ISOLATED SUBGROUPS IN FINITE GROUPS

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1. Introduction

Suppose that p is a prime, P is a finite group and $S \in \mathrm{Syl}_p(P)$. Then P is *p-minimal* if S is not normal in P and S is contained in a unique maximal subgroup of P. Now suppose that G is a finite group and $S \in \mathrm{Syl}_p(G)$. The set of subgroups of G which contain S and are p-minimal will be denoted by $\mathcal{P}_G(S)$. For $P \in \mathcal{P}_G(S)$ we put

$$L_G(P,S) := \langle \mathcal{P}_G(S) \setminus \{P\}, N_G(S)\rangle,$$

and next introduce isolated p-minimal subgroups.

Definition 1.1. *A p-minimal subgroup P in $\mathcal{P}_G(S)$ is A-isolated where A is a normal p-subgroup of $L_G(P,S)$ if $A \not\leq O_p(P)$. We say that $P \in \mathcal{P}_G(S)$ is an isolated subgroup of G if it is A-isolated for some A.*

Clearly P is an isolated subgroup of G if and only if $O_p(L_G(P,S)) \not\leq O_p(P)$. As a consequence $G \neq L_G(P,S)$. Thus, as $G = \langle \mathcal{P}_G(S), N_G(S)\rangle$ (see Lemma 3.1), any minimal generating set of G using subgroups from $\mathcal{P}_G(S)$, together with $N_G(S)$, must include P.

The archetypical p-minimal subgroup is to be found in groups of Lie type defined in characteristic p. If G is such a group and R is a minimal parabolic subgroup of G, then $P = O^{p'}(R)$ is a p-minimal subgroup of G. For such a P, $L := L_G(P,S)$ is the unique maximal parabolic subgroup of G not containing R and $O_p(L) \not\leq O_p(P)$, and so in fact P is an isolated subgroup of G. This in itself makes isolated subgroups worthy of study yet there are other compelling reasons for instigating a study of this type of subgroup. Such subgroups arise in the on-going work whose aim is to classify finite $\mathcal{K}_p$-proper groups of local characteristic p. We return to this important aspect of isolated subgroups shortly.

Our first theorem includes a statement about a special class of p-minimal subgroups which we call narrow subgroups. Before defining narrow subgroups, we first recall that a group G acts imprimitively on a $\mathrm{GF}(p)G$-module V provided there exists a non-trivial decomposition of V into a direct sum of subspaces $W_1, \dots, W_k$ which are permuted by the action of G. A group which does not act imprimitively on a module V is said to act primitively on V.

Definition 1.2. *Suppose that P is a p-minimal group, $S \in \mathrm{Syl}_p(P)$ and let M be the unique maximal subgroup of P containing S. Set $E := O^p(P)F/F$ where F is the* *core of M in P. Then we say that P is* narrow *if either E is a simple group or E is elementary abelian and M acts primitively on E.*

Again examples of narrow, isolated subgroups appear naturally in Lie type groups of characteristic p.

Our first theorem shows that finite groups with isolated subgroups have a restricted structure. Its proof consists of an analysis of the over groups of Sylow p-subgroups and relies only on results from classical group theory. This contrasts with the proof of Theorem 1.7 where we are forced to draw upon specific structural properties of certain finite simple groups.

Theorem 1.3. *Suppose that p is a prime, G is a finite group, $S \in \mathrm{Syl}_p(G)$ and $P \in \mathcal{P}_G(S)$ is an isolated subgroup of G. Set $L := L_G(P, S)$ and $Y := \langle O^p(P)^G\rangle$. Then $G = YL$, P is an isolated subgroup of YS and either*
(a) $Y = O^p(P)$; or
(b) $Y/O_p(Y)$ is a central product of quasisimple groups which are transitively permuted by S under conjugation and, if P is narrow, then $Y/O_p(Y)$ is quasisimple. Furthermore, $Y/O_p(Y)$ has order divisible by p.

Of course, if G is a quasisimple group and G has an isolated subgroup, then $Y = G$ and alternative (b) of Theorem 1.3 holds. If G is a p-minimal group and $S \in \mathrm{Syl}_p(G)$, then $L_G(G, S)$ is the unique maximal subgroup of G containing S. Hence, if $O_p(L_G(G, S)) > O_p(G)$, G is an isolated subgroup of G and the possibility in Theorem 1.3(a) trivially holds. As an indication of the limitations of Theorem 1.3, we could take $G = H_1 \times H_2$ with P an isolated subgroup of H_1S with $S \leq P$ where $S \in \mathrm{Syl}_p(G)$. Then P is an isolated subgroup of G (see Lemma 3.5) and of course $Y \leq H_1$. So the direct factor H_2 is invisible as far as Y is concerned.

For a finite group G and p a prime, let $F_p^*(G)$ denote the inverse image in G of $F^*(G/O_p(G)))$, the generalized Fitting subgroup of $G/O_p(G)$. Then we have the following corollary of Theorem 1.3.

Corollary 1.4. *Suppose that G is a finite group and $P \in \mathcal{P}_G(S)$ is an isolated subgroup of G where $S \in \mathrm{Syl}_p(G)$, p a prime.*
(a) $O^p(P) \leq F_p^*(G)$.
(b) If $Y := \langle O^p(P)^G\rangle$ is soluble, then $Y = O^p(P)$ and there is a prime $t \neq p$ such that $Y/O_p(Y)$ is a t-group of class at most 2.

Again casting an eye over the Lie type groups in characteristic p we see that every p-minimal subgroup is an isolated subgroup. This leads us to give

Definition 1.5. *Let G be a finite group, p a prime and $S \in \mathrm{Syl}_p(G)$. Then G is called completely isolated if P is isolated for all $P \in \mathcal{P}_G(S)$.*

For $P_1, P_2 \in \mathcal{P}_G(S)$ write $P_1 \sim P_2$ whenever $\langle O^p(P_1)^G\rangle = \langle O^p(P_2)^G\rangle$. Clearly $\sim$ is an equivalence relation on $\mathcal{P}_G(S)$. Our second theorem describes the structure of completely isolated finite groups.

Theorem 1.6. *If G is a completely isolated finite group, then $G = F_p^*(G)N_G(S)$, where $S \in \mathrm{Syl}_p(G)$. Moreover, $F^*(G/O_p(G))$ is a central product of the subgroups* *generated by the $\sim$ equivalence classes of $\mathcal{P}_{G/O_p(G)}(S/O_p(G))$ and $F(G/O_p(G))$ has class at most two.*

For p a prime, we define

$$\mathcal{L}_1(p) \;:=\; \{O^p(X) \mid X \cong H/Z \text{ where } H \text{ is a universal rank 1 Lie type group defined in characteristic } p \text{ and } Z \le Z(H)\}.$$

Observe that some of the groups in $\mathcal{L}_1(p)$ are smaller than we might have expected. For example, $O^2(\mathrm{SL}_2(2))$ has order 3 and $O^2({}^2\mathrm{B}_2(2))$ has order 5. So, to be explicit we list the members of $\mathcal{L}_1(p)$. For $p \ge 5$, we have

$$\mathcal{L}_1(p) = \{\mathrm{SL}_2(p^a), \mathrm{PSL}_2(p^a), \mathrm{SU}_3(p^a), \mathrm{PSU}_3(p^a) \mid a \ge 1\}.$$

Because of the non-simplicity of $\mathrm{SL}_2(3)$ and ${}^2\mathrm{G}_2(3)$, we have

$$\mathcal{L}_1(3) = \{\mathrm{Q}_8, 2^2, {}^2\mathrm{G}_2(3)' \cong \mathrm{SL}_2(8)\} \cup \{\mathrm{SL}_2(3^a), \mathrm{PSL}_2(3^a), \mathrm{SU}_3(3^{a-1}), {}^2\mathrm{G}_2(3^{2a-1}) \mid a \ge 2\}$$

and similarly, because of the non-simplicity of $\mathrm{SL}_2(2)$, $\mathrm{SU}_3(2)$ and ${}^2\mathrm{B}_2(2)$, we have

$$\mathcal{L}_1(2) = \{3, 5, 3_+^{1+2}, 3^2\} \cup \{\mathrm{SL}_2(2^a), \mathrm{SU}_3(2^a), \mathrm{PSU}_3(2^a), {}^2\mathrm{B}_2(2^{2a-1}) \mid a \ge 2\}.$$

The groups in $\mathcal{L}_1(p)$ will be discussed further in Section 3. Notice that if $O^p(P/O_p(P)) \in \mathcal{L}_1(p)$, $S \in \mathrm{Syl}_p(P)$, and $S > O_p(P)$, then P is a narrow unless $p = 2$ and $O^2(P/O_p(P)) \cong 3^2$ or 3_+^{1+2} in which case we have to be rather more careful. Thus Theorem 1.3 applies to narrow isolated subgroups with the property that $O^p(P/O_p(P)) \in \mathcal{L}_1(p)$ and tells us that when $O_p(G) = 1$, $Y := \langle O^p(P)^G \rangle$ is either quasisimple or equal to $O^p(P)$. In the latter case we have that either $Y = O^p(P)$ is quasisimple or Y is soluble and $O^p(P)$ is isomorphic to one of 3, 5, Q_8, 2^2, 3_+^{1+2} and 3^2.

Our next theorem concerns certain isolated subgroups of $\mathcal{CK}$-groups. We say that a finite simple group is a $\mathcal{K}$-group if it is isomorphic to a cyclic group of prime order, an alternating group of degree at least 5, a simple group of Lie type (including the Tits simple group) or one of the 26 sporadic simple groups. A group is a $\mathcal{CK}$-group, if each of its composition factors is a $\mathcal{K}$-group.

