The Other $\mathcal{P}(G, V)$-Theorem

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1 Introduction

Suppose that H is a finite group, $T \in Syl_p(H)$, p any prime, and V is an elementary abelian normal p-subgroup of H. Then an elementary Frattini argument shows that either $H = C_H(V)N_H(J(T))$ or $J(T) \not\leq C_H(V)$, where $J(T)$ denotes the Thompson subgroup of T.

In this paper we are interested in this second case. One of the questions is how $\overline{J(T)} := J(T)C_H(V)/C_H(V)$ is embedded in $\overline{H} := H/C_H(V)$. The first problem is to find suitable properties of the Thompson subgroup $J(T)$ that can be expressed in terms of $\overline{H}$ only. This is done in the following way.

Recall that $J(T)$ is generated by the elementary abelian subgroups A of maximal order of T. It is evident that $BC_V(B)$ is elementary abelian for every subgroup $B \leq A$. Hence by the maximality of $|A|$

$$|A| \geq |BC_V(B)| = |B||C_V(B)||V \cap B|^{-1} \geq |B||C_V(B)||C_V(A)|^{-1}.$$

This gives rise to the condition

$$(*) \qquad |B||C_V(B)| \leq |A||C_V(A)| \text{ for every } B \leq A.$$

Note that $B := C_A(V)$ yields an important special case of $(*)$:

$$(**) \qquad |V/C_V(A)| \leq |A/C_A(V)|.$$

Both conditions $(*)$ and $(**)$ can be phrased in terms of the factor group $\overline{H}$ and the $GF(p)\overline{H}$-module V just by replacing A by its image $\overline{A}$ in $\overline{H}$. Evidently A satisfies $(*)$ with respect to V and H iff $\overline{A}$ satisfies $(*)$ with respect to V and $\overline{H}$.

This consideration gives rise to the following definition in the more general set up of a finite group G and a finite dimensional $GF(p)G$-module V.

Definition 1.1 *Let A be a subgroup of G such that $A/C_A(V)$ is an elementary abelian p-group. Then A is an offender of G on V if $|V/C_V(A)| \leq |A/C_A(V)|$; and A is a best offender of G on V if $|B||C_V(B)| \leq |A||C_V(A)|$ for every $B \leq A$. The normal subgroup of G generated by the best offenders of G on V is denoted by $J_G(V)$.*

In the literature, at least in the case of a faithful $GF(p)G$-module V, the set of best offenders is denoted by $\mathcal{P}(G, V)$.

The classical result about $J_G(V)$ is the following:

The $\mathcal{P}(G, V)$-Theorem. *Suppose that V is a faithful finite dimensional $GF(p)G$-module and that K is a component of G. Then either $[J_G(V), K] = 1$ or $K \trianglelefteq J_G(V)$.*

This theorem was proved by Timmesfeld [Ti] for the case $p = 2$. Later Chermak [Ch] gave a proof for arbitrary p. Earlier proofs by Aschbacher [As] and Thompson (unpublished) used a $\mathcal{K}$-group hypothesis.

As we also want to allow certain solvable analogues of components, we need one further definition.

Definition 1.2 *A non-trivial subgroup K of $J_G(V)$ is a $J_G(V)$-component if K is minimal with respect to $K = [K, J_G(V)]$.*

The Other $\mathcal{P}(G,V)$-Theorem. *Suppose that V is a faithful finite dimensional $GF(p)G$-module and that $O_p(J_G(V)) = 1$. Then $[E,K] = 1$ and $[V,E,K] = 0$ for any two distinct $J_G(V)$-components E and K.*

In the case of a faithful $GF(p)G$-module with $O_p(J_G(V)) = 1$ we also show that every component of $J_G(V)$ is a $J_G(V)$-component (see 3.4) and that every $J_G(V)$-component, which is not a component, is isomorphic to $SL_2(p)'$, $p = 2$ or 3 (see 3.2).

2 Offenders

In this section G is a finite group, p is a prime, and V is a finite dimensional $GF(p)G$-module. Some of the arguments in this section got their inspiration from [CD].

Definition 2.3 *Let A be a subgroup of G.*

(a) A acts quadratically *on V if $[V,A,A] = 0$.*

(b) $j_A(V) := \frac{|A||C_V(A)|}{|V||C_A(V)|}$.

