{F}_4$	C_2	$n \geq 2$
$\mathsf{SL}_2(\mathbb{F}_2) \times \mathsf{SL}_n(\mathbb{F}_2)$	$\mathbb{F}_2^2 \otimes \mathbb{F}_2^n$	V	$\mathbb{F}_4$	C_2	$n \geq 3$
$3^{\cdot}\,\mathsf{Sym}(6)$	$\mathbb{F}_4^3$	V	$\mathbb{F}_4$	C_2	
$\mathsf{SL}_n(\mathbb{K}) \circ \mathsf{SL}_m(\mathbb{K})$	$\mathbb{K}^n \otimes \mathbb{K}^m$	V	*any*	1	$n, m \geq 3$
$\mathsf{SL}_2(\mathbb{K}) \circ \mathsf{SL}_m(\mathbb{K})$	$\mathbb{K}^2 \otimes \mathbb{K}^m$	V	$\mathbb{K} \neq \mathbb{F}_2$	1	$m \geq 2$
$\mathsf{SL}_n(\mathbb{F}_2) \wr C_2$	$\mathbb{F}_2^n \otimes \mathbb{F}_2^n$	V	$\mathbb{F}_2$	1	$n \geq 3$
$(C_2 \wr \mathsf{Sym}(4))'$	$\mathbb{F}_3^4$	V	$\mathbb{F}_3$	1	
$\mathsf{SU}_3(2)'$	$\mathbb{F}_4^3$	V	$\mathbb{F}_4$	1	
$\mathsf{F}^*(H) = \mathsf{Z}(H)K$ K *quasisimple*	?	V	?	1	

Moreover, in case $\mathsf{SU}_3(2)$ case, H is not generated by abelian, nearly quadratic subgroups.

(Here $\mathsf{Wr}(L, m)$ is the normal closure of $\mathsf{Sym}(m)$ in $L \wr \mathsf{Sym}(m)$)