Theorem 1.7. *Suppose that p is a prime, G is a finite $\mathcal{CK}$-group and $P \in \mathcal{P}_G(S)$. If $O^p(P/O_p(P)) \in \mathcal{L}_1(p)$, then either $O^p(P) \trianglelefteq G$ or $O_p(Z(L_G(P,S))) \le O_p(P)$.*

In other words Theorem 1.7 is asserting that if P is $O_p(Z(L_G(P,S)))$-isolated and $O^p(P/O_p(P)) \in \mathcal{L}_1(p)$, then $O^p(P)$ is normal in G. We note that if we take $G = \mathrm{J}_2$ (Janko's second simple group) we have a narrow subgroup P of G with $P \cong \mathrm{PSU}_3(3) \in \mathcal{L}_1(3)$ and $L_G(P,S) \cong 3{\cdot}\mathrm{PGL}_2(9)$. So $O_3(P) = 1$, $O_3(L_G(P,S))$ has order 3 and just misses being in $Z(L_G(P,S))$.

We comment that not every finite group possesses an isolated subgroup. However, as we have seen, if it does then it imposes global constraints on the structure of the group. Another example of this is the following corollary to [MPR, Corollary 1.2]. This asserts that a simple $\mathcal{CK}$-group with an isolated subgroup P satisfying $O^p(P/O_p(P)) \in \mathcal{L}_1(p)$ with $p \ge 11$ must be a Lie type group in characteristic p.

Isolated subgroups, as mentioned earlier, also surface in the project to classify finite $\mathcal{K}_p$-proper groups of local characteristic p. That is finite groups all of whose p-local subgroups H satisfy $F^*(H) = O_p(H)$ and have all simple sections from the list of finite simple groups itemized above. Let G to be such a group and let C be a maximal p-local subgroup containing $N_G(\Omega_1(Z(S)))$ where $S \in \mathrm{Syl}_p(G)$. Then in this context the $\widetilde{P}$!-Theorem [MMPS] asserts that in the presence of certain additional conditions there is an isolated subgroup $\widetilde{P}$ of G contained in $\mathcal{P}_C(S)$. Moreover, $O^p(\widetilde{P}/O_p(\widetilde{P})) \in \mathcal{L}_1(p)$. Thus all of our theorems apply to tell us that $Y := \langle O^p(\widetilde{P})^C\rangle O_p(C)/O_p(C)$ is a quasisimple group. In particular, as C is a $\mathcal{CK}$-group the results of [MPR] which determine the finite simple $\mathcal{K}$-groups possessing an isolated subgroup P with $O^p(P/O_p(P)) \in \mathcal{L}_1(p)$, can be applied to limit the possibilities for Y. For an account of the genealogy of this (still in progress) programme as well as an overview of its aims see [MSS].

Briefly this paper is organized as follows. In Section 2 we gather some elementary background results that we require in this paper. The following section contains a number of general properties of p-minimal groups ranging from their generational properties, their behaviour under quotients to the important compendium of structural properties given in Lemma 3.2. Towards the end of Section 3, in Lemmas 3.6 and 3.7, detailed information about p-minimal isolated subgroups with $O^p(P/O_p(P)) \in \mathcal{L}_1(p)$ is listed. These two results plus Corollary 3.9 are needed in our proof of Theorem 1.7. We begin Section 4 by noting a number of basic facts about isolated subgroups – Lemma 4.3 being a result we use frequently. After establishing further preparatory results such as Lemmas 4.4, 4.9 and 4.10, we then prove Theorems 1.3 and 1.6. The remainder of this section gives various consequences of Theorem 1.3 which will be of use in [MPR]. Our final section is devoted to a proof of Theorem 1.7.

If $\mathcal{X}$ is a collection of subsets of G, then $\bigcap \mathcal{X}$ means the intersection of all the subsets in $\mathcal{X}$. Assume that H is a subgroup of G. Then H^G will denote the set of all G-conjugates of H. Hence $\bigcap H^G$ is just the core of H in G. For p a fixed prime, we sometimes write Q_G for $O_p(G)$, the largest normal p-subgroup of G. Also we denote the preimage of $F(G/Q_G)$ by $F_p(G)$ and the preimage of $\Phi(G/Q_G)$ by $\Phi_p(G)$. The remainder of our group theoretic notation is standard and can be found in [As] or [KS]. Atlas [Atlas] names and conventions will be followed with a few exceptions which we now note. We shall use $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ to denote, respectively, the symmetric and alternating group of degree n. And $\mathrm{Dih}(n)$, $\mathrm{SDih}(n)$ stand, respectively, for the dihedral and semidihedral group of order n (so n is even).

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2. Preliminary Results

This section contains an assorted collection of background results. The first two concern subnormal subgroups while Lemma 2.3 is an elementary criterion for a p-subgroup of a group G to be contained in $O_p(G)$. Lemmas 2.4 and 2.6 look at the

Frattini subgroup and Lemma 2.7 examines $\Phi_p(O^p(G))$. And we end with Proposition 2.8 which gives a property of finite simple $\mathcal{K}$-groups with abelian Sylow p-subgroups.

Our first lemma records a well-known property of components.

Lemma 2.1. *Let G be a finite group, K a component of G and N a subnormal subgroup of G. Then either $K \leq N$ or $[K, N] = 1$.*

Proof. [As, 31.4]. $\square$

In the proof of Lemma 3.2 we shall need the following result.

Lemma 2.2. *Suppose that G is a finite group, M is a maximal subgroup of G and N is a subnormal subgroup of G with $N \leq M$. Then $N \leq \bigcap M^G$.*

Proof. We may assume that N is maximal, under inclusion, with the property that N is subnormal in G and $N \leq M$. Let $N = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_k = G$ be a subnormal chain from N to G. By Wielandt's Subnormality Theorem [KS, 6.7.1], $\langle N^M \rangle \trianglelefteq\trianglelefteq G$. Because of the maximality of N, $\langle N^M \rangle = N$ and so $N \trianglelefteq M$. Thus $N \trianglelefteq \langle M, N_1 \rangle$. Also, by the maximality of N, $N_1 \not\leq M$, whence $N \trianglelefteq G$, and this proves the lemma. $\square$

Lemma 2.3. *Suppose that G is a finite group, p is a prime, and A is a p-subgroup of G for which $[A, G] \leq N_G(A)$. Then $A \leq O_p(G)$ and $[A, G]$ is a p-group.*

Proof. Since
$$A \leq O_p([A, G]A) = O_p(\langle A^G \rangle) \trianglelefteq G,$$
$A \leq O_p(G)$ and $[A, G]$ is a p-group. $\square$

Lemma 2.4. *Suppose that G is a finite group and N is a normal subgroup of G. Then $\Phi(N) \leq \Phi(G)$.*

Proof. Assume that M is a maximal subgroup of G and that $\Phi(N)$ is not contained in M. Then, as M is maximal in G and $\Phi(N)$ is normal in G, $G = \Phi(N)M$. Thus
$$N = N \cap G = N \cap \Phi(N)M = \Phi(N)(N \cap M).$$
Thus $N = N \cap M$ and so $M \geq N \geq \Phi(N)$ which is not the case. Hence $\Phi(N) \leq \Phi(G)$ as claimed. $\square$

Lemma 2.5. *Suppose that G is a finite group. Then*

(a) $\Phi_p(G)$ is the intersection of all the maximal subgroups of G which contain a Sylow p-subgroup of G;
(b) $O_p(G) \in \mathrm{Syl}_p(\Phi_p(G))$; and
(c) if $O_p(O^p(G)) = 1$, then $\Phi_p(G)$ is nilpotent.

Proof. Let $F := \Phi_p(G)$ and H be the intersection of all the maximal subgroups of G which contain a Sylow p-subgroup of G. Obviously $F \le H$ and H is normal in G. Let $T \in \mathrm{Syl}_p(H)$. Then the Frattini Argument gives $G = N_G(T)H$. Since T has p' index in H, we infer that $N_G(T)$ contains a Sylow p-subgroup of G. Thus $N_G(T) = G$. Hence $T \le O_p(G)$ and so as $O_p(G) \le F$, we have $O_p(G) = T$. Assume that $F < H$. Then there exists a maximal subgroup M of G containing T which does not contain H. But then $G = MH$ and, as $M \cap H \ge T$, we have that M contains a Sylow p-subgroup of G which delivers the contradiction $H \le M$. Hence $F = H$ and $O_p(G) \in \mathrm{Syl}_p(F)$, so proving (a) and (b). Suppose that $O_p(O^p(G)) = 1$. Then $[O_p(G), F \cap O^p(G)] = 1$ and, since $F = O_p(G)(F \cap O^p(G))$ by (b), we get that F is nilpotent. Hence (c) holds. □

Lemma 2.6. *Suppose G is a finite group with H and K subgroups of G. If $G = HK$ and $H \cap K = [H, K] = 1$, then $\Phi(G) = \Phi(H)\Phi(K)$.*

Proof. Clearly we may assume $H \ne 1 \ne K$. Now every maximal subgroup M of G containing H, as $H \cap K = 1$, must be of the form HM_1 where M_1 is a maximal subgroup of K. Since a similar statement holds for maximal subgroups of G containing K, intersecting over all maximal subgroups of G which contain either H or K shows that $\Phi(G) \le \Phi(H)\Phi(K)$. Lemma 2.4 gives the reverse inclusion and so $\Phi(G) = \Phi(H)\Phi(K)$. □

Lemma 2.7. *Suppose that G is a finite group and p is a prime. Then $\Phi_p(O^p(G)) = O^p(G) \cap \Phi_p(G)$.*

Proof. Suppose that G is a minimal counterexample to the statement in the lemma. Set $X := O^p(G)$, $F := \Phi_p(G)$, $F_X := F \cap X$ and let $S \in \mathrm{Syl}_p(G)$. By the minimality of G, $Q_X = 1$.

Using Lemma 2.4, we get $\Phi_p(X) = \Phi(X) \le \Phi(G) \cap X \le F_X$. Hence $F_X \not\le \Phi(X)$.