(c) V is a simple $GF(p)G$-module, if $V \neq 0$ and V and 0 are the only G-submodules in V.

Lemma 2.4 *Let $A \leq G$ such that $A/C_A(V)$ is elementary abelian. Then the following hold:*

(a) $j_A(V) = \frac{|A/C_A(V)|}{|V/C_V(A)|}$.

(b) A is an offender on V iff $j_A(V) \geq 1$.

(c) If $[V,A] = 0$, then $j_A(V) = 1$.

(d) A is a best offender iff $j_B(V) \leq j_A(V)$ for all $B \leq A$.

Proof: (a) is obvious, and (b) and (c) follow from (a).

(d) By (a) $j_B(V) = j_{BC_A(V)}(V)$ and so we may assume $C_A(V) \leq B$. Then $C_A(V) = C_B(V)$ and so $j_B(V) \leq j_A(V)$ iff $|B||C_V(B)| \leq |A||C_V(A)|$. □

Lemma 2.5 *Let $A \leq G$ be a best offender on V and $B \leq G$ be an offender on V. Then the following hold:*

(a) A is an offender on V.

(b) B contains a best offender B^* on V with $j_B(V) \leq j_{B^*}(V)$ such that $[V, B^*] \neq 0$ or $B^* = B$.

(c) A is a best offender on every A-submodule of V.

Proof: (a): By 2.4(c),(d) $1 = j_1(V) \leq j_A(V)$, so by 2.4(b) A is an offender on V.
(b): Choose $B^* \leq B$ such that first $j_{B^*}(V)$ is maximal and then $|B^*|$ is maximal. If $[V, B^*] = 0$, then $j_{B^*}(V) = 1$ and thus by 2.4(b) also $j_B(V) = 1$. Now the maximal choice of $|B^*|$ yields $B^* = B$.
(c): Let W be an A-submodule of V and $A_0 \leq A$. Then

$$|A_0||C_W(A_0) + C_V(A)| \leq |A_0||C_V(A_0)| \leq |A||C_V(A)|$$

and thus

$$|A_0||C_W(A_0)||C_V(A)||C_W(A_0) \cap C_V(A)|^{-1} \leq |A||C_V(A)|.$$

Since $C_W(A_0) \cap C_V(A) = C_W(A)$ we get $|A_0||C_W(A_0)| \leq |A||C_W(A)|$. □

Lemma 2.6 *Let A and B be subgroups of G. Then*

$$j_{\langle A,B\rangle}(V) j_{A\cap B}(V) \geq j_A(V) j_B(V)$$

with equality iff $C_V(A \cap B) = C_V(A) + C_V(B)$ and $\langle A, B\rangle = AB$.

Proof: We may assume that $C_G(V) = 1$ since $j_A(V) = j_{AC_H(V)/C_H(V)}(V)$, so $|V| j_A(V) = |A||C_V(A)|$. Observe that

$$|\langle A, B\rangle| \geq |AB| \text{ and } |C_V(A \cap B)| \geq |C_V(A) + C_V(B)|.$$

Then

$$\begin{array}{rcl} |V|^2 j_{\langle A,B\rangle}(V) j_{A\cap B}(V) & = & |\langle A, B\rangle||C_V(\langle A, B\rangle)||A \cap B||C_V(A \cap B)| \\ & \geq & |AB||C_V(A) \cap C_V(B)||A \cap B||C_V(A) + C_V(B)| \\ & \geq & |A||B||C_V(A)||C_V(B)| \\ & = & |V|^2 j_A(V) j_B(V). \end{array}$$

□

Fundamental for the investigation of best offenders is a replacement property first proved by Thompson, the Thompson Replacement Theorem, and then generalized by Timmesfeld, the Timmesfeld Replacement Theorem. We will use the following version of the Timmesfeld Replacement Theorem with [KS] as a reference.

Lemma 2.7 *Let A be a best offender of G on V and W be a subgroup of V. Then for $A^* := C_A([W, A])$ the following hold:*

(a) A^* is a best offender on V with $j_A(V) = j_{A^*}(V)$.

(b) $C_V(A^*) = [W, A] + C_V(A)$.

(c) $[W, A^*] \neq 0$ if $[W, A] \neq 0$.

(d) $[W, A^*, A] = 0$.

Proof: Properties (a) and (b) can be found in [KS, 9.2.1 and 9.2.3], (c) is stated there differently, so we give a proof here.