Assume that M is a maximal subgroup of X and that M does not contain F_X. Then, as M is maximal in X and F_X is normal in X, $X = F_X M$. Since $F_X \cong F_X Q_G/Q_G \le \Phi_p(G)/Q_G = \Phi(G/Q_G)$, F_X is nilpotent of p'-order. Hence $N_{F_X}(F_X \cap M) > F_X \cap M$ which together with the maximality of M in X implies that $F_X \cap M$ is a normal subgroup of X. Furthermore, because of the maximality of M in X, $F_X/(F_X \cap M)$ has no non-trivial proper X-invariant subgroups. Hence $F_X/(F_X \cap M)$ is an elementary abelian t-group for some prime t with $t \ne p$. Let $R = \bigcap (F_X \cap M)^G$. Assume that $R > 1$. Then by induction we have $O^p(G/R) \cap \Phi_p(G/R) = \Phi_p(O^p(G/R))$. Furthermore, as $R \le \Phi_p(G)$, $\Phi_p(G/R) \ge \Phi_p(G)/R$. Hence, using $X/R = O^p(G/R)$, we have

$$F_X/R = \Phi_p(G)/R \cap X/R \le \Phi_p(G/R) \cap O^p(G/R) = \Phi_p(X/R).$$

But we know that M has index a power of t in X, so $M/R \ge \Phi_p(X/R)$ by Lemma 2.5(a). Thus $F_X/R \le \Phi_p(X/R) \le M/R$, which is a contradiction to our choice of M. Therefore, $R = 1$ and, in particular, F_X is an elementary abelian t-group. Notice that $M/(F_X \cap M)$ is a complement to $F_X/(F_X \cap M)$ in $X/(F_X \cap M)$. Choose K of minimal order in M such that $X = KF_X$. Then $K \cap F_X$ is normalized by both F_X and K and so is normal in X. Since $K \le M$, $K \cap F_X \ne F_X$. We claim that K is a complement to F_X in X. Suppose that $K \cap F_X > 1$. Then, as $R = 1$, there exists $g \in G$ such that $K^g \cap F_X \not\le F_X \cap M$. It follows that $K^g(F_X \cap M) = K^g(F_X \cap K^g)(F_X \cap M) = K^g F_X = X$ as $(F_X \cap M)(F_X \cap K^g)$ is normal in X. Therefore, as $K^g/(K^g \cap (F_X \cap M)) \cong X/(F_X \cap M)$ and $X/(F_X \cap M)$ splits over $F_X/(F_X \cap M)$, $K^g/(K^g \cap F_X \cap M)$ splits over $(K^g \cap F_X)/(K^g \cap F_X \cap M)$ and thus K contains a proper subgroup K_0 such that $K = K_0(K \cap F_X)$. But then $K_0 F_X = K F_X = X$ and we have a contradiction to the minimal choice of K. Therefore, X splits over F_X. By [As, 17.7] the total number of complements to F_X in X is $|F_X/C_{F_X}(X)||H^1(X, F_X)| = t^a$ for some integer a. Since S permutes (by conjugation) the complements to F_X in X and S is a p-group, we get that there exists a complement K to F_X in X which is normalized by S. So KS is a subgroup of G and $F_X KS = XS = G$. Since $KS \ge Q_G$, we infer that $KS = G$. As F_X is a p'-group, we get $F_X = 1$ and we have a contradiction to $F_X \not\le \Phi(X)$. This concludes the proof of Lemma 2.7 □

Proposition 2.8. *Suppose that p is a prime, G is a finite non-abelian simple $\mathcal{K}$-group and S is a Sylow p-subgroup of G. If S is abelian, then $N_G(S)$ acts irreducibly on $\Omega_1(S)$.*

Proof. See [GL, 12-1,pg 158] or [GLS3, 7.8.1]. □

3. p-minimal groups

The first result of this section is elementary to prove but is never-the-less a fundamental result for our work on p-minimal subgroups.

Lemma 3.1. *For G a finite group, p a prime and S a Sylow p-subgroup of G, $G = \langle \mathcal{P}_G(S) \rangle N_G(S)$.*

Proof. This is also proved in [PPS, 1.3]. By induction on $|G|$. If S is contained in a unique maximal subgroup of G, then either S is normal in G or $G \in \mathcal{P}_G(S)$ and the lemma holds. Otherwise we can choose different maximal subgroups M_1 and M_2 of G which contain S. Since $G = \langle M_1, M_2 \rangle$, applying induction to M_1 and M_2 gives $G = \langle \mathcal{P}_G(S) \rangle N_G(S)$. □

We now move on to study the structure of p-minimal groups – here is the main structural result for such groups.

Lemma 3.2. *Suppose that p is a prime, P is p-minimal and $S \in \mathrm{Syl}_p(P)$. Let M be the (unique) maximal subgroup of P containing S and set $F := \bigcap M^P$. Then the following hold.*

(a) $Q_P \in \mathrm{Syl}_p(F)$.
(b) $F = \Phi_p(P)$ *and, in particular, if* $O_p(O^p(P)) = 1$, *then* F *is nilpotent.*
(c) If N *is a subnormal subgroup of* P *contained in* M, *then* $N \cap S \le Q_P$.
(d) If $O^p(P)$ *is* p*-closed, then* P *is a* $\{t, p\}$*-group for some prime* $t \ne p$.
(e) For $N \trianglelefteq P$, *either* $O^p(P) \le N$ *or* $N \le F$. *In particular,* $P = O^{p'}(P)$.

(f) $O^p(P)/(F \cap O^p(P))$ *is a minimal normal subgroup of* $P/(F \cap O^p(P))$*.*
(g) If P *is soluble, then* $O^p(P)$ *is* p*-closed and* P *is a* $\{t, p\}$*-group for some prime* $t \neq p$*.*
(h) $O^p(P) = [O^p(P), P]$*.*

Proof. Parts (a) and (b) follow immediately from Lemma 2.5(a) and (b).

(c) From Lemma 2.2, $N \leq \bigcap M^P = F$. Hence by (a), $N \cap S \leq Q_P$.

(d) For r a prime divisor of $|O^p(P)|$ with $r \neq p$, $O^p(P)$ possesses a Sylow r-subgroup S_r such that $O_p(O^p(P))S_r$ is S-invariant. If (d) were false, then we would get $SS_r \leq M$ for all prime divisors r of $|O^p(P)|$, which then forces $M = P$. Thus (d) holds.

(e) Clearly $S \leq SN \leq P$ and so either $SN = P$ or $SN \leq M$. The former alternative yields $O^p(P) \leq N$ and the latter $N \leq \bigcap M^P = F$, as required. If $O^{p'}(P) \leq F$, then by (a) $Q_P = S$ a contradiction. Thus $O^{p'}(P) \not\leq F$ and so $P = SO^p(P) \leq O^{p'}(P)$.

(f) Note that $O^p(P) > F \cap O^p(P)$, as $O^p(P) \leq F$ would give $FS = P$ whereas $FS \leq M$. Then (f) follows immediately from (e).

(g) Suppose that P is soluble. Hence, by (f), $O^p(P)/(F \cap O^p(P))$ is a t-group, t a prime with $t \neq p$. Since F is p-closed by (a), $F \cap O^p(P)$ is also p-closed and so $O^p(P)$ is p-closed. Now (d) gives the result.

(h) As $[O^p(P), P] \trianglelefteq P$, applying (e) gives either $O^p(P) \leq [O^p(P), P]$ or $[O^p(P), P] \leq F$. So to prove (e) we must show that $[O^p(P), P] \leq F$ cannot occur. Suppose that $[O^p(P), P] \leq F$. Then $[O^p(P), M] \leq F$ and, as $O^p(P)M = P$, we infer that $M \trianglelefteq P$. Hence $M = F$ and so, by (a), $S = Q_P$. But then $S \trianglelefteq P$ contrary to P being p-minimal. Thus $[O^p(P), P] \leq F$ cannot hold. □

It is of course important to know about the quotients of a p-minimal group P. For N a normal subgroup of P, having $O^p(P) \leq N$ is the kiss of death as far as P/N being a p-minimal group (because a p-group is not p-minimal). However, as we see next, this is the only bad case.

Lemma 3.3. *Suppose that* p *is a prime,* P *is a* p*-minimal group and let* N *be a normal subgroup of* P *with* $O^p(P) \not\leq N$*. Put* $\overline{P} := P/N$*. Then*

(a) $\overline{P}$ *is a* p*-minimal group;*
(b) $P/\Phi_p(P) \cong \overline{P}/\Phi_p(\overline{P})$*; and*
(c) if P *is narrow, then so is* $\overline{P}$*.*

Proof. Let $S \in \mathrm{Syl}_p(P)$ and let M be the unique maximal subgroup of P containing S. By Lemma 3.2(e), as $O^p(P) \not\leq N$, $N \leq M$ and hence $\overline{M}$ is the unique maximal subgroup of $\overline{P}$ containing $\overline{S}$. Suppose that $\overline{S} \trianglelefteq \overline{P}$. Then, by the Frattini argument, $P = N_P(S)N \leq M$, which is impossible. Therefore $\overline{S}$ is not normal in $\overline{P}$ and so $\overline{P}$ is a p-minimal group. Clearly (b) follows from Lemma 3.2(b) and (c) follows from (a) and (b). □

Lemma 3.4. $\mathcal{P}_G(S) = \mathcal{P}_{O^{p'}(G)}(S)$ *and* $O^{p'}(G) = \langle \mathcal{P}_G(S) \rangle$*.*

Proof. Set $X := O^{p'}(G)$. The inclusion $\mathcal{P}_X(S) \subseteq \mathcal{P}_G(S)$ is obvious. Assume that $P \in \mathcal{P}_G(S)$. Then, by Lemma 3.2(e), $P = O^{p'}(P) \leq P \cap X$. Thus $\mathcal{P}_G(S) \subseteq \mathcal{P}_X(S)$ and so $\mathcal{P}_G(S) = \mathcal{P}_X(S)$. Since $\langle \mathcal{P}_X(S)\rangle = \langle \mathcal{P}_G(S)\rangle$ is normalized by $N_G(S)$, Lemma 3.1 implies that $\langle \mathcal{P}_X(S)\rangle$ is a normal subgroup of G which contains S. Thus $\langle \mathcal{P}_X(S)\rangle = O^{p'}(G)$. □

We next see that p-minimal subgroups are nicely located within central products.