Assume that $[W, A^*] = 0$. Then by (b) $W \leq [W, A] + C_V(A)$; in particular $[W, A] = [W, A, A]$. As $A/C_A(V)$ is a p-group, this last property gives $[W, A] = 0$.

Finally, by definition $[W, A, A^*] = 0$, and $[A, A^*, W] = 0$ since $A/C_A(V)$ is abelian. Hence the Three Subgroups Lemma implies (d). $\square$

Lemma 2.8 *Let A and B be quadratic offenders of G on V such that*

$$[A, B] \leq C_G(V) \text{ and } AC_G(V) \cap BC_G(V) = C_G(V).$$

Then $\langle A, B\rangle$ is a quadratic best offender on V, or there exists a quadratic best offender $X \leq \langle A, B\rangle$ with

$$j_X(V) > \max\{\mathrm{j_A(V), j_B(V)}\}.$$

Proof: Let $D := \langle A, B\rangle$. Then $D/C_D(V)$ is an elementary abelian p-group. We may assume that G is faithful on V, so D is elementary abelian and $A \cap B = 1$. From 2.6 we get that $j_D(V) \geq j_A(V)j_B(V)$ since $j_{A\cap B}(V) = 1$.

Assume first that $j_D(V) > max\{j_A(V), j_B(V)\}$. Then $j_D(V) > 1$, and by 2.5(b) there exists a best offender $D^* \leq D$ with $j_{D^*}(V) \geq j_D(V) > 1$; in particular $[V, D^*] \neq 0$. Now 2.7 gives the desired quadratic best offender $X \leq D^*$.

Assume now that $j_D(V) \leq max\{j_A(V), j_B(V)\}$. By 2.6

$$j_D(V) = j_D(V)j_1(D) \geq j_A(V)j_B(V),$$

so $j_D(V) = j_A(V)j_B(V)$ and again by 2.6 $C_V(A) + C_V(B) = C_V(A \cap B) = V$. This yields $[V, A] = [C_V(B), A]$ and $[V, A, B] = 0$, and the quadratic action of AB follows. $\square$

Lemma 2.9 *Let $A \leq G$ and $L \leq G$ wih $L = [L, A]$. Then $C_A(L) \cap C_A([V, L, A]) \leq C_A([V, L])$.*

Proof: Note that $O^p(L) = L \leq \langle A^L\rangle$ since $L = [L, A]$. Thus, we may assume that $V = [V, L]$, in particular $[V, A, A_0] = 0$. Let $A_0 := C_A(L) \cap C_A([V, L, A])$. Then

$$[V, L, A_0] \leq [V, \langle A^L\rangle, A_0] = \langle [V, A, A_0]^L\rangle = 0.$$

$\square$

Lemma 2.10 *Let A be a best offender of G on V, $L \trianglelefteq G$, and W a simple L-submodule of V. Suppose that $[L, A] \not\leq C_L(W)$. Then W is A-invariant.*

Proof: Note that $[L, A] \not\leq C_G(W)$ implies $[W, A] \neq 0$. Hence by 2.7 there exists a quadratic best offender $A^* \leq A$ such that $[W, A^*] \neq 0$ and $[W, A^*, A] = 0$. Then

$$[W, N_{A^*}(W)] \leq W \cap C_V(A) \leq W \cap W^b \text{ for every } b \in A.$$

Since $W \cap W^b$ is an L-module and W is simple, we conclude that either W is A-invariant or $[W, N_{A^*}(W)] = 0$. In the first case we are done, so we may assume that $[W, N_{A^*}(W)] = 0$. In particular

$$W \cap W^a = 0 \text{ for every } a \in A^* \setminus C_{A^*}(W). \tag{1}$$

Pick $c \in A^* \setminus C_{A^*}(W)$. Then

$$U := W \oplus W^c = W \oplus [W, c] = W^c \oplus [W, c]. \tag{2}$$

and

$$[W, c] \leq U \cap C_V(A) \leq U \cap U^b \text{ for every } b \in A. \tag{3}$$