Lemma 3.5. *Assume that G has subgroups H_1 and H_2 such that $[H_1, H_2] = 1$ and $G = H_1H_2$. Then*

$$\mathcal{P}_G(S) = \mathcal{P}_{H_1S}(S) \cup \mathcal{P}_{H_2S}(S).$$

Proof. For $i = 1, 2$, set $S_i := S \cap H_i$ and let $P \in \mathcal{P}_G(S)$. Then, by Lemma 3.2(e), $P = \langle S_1^P\rangle\langle S_2^P\rangle$. Let M be the unique maximal subgroup of P containing S. We may assume $\langle S_1^P\rangle \not\leq M$. So, using Lemma 3.2(e), $O^p(P) \leq \langle S_1^P\rangle \leq H_1$ and $P \leq H_1S$. Thus $P \in \mathcal{P}_{H_1S}(S)$. □

We continue this section by illuminating, in the next two lemmas, the structure of narrow groups with $O^p(P/Q_P) \in \mathcal{L}_1(p)$.

Lemma 3.6. *Suppose that p is a prime, P is a narrow group and $O^p(\overline{P}) \in \mathcal{L}_1(p)$ where $\overline{P} = P/Q_P$. Then exactly one of the following holds.*

(a) $p^a \geq 4$ and $\overline{P}$ is isomorphic to $\mathrm{SL}_2(p^a)$ or $\mathrm{PSL}_2(p^a)$ perhaps extended by field automorphisms of order a power of p.
(b) $p^a \geq 3$ and $\overline{P}$ is isomorphic to $\mathrm{SU}_3(p^a)$ or $\mathrm{PSU}_3(p^a)$ perhaps extended by field automorphisms of order a power of p.
(c) $p = 2$, $a > 1$ and $\overline{P}$ is isomorphic to ${}^2\mathrm{B}_2(2^a)$.
(d) $p = 3$, $a > 1$ and $\overline{P}$ is isomorphic to ${}^2\mathrm{G}_2(3^a)$ perhaps extended by field automorphisms of order a power of 3.
(e) $p = 2$, $O^2(\overline{P}) \cong 3$ and $\overline{P} \cong \mathrm{SL}_2(2) \cong \mathrm{Sym}(3)$.
(f) $p = 3$, $O^3(\overline{P}) \cong \mathrm{Q}_8$ or 2^2 and, respectively, $\overline{P} \cong \mathrm{SL}_2(3) \cong 2^{\cdot}\,\mathrm{Alt}(4)$ or $\mathrm{PSL}_2(3) \cong \mathrm{Alt}(4)$.
(g) $p = 2$, $O^2(\overline{P}) \cong 3^{1+2}_+$ or 3^2 and $P/O^2(P)Q_P \cong \mathrm{SDih}(16)$, 8 or Q_8.
(h) $p = 2$, $O^2(\overline{P}) \cong 5$ and $\overline{P} \cong \mathrm{Dih}(10)$ or ${}^2\mathrm{B}_2(2)$.
(i) $p = 3$ and $\overline{P}$ is isomorphic to ${}^2\mathrm{G}_2(3)' \cong \mathrm{SL}_2(8)$ or ${}^2\mathrm{G}_2(3) \cong \mathrm{SL}_2(8) : 3$.

Proof. The structure of $O^p(\overline{P})$ follows directly from the structure of the rank one Lie type groups and then the rest follows from the structure of the automorphism groups of $O^p(\overline{P})$ and the fact that $P = O^{p'}(P)$ is p-minimal. However, we note in detail the special case of $\overline{P} \cong \mathrm{SU}_3(2)$ or $\mathrm{PSU}_3(2)$. In these cases we have that $O^2(P) \cong 3^{1+2}_+$ or 3^2 respectively. Thus $\mathrm{Out}(O^2(\overline{P})) \cong \mathrm{GL}_2(3)$ and so $X := P/O^2(P)Q_P$ is isomorphic to a subgroup of $\mathrm{SDih}(16)$ (which is isomorphic to a Sylow 2-subgroup of $\mathrm{GL}_2(3)$). Since $\overline{P}$ is 2-minimal, we have that X acts irreducibly on $O^2(\overline{P})/\Phi(O^2(\overline{P})) \cong 3^2$. Thus $X \cong \mathrm{SDih}(16)$, 8, Q_8, $\mathrm{Dih}(8)$ or 4. However, the latter two groups act imprimitively on $O^2(\overline{P})/\Phi(O^2(\overline{P}))$ and so these two possibilities do not arise in (g). □

Lemma 3.7. *Suppose that p is a prime, P is narrow and that $O^p(\overline{P}) \in \mathcal{L}_1(p)$ where $\overline{P} := P/Q_P$. Let $\overline{S} \in \mathrm{Syl}_p(\overline{P})$, $\overline{M}$ be the unique maximal subgroup of $\overline{P}$ containing $\overline{S}$ and $\overline{R} := N_{O^p(\overline{P})}(\overline{S} \cap O^p(\overline{P}))$.*

(a) If $\overline{P}$ has abelian Sylow p-subgroups, then either
  (i) $\overline{P} \cong \mathrm{PSL}_2(p^a)$ or $\mathrm{SL}_2(p^a)$ for some $a \geq 1$;
  (ii) $p = 3$ and $\overline{P} \cong {}^2\mathrm{G}_2(3)' \cong \mathrm{SL}_2(8)$;
  (iii) $p = 2$ and $\overline{P} \cong {}^2\mathrm{B}_2(2)$ or $\mathrm{Dih}(10)$; or
  (iv) $p = 2$ and $\overline{P} \cong 3^{1+2}_+ : 8$ or $3^2 : 8$.

(b) If $\overline{P}$ has cyclic Sylow p-subgroups, then $\overline{P} \cong \mathrm{PSL}_2(p)$, $\mathrm{SL}_2(p)$, ${}^2\mathrm{G}_2(3)'$, ${}^2\mathrm{B}_2(2)$, $\mathrm{Dih}(10)$, $3^{1+2}_+ : 8$, and $3^2 : 8$.

(c) If $\overline{D} \leq \overline{S}$ is normal in $\overline{M}$ with $\overline{D}$ abelian but not elementary abelian, then $\overline{D}$ is cyclic and either $\overline{P}$ is soluble, or $p = 3$ and $\overline{P}' \cong {}^2\mathrm{G}_2(3)'$.

(d) $\overline{R}$ is soluble.

(e) Either $C_{\overline{S}}(\overline{R}) = 1$ or $\overline{P}$ is soluble.

(f) If $\overline{R}$ normalizes a non-trivial cyclic subgroup of $\overline{S}$, then either $|\overline{S}| \leq p^3$, or $p = 2$ and $\overline{S} \cong \mathrm{SDih}(16)$.

(g) If P is soluble, then $p \leq 3$.

Proof. Suppose first that $\overline{P}$ is soluble. Then Lemma 3.6 implies that $p = 2$ or 3 and gives an explicit description of $\overline{P}$. With this information it is easy to verify all the claims in the lemma in this case. So assume that $\overline{P}$ is not soluble. Let $\overline{X} = O^p(\overline{P})$ and $\overline{T} = \overline{S} \cap \overline{X}$. If $\overline{X} = {}^2\mathrm{G}_2(3)' \cong \mathrm{SL}_2(8)$. Then $\overline{P}$ has cyclic Sylow 3-subgroups if and only if $\overline{P} = \overline{X}$. In this case we have $\overline{S}$ is cyclic of order 9 and is inverted by $\overline{R}$. Thus (a), (b), (c) and (d) hold in this case. Since $\overline{S}$ is either cyclic or extraspecial, and $\overline{R}$ inverts $\overline{T}$, $C_{\overline{S}}(\overline{R}) = 1$. Hence (e) is true and, as $|\overline{S}| \leq 3^3$, (f) holds as well. So we may assume that $\overline{X} \not\cong {}^2\mathrm{G}_2(3)'$. Hence we now consider the cases when $\overline{X}$ is not soluble and

$$\overline{X} \in \{\mathrm{SL}_2(p^a), \mathrm{PSL}_2(p^a), \mathrm{SU}_3(p^a), \mathrm{PSU}_3(p^a), {}^2\mathrm{B}_2(2^{2a+1}), {}^2\mathrm{G}_2(3^{2a+1}) \mid a \geq 1\}.$$

In particular, we note that p divides $|\overline{X}|$. If $\overline{P}$ has abelian Sylow p-subgroups, then so does $\overline{X}$. Thus the candidates for the groups $\overline{P}$ with abelian Sylow p-subgroups are as listed in (a) and (b). Now for (a) we note that if $\overline{P} \neq \overline{X}$, then, by Lemma 3.6 (a) and (i), $\overline{P}$ includes field automorphisms of $\mathrm{SL}_2(p^a)$. Since the field automorphism of $\mathrm{SL}_2(p^a)$ of order p centralizes a subgroup of $\overline{X}$ which has Sylow p-subgroups of order $p^{a/p}$, we have $\overline{S}$ is abelian if and only if $\overline{P} = \overline{X}$. Hence (a) holds and (b) follows from (a).

Suppose that $\overline{D} \leq \overline{S}$ where $\overline{D}$ is abelian and is normalized by $\overline{M} = \overline{RS}$. We will prove (c) by showing that $\overline{D}$ is elementary abelian. Since $[\overline{R}, \overline{D}] \leq \overline{R} \cap \overline{D} \leq \overline{T}$, we infer that $\overline{D} \leq \overline{T}$ as in all the cases we are considering the field automorphisms of $\overline{X}$ do not centralize $\overline{R}/\overline{T}$. If $\overline{X} \cong \mathrm{SL}_2(p^a)$ or $\mathrm{PSL}_2(p^a)$, we get $\overline{D}$ is elementary abelian. Now suppose that $\overline{X} \cong \mathrm{SU}_3(p^a)$ or $\mathrm{PSU}_3(p^a)$. We use [H, II 10.12] to deduce the required facts about the structure of $\overline{X}$. If $\overline{D} \not\leq Z(\overline{T})$, then, as $\overline{R}$ acts irreducibly on $\overline{T}/Z(\overline{T})$, we have $\overline{T} = \overline{D}Z(\overline{T})$. Since $Z(\overline{T}) = \Phi(\overline{T})$, this means $\overline{D} = \overline{T}$ is not abelian which is impossible. Thus $\overline{D} \leq Z(\overline{T})$ is elementary abelian. In the case when

$\overline{X} = {}^2\mathrm{B}_2(2^{2a+1})$, a similar argument to the one above shows that $\overline{D}$ is elementary abelian . Suppose that $\overline{X} \cong {}^2\mathrm{G}_2(3^{2a+1})$ with $a \geq 1$. We use [HB3, 13.2] or [W] to extract facts about the structure of $\overline{T}$. This time $\overline{R}$ acts irreducibly on $\overline{T}/\Phi(\overline{T})$, $\Phi(\overline{T})/Z(\overline{T})$ and on $Z(\overline{T})$ and each of these groups has order 3^{2a+1}. Furthermore $\Phi(\overline{T})$ is elementary abelian. If $\overline{D} \not\leq \Phi(\overline{T})$, then $\overline{T} = \overline{D}\Phi(\overline{T})$ and so $\overline{D} = \overline{T}$, again a contradiction. Thus $\overline{D} \leq \Phi(\overline{T})$ and so $\overline{D}$ is elementary abelian. This completes the proof of (c). In particular, we note from the above proof that the irreducible action of $\overline{R}$ implies that $\overline{D} \geq Z(\overline{T})$ for all abelian subgroups of $\overline{S}$ normalized by $\overline{R}$. Thus if $\overline{R}$ normalizes a non-trivial cyclic subgroup $\overline{D}$ of $\overline{S}$, we must have $\Omega_1(\overline{D}) = Z(\overline{T})$ has order p. It follows that $|\overline{S}| = |\overline{T}| \leq p^3$ and thus (f) holds.

We know from the above sources that $\overline{R}/\overline{T}$ is a cyclic group. Thus $\overline{R}$ is soluble and (d) holds.

Finally for part (e) we cite [DS, (5.1)(e)]. □

For a p-group S, p a prime, we use $J(S)$ to denote the elementary abelian version of the Thompson subgroup of S. That is, $J(S) := \langle \mathfrak{A}_e(S) \rangle$ where $\mathfrak{A}_e(S)$ is defined to be the set of elementary abelian subgroups of S which have maximal rank.

Theorem 3.8. *Suppose that p is a prime, P is a p-minimal group, $C_P(Q_P) \leq Q_P$, $S \in \mathrm{Syl}_p(P)$ and $P > \langle C_P(\Omega_1(Z(S))), N_P(J(S)) \rangle$. Let $V := \Omega_1(Z(Q_P))$ and $\overline{P} := P/C_P(V)$. Then there exist subgroups $E_1, \ldots, E_n$ of P such that*

(a) S acts transitively on $\{E_1, \ldots, E_n\}$ by conjugation;
(b) $[\overline{E_i}, \overline{E_j}] = 1$ for $i \neq j$;
(c) either $\overline{E_i} \cong \mathrm{SL}_2(p^n)$ for some $n \geq 1$ or $p = 2$ and $\overline{E_i} \cong \mathrm{Sym}(2^m + 1)$ for some $m \geq 1$; and
(d) $V/C_V(E_i)$ is a corresponding natural $\overline{E}_i$-module and, for $i \neq j$, $V/C_V(E_i)$ is centralized by E_j.

Proof. This is the $C^{**}(G,T)$-Theorem from [BHS]. □

Corollary 3.9. *Suppose that P is as in Theorem 3.8 (and use the notation there). Let M be the unique maximal subgroup of P containing S, and set $E := \langle E_i \mid 1 \leq i \leq n \rangle$. Assume there is a non-trivial subgroup A of V which is centralized by M and such that $A \cap C_V(E) = 1$. Then $p = 2$, $\overline{E_i} \cong \mathrm{Sym}(2^m + 1)$ and $AC_V(E)/C_V(E) = C_{V/C_V(E)}(S)$ has order 2.*

Proof. As A is centralized by S and S is transitive on $\{E_1, \ldots, E_n\}$, $A \cap C_V(E_1) \leq A \cap C_V(E) = 1$. Therefore, $|A| = |AC_V(E_1)/C_V(E_1)|$. Now $M \cap E_1$ is the unique maximal subgroup of E_1 containing $S \cap E_1 \in \mathrm{Syl}_p(E_1)$ and so $C_{V/C_V(E_1)}(M \cap E_1) = 1$ whenever $V/C_V(E_1)$ is a natural module for $\overline{E_1} \cong \mathrm{SL}_2(p^n)$ with $p^n \neq 2$. Since $AC_V(E_1)/C_V(E_1) \leq C_{V/C_V(E_1)}(M \cap E_1)$, we infer that $p = 2$, $\overline{E_1} \cong \mathrm{Sym}(2^m + 1)$ and then that $|AC_V(E_1)/C_V(E_1)| = 2$ by direct calculation in the natural $\mathrm{Sym}(2^m + 1)$-module. □

4. Finite groups with isolated subgroups

Throughout this section G is assumed to be a finite group, p a prime and $S \in Syl_p(G)$. Our first three results give some elementary properties of isolated subgroups, Lemma 4.2(a) having already been mentioned in Section 1.

Lemma 4.1. *Suppose that $P \in \mathcal{P}_G(S)$ is an A-isolated subgroup of G. Then P is an A-isolated subgroup of $O^{p'}(G)$.*

Proof. Set $X := O^{p'}(G)$. Suppose that $P \in \mathcal{P}_G(S)$ is A-isolated. Then, by Lemma 3.4, $P \in \mathcal{P}_X(S)$, A is normal in $L_X(P,S)$ and, of course, $A \not\leq Q_P$. Thus P is A-isolated in X. □

Lemma 4.2. *Let $P \in \mathcal{P}_G(S)$ be an isolated subgroup of G. Then*
(a) $P \not\leq L_G(P,S)$ and $L_G(P,S) \neq G$; and
(b) $N_G(S) \leq N_G(P)$.

Proof. Set $L := L_G(P,S)$. If $P \leq L$, then $Q_L \leq Q_P$, contrary to P being an isolated subgroup. So $P \not\leq L$, and (a) holds. For $g \in N_G(S)$, $P^g \in \mathcal{P}_G(S)$ and, as $N_G(S) \leq L$ and $P \not\leq L$, $P^g \not\leq L$. Since P^g is also p-minimal, $P = P^g$ and the lemma holds. □

The next lemma is the initial structural result about groups which possess an isolated subgroup.

Lemma 4.3. *Assume that $P \in \mathcal{P}_G(S)$ is an A-isolated subgroup, and set $L := L_G(P,S)$.*
(a) Suppose that $S \leq H \leq G$ and $H \not\leq L$. Then $P \leq H$, $L_H(P,S) = L \cap H$ is a maximal subgroup of H and P is an A-isolated subgroup of H.
(b) $P \cap L$ is the unique maximal subgroup of P containing S.
(c) L is a maximal subgroup of G.

Proof. From Lemma 3.1, $H = \langle \mathcal{P}_H(S) \rangle N_H(S)$ and so, as $H \not\leq L$, $P \in \mathcal{P}_H(S)$. Thus $P \leq H$. By Lemma 4.2(a) and the definition of L, $\mathcal{P}_H(S) \setminus \{P\} = \mathcal{P}_{L\cap H}(S)$. Since $N_H(S) \leq L \cap H$, using Lemma 3.1 gives

$$L_H(P,S) = \langle \mathcal{P}_H(S) \setminus \{P\} \rangle N_H(S) = L \cap H.$$

If $L \cap H < K \leq H$, then clearly $K \not\leq L$ and so, as $S \leq K$, $P \leq K$. Hence $\mathcal{P}_K(S) = \mathcal{P}_H(S)$. Therefore, by Lemma 3.1,

$$K = \langle \mathcal{P}_K(S) \rangle N_K(S) = \langle \mathcal{P}_H(S) \rangle N_H(S) = H.$$

Thus $L \cap H$ is a maximal subgroup of H. Since A is a normal p-subgroup of $L \cap H = L_H(P,S)$, P is an A-isolated subgroup of H and this proves (a). Parts (b) and (c) follow from (a) by taking, respectively, $H = P$ and $H = G$. □

We now obtain further restrictions on the structure of P.

Lemma 4.4. *Suppose that $P \in \mathcal{P}_G(S)$ is an A-isolated subgroup of G and M is the unique maximal subgroup of P containing S. Put $X := O^p(P)$ and $F := \bigcap M^P$. Then*

(a) $XA = \langle A^P \rangle$;
(b) $[XA, X \cap F] \leq O_p(X)$;
(c) $X/O_p(X)$ *is either a central product of quasisimple groups transitively permuted by* S *or there is a prime* t *such that* $X/O_p(X)$ *is a* t*-group of class at most* 2*; and*
(d) $\Phi_p(X) = X \cap F$.

Proof. Since F is p-closed by Lemma 3.2(a) and $A \not\leq Q_P$, Lemma 3.2(e) implies that $O^p(P) \leq \langle A^P \rangle$. So (a) holds.

Set $L := L_G(P, S)$. Then $(F \cap X)S \leq M = P \cap L$ by Lemma 4.3(b). Thus $F \cap X$ normalizes A. Hence, by Lemma 3.2(a),

$$[A, F \cap X] \leq Q_L \cap F \cap X \leq Q_P \cap X = O_p(X).$$

Hence (b) follows from (a).