Assume that $[W, c]$ is L-invariant. Then $[W, c, A] = 0$ implies that also $[W, c, [L, A]] = 0$. The decomposition (2) shows that $[W, [L, A]] = 0$, a contradiction. Thus we have

$$[W, c] \text{ is not } L\text{-invariant.} \tag{4}$$

Let $b \in A$. Since $U \cap U^b$ is an L-module and W is simple, we get from (3) that either $U \cap U^b = [W, c]$ or $U = U^b$. The first case contradicts (4), so U is A-invariant. But then, as $[W, A, c] = [W, c, A] = 0$, we get that

$$C_U(A^*) = C_U(c) = [W, c] \text{ and } |U/C_U(A^*)| = |W|.$$

By 2.5(c) A^* is a best offender on U, so $|W| = |U/C_U(A^*)| \leq |A^*/C_{A^*}(U)|$. Observe that $|U^\sharp| = |W|^2 - 1 = (|W| + 1)|W^\sharp|$. On the other hand, by (1) any two distinct A^*-conjugates of W intersect trivially, so there are at most $|W| + 1$ such conjugates. We conclude that $|U/C_U(A^*)| = |W| = |A^*|$, and the A^*-conjugates of W together with $[W, c]$ form a partition of U. Since all these conjugates of W are L-invariant, also $[W, c]$ is L-invariant. This contradicts (4). □

3 $J_G(V)$-Components

As in the last section G is a finite group and V is a finite dimensional $GF(p)G$-module.

Lemma 3.1 *Suppose that V is a faithful $GF(p)G$-module. Let $L \trianglelefteq J_G(V)$ with $O_p(L) = 1$ and $[L, J_G(V)] \neq 1$. Then L contains a $J_G(V)$-component of G.*

Proof: Let $J := J_G(V)$, and let $K \trianglelefteq J$ be minimal with respect to $K \leq L$ and $[K, J] \neq 1$. It suffices to show that $K = [K, J]$.

Assume that $K \neq [K, J]$. Then the minimality of K gives $[K, J, J] = 1$ and thus $[K, K, K] = 1$. Hence K is nilpotent. Again the minimality of K shows that K is an r-group, r a prime. Moreover $r \neq p$ since $O_p(L) = 1$. As J is generated by p-elements we get that $J = O^r(J)$. Thus, $[K, J, J] = 1$ implies $[K, J] = 1$, a contradiction. □

In the next lemma we will use a result of Glauberman [Gl] as it is stated in [KS].

Lemma 3.2 *Suppose that V is a faithful $GF(p)G$-module. Let R be a $J_G(V)$-component with $O_p(R) = 1$ that is not a component of $J_G(V)$. Then the following hold:*

(a) $p = 2$ or 3.

(b) $R \cong SL_2(p)'$ and $|[V, R]| = p^2$.

(c) For every $J_G(V)$-component K with $K \neq R$, $[R, K] = 1$ and $[V, R, K] = 0$.

Proof: Set $J := J_G(V)$ and $\widetilde{J} := J/Z(J)$. By 3.1 $\widetilde{R}$ is a minimal normal subgroup of $\widetilde{J}$. Hence, either $\widetilde{R}$ is the product of components of $\widetilde{J}$ or $\widetilde{R}$ is an elementary abelian q-group.

Assume first that $\widetilde{R}$ is the product of components. Then an elementary argument shows that R' is the product of components of J (see [KS, 6.5.1]). By the $\mathcal{P}(G,V)$-Theorem each of these components is normal in J, so by 3.1 R is a component, which is not the case.

Assume now that $\widetilde{R}$ is a q-group. Then R is nilpotent and thus also R is a q-group. Moreover $q \neq p$ since $O_p(R) = 1$, so $R \leq O_{p'}(G)$. Pick $T_0 \in Syl_p(C_J(R))$, and set

$$J_0 := J_{N_J(T_0)}(V),\ W := C_V(T_0) \text{ and } \overline{J}_0 := J_0/C_{J_0}(W).$$

Observe that $J = J_0C_G(R)$, so R is also a $J_{J_0}(V)$-component. By the $P \times Q$-Lemma R acts faithfully on W. Also $C_{J_0}(R)/J_0 \cap T_0$ is a p'-group. Let X be the inverse image of $C_{\overline{J}_0}(\overline{R})$ in J_0. Then $[R,X] \leq C_R(W)$, and the faithful action of R gives $X \leq C_{J_0}(R)$, so $\overline{X}$ is a p'-group.