From Lemma 3.2(f) $X/(X \cap F)$ is a minimal normal subgroup of $P/(X \cap F)$. Hence $X/(X \cap F)$ is isomorphic to a direct product of simple groups. If $X/(X \cap F)$ is an elementary abelian t-group (t a prime), then X is p-closed, whence $X/O_p(X)$ is a t-group by Lemma 3.2(d). Since $[X, X \cap F] \leq O_p(X)$, $X/O_p(X)$ will also have class at most two. In the case when $X/(X \cap F)$ is a direct product of non abelian simple groups, $[X, X \cap F] \leq O_p(X)$ and P being p-minimal force $X/(X \cap F)$ to be a central product of quasisimple groups transitively permuted by S. So (c) holds.

Finally, combining Lemmas 2.7 and 3.2(b) gives

$$\Phi_p(X) = X \cap \Phi_p(P) = X \cap F.$$

□

Lemma 4.5. *Suppose that* $P \in \mathcal{P}_G(S)$ *is an* A*-isolated subgroup of* G.
(a) If $A \leq T \trianglelefteq S$*, then* $N_G(T) \leq L_G(P, S)$.
(b) If $S \leq H \leq G$ *and* $Q_H \leq Q_P$*, then* $P \leq H$.

Proof. Put $L := L_G(P, S)$. Assume that $A \leq T \trianglelefteq S$ and $N_G(T) \not\leq L$. Since $S \leq N_G(T)$, applying Lemma 4.3(a) yields $P \leq N_G(T)$. But then

$$A \leq T \leq O_p(N_G(T)) \leq Q_P,$$

which is not the case. Thus $N_G(T) \leq L$, and we have (a). Now assume that $S \leq H \leq G$ with $Q_H \leq Q_P$. If we have $H \leq L$ then

$$A \leq Q_L \leq Q_H \leq Q_P,$$

contrary to P being an A-isolated subgroup. Therefore $H \not\leq L$ and, again by Lemma 4.3(a), $P \leq H$, so proving the lemma. □

As we shall see in Lemma 4.6 the property of having an isolated subgroup is preserved when taking certain quotients.

Lemma 4.6. *Suppose that* $P \in \mathcal{P}_G(S)$ *and let* N *be a normal subgroup of* G *with* $O^p(P) \not\leq N$. *Put* $L := L_G(P, S)$, $Y := \langle O^p(P)^G \rangle$ *and* $\overline{G} := G/N$. *Then the following hold.*
(a) $P \cap N \leq \Phi_p(P)$.

(b) $G = LY$.
(c) If P *is an isolated subgroup of* G*, then* $N \leq \bigcap L^G$ *and*

$$\bigcap L^G = C_G(\langle Q_L^G\rangle/Q_G) = C_L(YQ_G/Q_G).$$

(d) If P *is an* A*-isolated subgroup of* G*, then* $\overline{P}$ *is an* $\overline{A}$*-isolated subgroup of* $\overline{G}$*.*

Proof. Part (a) follows immediately from Lemmas 3.2(b) and 3.2(e), and part (b) follows from Lemma 3.1 as LY contains $N_G(S)$ and all the p-minimal subgroups containing S.

Set $Z := \bigcap L^G$. Since $O^p(P) \not\leq N$, $P \not\leq NS$ and so $NS \leq L$ by Lemma 4.3(a). Hence $N \leq Z$. Put $J := C_G(\langle Q_L^G\rangle/Q_G)$ and $K := C_L(YQ_G/Q_G)$. Because P is an isolated subgroup of G, Lemma 4.4(a) gives $O^p(P) \leq \langle Q_L^P\rangle$ and so $Y \leq \langle Q_L^G\rangle$. Since $Q_L \trianglelefteq JS$, $P \not\leq JS$ and so $JS \leq L$. Therefore $J \leq K$. Because $Z \leq L$, Z normalizes Q_L and so $[Q_L, Z] \leq Q_L \cap Z = Q_Z \leq Q_G$. So we have $Z \leq J \leq K$. Since $Q_G \leq K$, Y normalizes K and so $K \trianglelefteq \langle Y, L\rangle = G$. So, as $K \leq L$, $K \leq Z$. Thus $Z = J = K$ and (c) holds.

Since $\overline{P} = PN/N \cong P/(P \cap N)$, Lemma 3.3(a) implies that $\overline{P} \in \mathcal{P}_{\overline{G}}(\overline{S})$. By (c) we also have $\overline{P} \not\leq \overline{L}$. Now suppose that $\overline{R} \in \mathcal{P}_{\overline{G}}(\overline{S})$ with $\overline{R} \not\leq \overline{L}$. Then, as $N \leq L$ by (c), $R \not\leq L$. Since $R \geq NS \geq S$, we infer from Lemma 4.3(a) that $P \leq R$ and that, as $\overline{R}$ is p-minimal, $L \cap R$ is the unique maximal subgroup of R containing NS. But $P \geq S$ and so $PN \geq NS$. Since $P \not\leq L$, we deduce that $PN = R$ and that $\overline{R} = \overline{P}$. Hence, as $N_{\overline{G}}(\overline{S}) = \overline{N_G(S)} \leq \overline{L}$, $\overline{L} = L_{\overline{G}}(\overline{P}, \overline{S})$. Finally assume that $\overline{A} \leq Q_{\overline{P}}$. Let K denote the inverse image in G of $Q_{\overline{P}}$. Evidently $\langle A^P\rangle \leq K$. By Lemma 4.4(a), $O^p(P) \leq \langle A^P\rangle$. So $O^p(P) \leq K$ and consequently, as K/N is a p-group, we obtain $O^p(P) \leq N$ whereas $O^p(P) \not\leq N$. Therefore $\overline{P}$ is $\overline{A}$-isolated in $\overline{G}$ and so (d) holds. □

Lemma 4.7. *Suppose that* $P \in \mathcal{P}_G(S)$ *is an* A*-isolated subgroup of* G*. If* $O^p(G)$ *is* p*-closed, then* $G = L_G(P, S)P$ *and* $\langle A^G\rangle = \langle A^P\rangle$*.*

Proof. We set $L := L_G(P, S)$, and prove that $G = LP$. Because of Lemma 4.6(d), without loss of generality we may assume that $Q_G = 1$. So $O^p(G)$ is a p'-group, and hence $O^p(P)$ is also a p'-group. Therefore, by Lemma 3.2(d), P is a $\{t, p\}$-group for some prime $t \neq p$. For each prime r dividing $|O^p(G)|$, $O^p(G)$ has an S-invariant Sylow r-subgroup, denoted S_r, and we may select S_t so that $P \leq SS_t$. Therefore $P = (P \cap S_t)S$. If r is a prime divisor of $|O^p(G)|$ with $r \neq t$, then evidently $P \not\leq SS_r$ and consequently $SS_r \leq L$ by Lemma 4.3(a). Thus $O^p(G) = (O^p(G) \cap L)S_t$. Since $SS_t \not\leq L$, Lemma 4.3(a) implies that $SS_t \cap L = S(S_t \cap L)$ is a maximal subgroup of SS_t. Now $N_{S_t}(S_t \cap L)$ is S-invariant and so $SN_{S_t}(S_t \cap L)$ is a subgroup of SS_t. Thus $S_t \cap L \trianglelefteq S_t$ and S acts irreducibly on $S_t/(S_t \cap L)$. As $S_t \cap P \not\leq S_t \cap L$, this gives $S_t = (S_t \cap L)(S_t \cap P)$. Hence we have

$$\begin{aligned} G &= O^p(G)S = (O^p(G) \cap L)S_tS \\ &= (O^p(G) \cap L)(S_t \cap L)(S_t \cap P)S = LP. \end{aligned}$$

Finally it follows that $\langle A^G\rangle = \langle A^{LP}\rangle = \langle A^P\rangle$ and this completes the proof of Lemma 4.7. □

Corollary 4.8. *Suppose that $P \in \mathcal{P}_G(S)$ is an A-isolated subgroup of G. If $Y := \langle O^p(P)^G \rangle$ is p-closed, then $Y = O^p(P)$.*

Proof. We have that P is an A-isolated subgroup of $H := SY$. Since $O^p(H) \le Y$ and Y is p-closed, $O^p(H)$ is p-closed. Therefore as A is normalized by $L := L_G(P,S)$, Lemmas 4.6(b) and 4.7 imply

$$\langle A^P \rangle = \langle A^{SY} \rangle = \langle A^{LY} \rangle = \langle A^G \rangle.$$

By Lemma 4.4(a), $O^p(P)$ is a characteristic subgroup of $\langle A^P \rangle$ and so $O^p(P)$ is normal in G. Thus $Y = \langle O^p(P)^G \rangle = O^p(P)$, as claimed. □

Lemma 4.9. *Suppose that $P \in \mathcal{P}_G(S)$ is an A-isolated subgroup of G, and set $L := L_G(P,S)$, $Z := \bigcap L^G$ and $Y := \langle O^p(P)^G \rangle$.*

(a) If $N \trianglelefteq G$ and $N \not\le L$, then $Y \le N$.

(b) If N is a proper characteristic subgroup of Y, then $N \le Z$ and $[Y,N] \le Q_G$.

(c) Either $Y = O^p(P)$ and there exists a prime $t \ne p$ such that YQ_G/Q_G is a t-group of class at most 2, or YQ_G/Q_G is a central product of quasisimple groups.

(d) Assume that $Y \ne O^p(P)$, and let $K \le Y$ be such that KQ_G/Q_G is a component in YQ_G/Q_G. Then $YQ_G/Q_G = \langle K^S \rangle Q_G/Q_G$ and $K \cap P \not\le L$.

(e) $Y \le F_p^*(G)$.

Proof. (a) Since $N \trianglelefteq G$ and $N \not\le L$, $S \le NS \le G$ and $NS \not\le L$. Thus $P \le NS$ by Lemma 4.3(a) and so $O^p(P) \le O^p(NS) \le N$. Therefore $Y = \langle O^p(P)^G \rangle \le N$.

(b) Because $Y \not\le N$, part (a) implies that $N \le L$. Hence $N \le \bigcap L^G = Z$. Thus, by Lemma 4.6(c), $[N,Y] \le Q_G$.