We have shown that $C_{\overline{J}_0}(O_{p'}(\overline{J}_0)) \leq C_{\overline{J_0}}(\overline{R}) = \overline{X} \leq O_{p'}(\overline{J}_0)$. In addition, by 2.5(c) $\overline{J}_0$ is generated by best offenders on W. Hence W and $\overline{J}_0$ satisfy the hypothesis of Theorem 9.3.7 in [KS]. It follows that either

$$p = 2,\ R \cong C_3 \text{ and } |[W,R]| = 4, \text{ or}$$

$$p = 3,\ R \cong Q_8 \text{ and } |[W,R]| = 9.$$

Let $A \leq J_0$ be a best offender on V such that $[R,A] \neq 1$. Clearly $R = [R,A]$ and $|A/C_A(R)| = p$. Moreover, according to 2.7 there exists a best offender $A^* \leq A$ such that $[V,R,A] \leq C_V(A^*)$ and $[V,R,A^*] \neq 1$. Hence 2.9 shows that $[R,A^*] \neq 1$, and we may assume that A acts quadratically on $[V,R]$. But then again 2.9 shows that that $C_A(R) = C_A([V,R])$ and $|A/C_A([V,R])| = p$. As A is an offender on $[V,R]$ by 2.5(c), we get $|[V,R]/C_{[V,R]}(A)| = p$. This gives $[V,R] = [W,R]$, and (a) and (b) hold.

Now let K be any other $J_G(V)$-component. As we have seen above K is either a component or has a structure as R. In the first case the fact that $GL_2(p)$ is solvable for $p \leq 3$ shows that $[V,R,K] = 0$ and so also $[R,K] = 1$. In the second case we can choose a best offender B with $K = [K,B]$ such that $\langle A,B\rangle$ is a p-group. Then $C_{\langle A,B\rangle}(RK)$ is a normal p-subgroup of $RK\langle A,B\rangle$ and so centralizes $[V,RK]$. Hence the above result from [KS] applies to $RK\langle A,B\rangle/C_{RK\langle A,B\rangle}([V,RK])$. Then

$$[R,K] \leq C_{R\cap K}([V,RK]) = 1$$

and $[V,R,K] = 0$. Hence (c) also holds in this case. □

Lemma 3.3 *Suppose that V is a faithful $GF(p)G$-module and that K is a $J_G(V)$-component with $O_p(K) = 1$. Then there exists a best offender A of G such that $[K,A] = K$ and A is quadratic on $[V,K]$.*

Proof: By 3.1 there exists a best offender B such that $[K,B] \neq 1$. Hence 3.2 gives $[K,B] = K$ and thus $[V,K,B] \neq 0$. Now 2.7 with $W := [V,K]$ gives a best offender $A \leq B$ satisfying $[W,B,A] = 0$ and $[W,A] \neq 0$. The first property shows that A is quadratic on W. It remains to prove $K = [K,A]$.

Assume that $[K,A] \neq K$. Then again 3.2 yields $[K,A] = 1$ and thus by 2.9 $W = [W,K] \leq C_W(A)$, a contradiction. □

Lemma 3.4 *Suppose that V is a faithful $GF(p)G$-module and that K is a component of $J_G(V)$ with $O_p(K) = 1$. Then K is a $J_G(V)$-component.*

Proof: Let $J := J_G(V)$. By the $\mathcal{P}(G,V)$-Theorem K is normal in J, so $K = [K,J]$. Hence, either K is a $J_G(V)$-component, or there exists a $J_G(V)$-component $R < K$. Suppose that the second case holds. Then $R \leq Z(K)$ since K is a component. Thus R is a $J_G(V)$-component that is not a component. Hence 3.2 implies that $|[V,R]| = p^2$. But then $GL([V,R])$ is solvable and as above $[V,R,K] = 0$; particular $[V,R,R] = 0$. This is impossible since R is a non-trivial p'-group. $\square$

Lemma 3.5 *Suppose that V is a faithful $GF(p)G$-module and that K is a $J_G(V)$-component with $O_p(K) = 1$. Then there exists a best offender A in G such that $[K,A] = K$ and A is quadratic on $[V,K]$.*

Proof: By 3.3 there exists a best offender A such that $[K,A] \neq 1$ and A is quadratic on $[V,A]$. It remains to prove $K = [K,A]$.