For the proof of parts (c) and (d), by Lemma 4.6(d), we may suppose without loss of generality that $Q_G = 1$.

(c) Suppose that $E(Y) \ne Y$. Then $[E(Y),Y] = 1$ by (b) and so $E(Y) = 1$. Hence $F^*(Y) = F(Y)$. Since $C_Y(F^*(Y)) \le F^*(Y)$ using (b) again we see that $Y = F^*(Y) = F(Y)$. So Y is nilpotent and in particular is p-closed. Applying Corollary 4.8 gives $Y = O^p(P)$. Therefore either $Y = E(Y)$ or $Y = O^p(P)$. Finally, in the latter case, Lemma 4.4(c) implies that Y is a t-group of class at most 2 for some prime $t \ne p$.

(d) Assume that $Y \ne O^p(P)$ and let K be a component of Y. Then $K \trianglelefteq Y$ by (c). Hence, as $G = YL$,

$$\langle K^G \rangle = \langle K^{YL} \rangle = \langle K^L \rangle.$$

Since $[Z,Y] = 1$ by Lemma 4.6(c), we infer that $K \not\le L$. Therefore $\langle K^S \rangle \ge O^p(P)$ by Lemma 4.3(a). If K_1 is a component of Y which is not contained in $\langle K^S \rangle$, then using (b) we get

$$O^p(P) \le \langle K^S \rangle \cap \langle K_1^S \rangle \le Z(Y) \le Z \le L,$$

which is impossible. Therefore $Y = \langle K^S \rangle$ and the first part of (d) holds.

Assume that $K \cap P \le L$, and argue for a contradiction. So $K \cap P \le L \cap P$ which is the unique maximal subgroup of P containing S. Since $(K \cap P) \trianglelefteq\trianglelefteq P$, Lemma 3.2(c) gives $K \cap S = (K \cap P) \cap S \le Q_P$. Hence, as $Y = \langle K^S \rangle$, $Y \cap S = Y \cap Q_P \trianglelefteq P$ and, in particular, $O^p(P)$ is p-closed. Set $R := N_Y(Y \cap S)S$. Then $O^p(R)$ is p-closed and

$R \geq P$. Therefore, $R = (R \cap L)P$ and $\langle A^R \rangle = \langle A^P \rangle \leq P$, using Lemma 4.7. Suppose that $A \leq N_G(K)$. Then

$$[A, K \cap R] \leq K \cap \langle A^R \rangle \leq K \cap P \leq L$$

which implies that $[A, K \cap R]$ normalizes A. It follows from Lemma 2.3 that $[A, K \cap R]$ is a p-group. Consequently, as $K \cap R = N_K(K \cap S)$ is p-closed,

$$[A, K \cap R] \leq Q_P \cap K.$$

Thus $[A, R] = \langle [A, K \cap R]^S \rangle \leq Q_P$. But then $\langle A^R \rangle = \langle A^P \rangle \geq O^p(P)$ is a p-group, a contradiction. Hence we conclude that $A \not\leq N_G(K)$. So $Q_L \not\leq N_G(K)$. Since $[K \cap L, Q_L]$ is a p-group we infer that $(K \cap L)Z(Y)/Z(Y)$ is a p-group which yields that $K \cap L$ is nilpotent. Hence $K \cap S = O_p(K \cap L)$ and therefore $Y \cap S = O_p(Y \cap L)$. Thus

$$Y \cap Q_P = Y \cap S \trianglelefteq \langle P, Y \cap L \rangle = Y,$$

and we conclude that Y is p-closed. Now a final application of Corollary 4.8 gives $Y = O^p(Y) = O^p(P)$. By assumption $Y \neq O^p(P)$ and so this is the desired contradiction. Therefore $K \cap P \not\leq L$.

Part (e) follows from (c). □

Lemma 4.10. *Suppose that $P \in \mathcal{P}_G(S)$ is A-isolated and narrow and set $Y := \langle O^p(P)^G \rangle$. Then either $Y = O^p(P)$ or YQ_G/Q_G is quasisimple.*

Proof. As usual we set $L := L_G(P, S)$. We assume that $Y \neq O^p(P)$ and aim to show that YQ_G/Q_G is quasisimple. Because of Lemma 4.6(d) there is no loss in assuming that $\bigcap L^G = 1$. By Lemma 4.9(c), $Z(Y)Q_Y < Y$ and so, by Lemma 4.9(b), $Z(Y)Q_Y \leq \bigcap L^G = 1$. Thus we seek to show that Y is simple. Put $F := \Phi_p(P)$ and $X := O^p(P)$. By Lemma 4.9(c) and 4.9(d), Y is semisimple and for any component K of Y, $Y = \langle K^S \rangle$ and $K \cap P \not\leq L$. In particular, if K is normalized by S, then $Y = \langle K^S \rangle = K$ is simple and we are done. Hence we assume that K is not normalized by S and look for a contradiction.

Suppose first that XF/F is a non-abelian simple group. Observe that $(K \cap P)F/F \cap XF/F \neq 1$. For if $(K \cap P)F/F \cap XF/F = 1$, then $(K \cap P)F/F$ is a p-group and so, as a result, $(K \cap P)F \leq FS \leq L$ whereas $K \cap P \not\leq L$. Then, as $X \leq Y$, $(K \cap P)F/F$ is normalized by XF/F and so $(K \cap P)F \geq XF$. Now selecting $s \in S$ such that $K^s \neq K$, we have

$$X = [X, X] \leq [XF, XF] \leq [(K \cap P)F, (K^s \cap P)F] \leq [K, K^s]F = F,$$

which is a contradiction as X is not contained in F. Therefore, XF/F is an elementary abelian t-group for some prime $t \neq p$ and P is a $\{t, p\}$-group. Moreover, by Lemma 3.2(f), SF is a maximal subgroup of P. We have that $O^p(K \cap P) \leq O^p(P) = X$ and so, as $K \cap P \not\leq L$, $X = \langle O^p(K \cap P)^S \rangle$. Set $D := O^p(K \cap P)$ and note that, as Y is semisimple,

$$X = \bigtimes_{T \in D^S} T.$$

By Lemma 4.4(d), $\Phi_p(X) = X \cap F$. On the other hand, by Lemma 2.6, $\Phi_p(X) = \bigtimes_{T\in D^S} \Phi_p(T)$ and so we conclude that

$$XF/F \cong X/(X \cap F) = \bigtimes_{T\in D^S} T/\Phi_p(T).$$

Since P is narrow, it must be that $D^S = \{D\}$ and so $X = D$. Hence K is normalized by S, a contradiction. □

It is now a simple matter to deduce Theorems 1.3 and 1.6 and Corollary 1.4.

Proof of Theorem 1.3. Combining Corollary 4.8, Lemma 4.9(c), (d) and (e) together with Lemma 4.10 yields Theorem 1.3. For part (b) we note that p divides $|Y/Q_Y|$ else Y would be p-closed and so, by part (a), $Y = O^p(P)$. □

Proof of Corollary 1.4. This follows from Theorem 1.3 and Lemma 4.9(c). □

Proof of Theorem 1.6. By Lemma 4.9(e), $\langle \mathcal{P}_G(S)\rangle \le F_p^*(G)$ and hence, by Lemma 3.1, $G = F_p^*(G)N_G(S)$. The next lemma completes the proof of Theorem 1.6. □

Lemma 4.11. *Suppose that $Q_G = 1$ and $P_1, P_2 \in P_G(S)$ are both isolated subgroups of G. Then either $\langle O^p(P_1)^G\rangle = \langle O^p(P_2)^G\rangle$ or $[\langle O^p(P_1)^G\rangle, \langle O^p(P_2)^G\rangle] = 1$.*

Proof. For $i = 1, 2$, set $Y_i = \langle O^p(P_i)^G\rangle$ and $L_i = L_G(P_i, S)$. If Y_1 and Y_2 are products of components of G then Theorem 1.3 guarantees that either $Y_1 = Y_2$ or $[Y_1, Y_2] = 1$. So suppose that $Y_1 = O^p(P_1)$ and without loss that $P_1 \ne P_2$. Then $P_1 \le L_2$ and so

$$[Q_{L_2}, Y_1] = [Q_{L_2}, O^p(P_1)] \le O_p(O^p(P_1)) \le Q_G = 1.$$

Since $O^p(P_2) \le \langle Q_{L_2}^{P_2}\rangle$ by Lemma 4.4(a), we get $Y_2 \le \langle Q_{L_2}^{G}\rangle$ and conclude that $[Y_1, Y_2] = 1$. □

In the companion paper [MPR] we will investigate specific simple groups with an eye to showing that they have or do not have an isolated narrow subgroup. The remaining results in this section will be applied to proper subgroups of such groups. And in all these results for $P \in \mathcal{P}_G(S)$ an isolated subgroup of G we set $L := L_G(P, S)$ and $Y := \langle O^p(P)^G\rangle$.