If K is a component, then $[K,A] \neq 1$ implies $[K,A] = K$. If K is not a component, then by 3.2(a),(b) $R \cong C_3$ or Q_8 and again $[K,A] \neq 1$ implies $[K,A] = K$. $\square$

Lemma 3.6 *Suppose that V be a faithful $GF(p)G$-module and $O_p(J_G(V)) = 1$. Let K and L be two distinct components of $J_G(V)$. Then*

$$[V,K,L] = [V,L,K] = 0.$$

Proof: We apply 3.4 and 3.5. Then there exist best offenders A and B such that $[K,A] = K$ and $[L,B] = L$, and A and B act quadratically on $[V,K]$ and $[V,L]$, respectively. Moreover since K and L are normal in $J_G(V)$, we may assume that $\langle A,B\rangle$ is a p-group.

By way of contradiction we assume that $[V,K,L] \neq 0$. Set $G_0 := KL\langle A,B\rangle$. Then there exists a G_0-submodule W of V that is minimal with respect to $[W,K,L] \neq 0$. Since $[K,L] = 1$ also $[W,L,K] \neq 0$, and the situation is symmetric in K and L.

Suppose that $W \neq [W,K]$. Then the minimality of W gives $[W,K,K,L] = 0$. On the other hand $[W,K] = [W,K,K]$, so $[W,K,L] = 0$, a contradiction. Hence we have $W = [W,K]$ and by symmetry also $W = [W,L]$. Thus A and B are quadratic on W, and by 2.5(c) A and B are quadratic best offenders on W.

Let U be a maximal $GF(p)G_0$-submodule of W and set $\hat{W} := W/U$. Then $\hat{W}$ is a simple $GF(p)G_0$-module. By 2.9 $[W,C_A(K)] = 0$ and similarly $[W,C_B(L)] = 0$. Thus we have

$$(*) \qquad C_A(\hat{W}) = C_A(K) = C_A(W) \text{ and } C_B(\hat{W}) = C_B(L) = C_B(W).$$

Since A and B are quadratic best offenders on W, $(*)$ shows that A and B are also quadratic offenders on $\hat{W}$. Hence there exist quadratic best offenders $\hat{A}$ and $\hat{B}$ on $\hat{W}$ in $\langle A,B\rangle$ such that $[K,\hat{A}] = K$ and $[L,\hat{B}] = L$. In addition, we choose $\hat{A}$ such that $j_{\hat{A}}(\hat{W})$ is maximal with that property.

Let $\hat{X}$ be a simple K-submodule of $\hat{W}$. Assume that $[K,\hat{B}] \neq 1$. Then 2.10 shows that $\hat{X}$ is normalized by every L-conjugate of $\hat{B}$, so L normalizes $\hat{X}$. As $\hat{X}$ is a simple K-module, Schur's Lemma shows that $End_K(\hat{X})$ is a finite division ring and then Wedderburn's Theorem that $L/C_L(\hat{X})$ is cyclic. This shows that $[\hat{X},L] = 0$ since L is perfect. Hence $C_{\hat{W}}(L)$ is a non-trivial G_0-submodule of $\hat{W}$. As $\hat{W}$ is a simple G_0-module, we conclude that $[\hat{W},L] = 0$, a contradiction.

We have shown that $[K,\hat{B}] = 1$ and similarly $[L,\hat{A}] = 1$. Observe that $C_{\langle A,B\rangle}(KL)$ is a normal p-subgroup of G_0 and hence centralizes $\hat{W}$, so

$$[\hat{A},\hat{B}] \leq C_{G_0}(\hat{W}) \text{ and } \hat{A}C_{G_0}(\hat{W}) \cap \hat{B}C_{G_0}(\hat{W}) = C_{G_0}(\hat{W}).$$

Now 2.8 and the maximality of $j_{\hat{A}}(\hat{W})$ show that $\langle \hat{A}, \hat{B} \rangle$ is a quadratic best offender on $\hat{W}$. But the above argument with $\langle \hat{A}, \hat{B} \rangle$ in place of $\hat{B}$ implies that $[K, \langle \hat{A}, \hat{B} \rangle] = 1$, which contradicts $[K, \hat{A}] = K$. □

The proof of The Other $\mathcal{P}(G, V)$-Theorem. Let E and K be distinct $J_G(V)$-components. Then by 3.2(b) and 3.6 $[V, E, K] = 0 = [V, K, E]$. So by the Three Subgroups Lemma, $[E, K] = 1$.

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