Lemma 4.12. *Suppose that $P \in \mathcal{P}_G(S)$ is an isolated subgroup in G. Then either P is soluble or $O^p(P)Q_P/Q_P$ is a central product of quasisimple groups. If, additionally, P is narrow, then either P is soluble or $O^p(P)Q_P/Q_P$ is quasisimple.*

Proof. Let $M := L \cap P$. Then, by Lemma 4.3(b), M is the unique maximal subgroup of P containing S. Put $F := \bigcap M^P$. Then F is normal in P and contained in L. Therefore $[Q_L, F] \le Q_L \cap F \le S \cap F = Q_P$. Thus F/Q_P is centralized by $\langle Q_L^P\rangle \ge O^p(P)$ and so $FQ_P/Q_P \le Z(O^p(P)Q_P/Q_P)$. Since $P \in \mathcal{P}_G(S)$, we have that either P is soluble or $O^p(P)Q_P/Q_P$ is a central product of quasisimple groups. If P is narrow, the latter case implies that $O^p(P)Q_P/Q_P$ is quasisimple. □

Lemma 4.13. *Suppose that $O^p(G)$ is quasisimple and set $\overline{G} := G/Z(O^p(G))$. If $P \in \mathcal{P}_G(S)$ is an A-isolated subgroup of G, then $\overline{P}$ is an $\overline{A}$-isolated subgroup of $\overline{G}$.*

Proof. Set $Z := Z(O^p(G))$ and $L := L_G(P, S)$. If $O^p(P) \not\leq Z$, then the result follows from Lemma 4.6(d). So assume $O^p(P) \leq Z$. Then $[O^p(P), O^p(L)] = 1$ which, as $G = \langle L, O^p(P)\rangle$, means that $O^p(L) \trianglelefteq G$. But $O^p(L) \leq O^p(G)$ and $O^p(G)$ being quasisimple implies that $O^p(L) \leq Z$ or $O^p(L) = O^p(G)$. Both possibilities are impossible as, in the first case, we get that $G = SZ$ which is soluble and, in the second case, we get $P = O^p(P)S \leq O^p(G)S = O^p(L)S = L$. □

Lemma 4.14. *Suppose that $Q_G = 1$, $F(G) = C_G(E(G))$ and G operates transitively by conjugation on the set of components of G. If $P \in \mathcal{P}_G(S)$ is narrow and isolated in G, then $E(G) = Y$ is quasisimple.*

Proof. Assume that P is a narrow isolated subgroup of G. Suppose first that Y is soluble. Then Corollary 1.4(b) implies that $Y = O^p(P)$. As $G = LY$, we have that $Q_LY \trianglelefteq G$ and this subgroup is also soluble. Hence Q_LY centralizes $E(G)$. But then $Q_LY \leq C_G(E(G))$ which is nilpotent by assumption. Thus Q_LY is nilpotent and this contradicts $Q_L \not\leq Q_P$. Thus Y is not soluble and so Lemma 4.10 implies that Y is a component of G. Since G acts transitively on the set of components of G, we obtain $Y = E(G)$ and this proves the lemma. □

Lemma 4.15. *Suppose that $Q_G = 1$, N is a normal subgroup of G with N soluble and $C_G(N) \leq N$. If $P \in \mathcal{P}_G(S)$ is isolated in G, then $O^p(P) \trianglelefteq G$ and P is soluble.*

Proof. Suppose that $N \leq L$. Then $O^p(P) \not\leq N$ and hence $Y \not\leq N$. However, Lemma 4.6(c) implies that $Y \leq C_G(N) \leq N$, which is a contradiction. Therefore $N \not\leq L$. Hence $P \leq NS$ by Lemma 4.3(a). But then $O^p(P) \leq N$ and, as N is soluble, Lemma 4.9(c) gives the result. □

Lemma 4.16 exploits the fact that, as observed in Lemma 4.2(b), an isolated p-minimal subgroup is normalized by $N_G(S)$.

Lemma 4.16. *Suppose that $P \in \mathcal{P}_G(S)$ is an isolated subgroup of G. Assume that one of the following holds.*

(a) $N_G(S)$ acts irreducibly on S.
(b) $N_G(S) = L$.
(c) $N_G(S)$ is contained in a unique maximal subgroup of G.

Then $P \trianglelefteq G$.

Proof. Suppose (a) holds. Since $N_G(S) \leq L$, we get that $Q_L = S$ and so (b) holds. Now suppose (b) holds. Then, by Lemma 4.3(c), $N_G(S)$ is a maximal subgroup of G and hence (c) holds. So we may assume that (c) holds. Since $N_G(S) \leq L$, we get that L is the unique maximal subgroup of G containing $N_G(S)$. Because $N_G(S)P \leq N_G(P)$ and $P \not\leq L$, we must have $G = N_G(P)$ and so $P \trianglelefteq G$. □

Lemma 4.17. *Suppose that G is an almost simple group, and set $X := F^*(G)$. If $P \in \mathcal{P}_G(S)$ is A-isolated, then $P \in \mathcal{P}_{XS}(S)$ is A-isolated.*

Proof. Note that if $H \geq XS$, then $O_p(H) \leq C_G(X) = C_G(F^*(G)) = 1$. Thus $L_G(P,S) \ngeq XS$, whence $P \leq XS$. □

5. The Centre of $L_G(P,S)$

The main purpose of this section is to prove Theorem 1.7.

Theorem 5.1. *Suppose that p is a prime and G a finite $\mathcal{CK}$-group. If $P \in \mathcal{P}_G(S)$ with $O^p(P/Q_P) \in \mathcal{L}_1(p)$, then either $O^p(P) \trianglelefteq G$ or $O_p(Z(L_G(P,S))) \leq Q_P$.*

Proof. Suppose the theorem is false and let G be a minimal counterexample. Again set $L := L_G(P,S)$ and $Y := \langle O^p(P)^G\rangle$. So we have $O^p(P)$ is not normal in G and $D := O_p(Z(L)) \nleq Q_P$. Thus P is D-isolated. The minimality of G gives

5.1.1. $Q_G = 1$.

Thus, by Theorem 1.3,

5.1.2. *Y is a central product of quasisimple simple groups each of which has order divisible by p.*

Since $Q_P \leq L$, we have $[Q_P, D] = 1$ and so, by Lemma 4.4(a),

5.1.3. $[Q_P, O^p(P)] = 1$.

We next show that

5.1.4. *P is soluble and $O^p(P)$ is a p'-group.*

Since $P \cap L$ centralizes DQ_P/Q_P, [DS, (5.1)e] implies that P is soluble. Hence (5.1.4) now follows from (5.1.3).

5.1.5. $D \cap Q_P = 1$.

By (5.1.3) $D \cap Q_P$ is normalized by $\langle O^p(P), L\rangle = G$. Thus (5.1.1) gives (5.1.5).

Pick $1 \neq a \in D$ of order p and set $A := \langle a\rangle$. Then, by (5.1.5), $A \nleq Q_P$. Since L is a maximal subgroup of G, (5.1.1) implies

5.1.6. $N_G(A) = L = C_G(A)$.

5.1.7. *A is not weakly closed in G (with respect to S).*

Suppose that A is weakly closed in G. If there exists $g \in G$ such that $a^g \in S$, then $A^g \leq S$ whence $A = A^g$. So $g \in N_G(A)$ and hence, by (5.1.6), $a^g = a$. Therefore a is weakly closed in G and, in particular, in AY (with respect to $S \cap AY$). Now, as G is a $\mathcal{CK}$-group, [GLS3, Remark 7.8.3] implies that $[A, Y]$ is a p'-group. Hence by (5.1.2), $[A, Y] \leq Z(Y)$ and so the Three Subgroup Lemma and (5.1.2) once more give $[A, Y] = 1$. Hence, using (5.1.6), $O^p(P) \leq Y \leq L$, a contradiction and so (5.1.7) holds.

By (5.1.7) there exists a p-subgroup R of G with $A \leq R$ and $N_G(R) \not\leq N_G(A) = L$. So there also exists a subgroup H of G with $H \not\leq L$ and $A \leq Q_H$. Choose such an H with $|H \cap L|_p$ maximal and then with $|H|$ minimal. Let $T \in \mathrm{Syl}_p(H \cap L)$. Since $A \trianglelefteq L$ we may assume that $T \leq S$. If $T = S$, then, as $H \not\leq L$, $P \leq H$ and then $A \leq Q_H \leq Q_P$, a contradiction. Therefore,

5.1.8. $S > T$.

Next we show

5.1.9. *Let U be a p-group with $T \leq U$. Then $U \leq N_G(U) \leq L$. In particular, $T \in \mathrm{Syl}_p(H)$ and $H \neq N_H(T)$.*

We have $A \leq T \leq O_p(N_G(U \cap L))$ and either $U \cap L > T$ or, by (5.1.8),

$$|N_G(U \cap L) \cap L|_p > |U \cap L| = |T|.$$

Hence $|N_G(U \cap L) \cap L|_p > |T| = |H \cap L|_p$ and so, by the maximal choice of H, $N_G(U \cap L) \leq L$. In particular, $U = U \cap L$ and (5.1.9) holds.

5.1.10. *$L \cap H$ is the unique maximal subgroup of H containing T. In particular, $H \in \mathcal{P}_H(T)$.*

Indeed let $T \leq M < H$. Then $A \leq Q_H \leq Q_M$, $|M \cap L|_p = |T| = |H \cap L|_p$ and thus the minimal choice of H implies $M \leq L$. So $M \leq L \cap H$. By (5.1.9), $T \ntrianglelefteq H$ and so $H \in \mathcal{P}_H(T)$.

5.1.11. $C_G(O^p(H)) \cap Z(S) = 1$.

Suppose that $E := C_G(O^p(H)) \cap Z(S) > 1$ and put $C := C_G(E)$. Then, by (5.1.1), $G > C \geq \langle O^p(H), S\rangle$. Note that $S \leq C$ and $C \not\leq L$. Therefore, by Lemma 4.3(a), $P \leq C$. Also $L \cap C = L_C(P, S)$ and $D \leq Z(L \cap C)$. Thus $O_p(Z(L \cap C)) \not\leq Q_P$ and hence $O^p(P) \trianglelefteq C$ by induction. We now have $C = (L \cap C)O^p(P)$. So

$$\langle A^{O^p(H)}\rangle \leq \langle A^C\rangle = \langle A^{(L\cap C)O^p(P)}\rangle = \langle A^{O^p(P)}\rangle \leq O^p(P)A.$$

Since also $\langle A^{O^p(H)}\rangle \leq Q_H$, we get that $\langle A^{O^p(H)}\rangle \leq O^p(P)A \cap Q_H = A$ by (5.1.4). But then, by (5.1.6), $O^p(H) \leq N_G(A) = L$, a contradiction.

By the maximal choice of H, $T \in \mathrm{Syl}_p(N_G(Q_H))$. Hence $Z(S) \leq T \leq H$. Since $H \in \mathcal{P}_H(T)$ by (5.1.10), Lemma 3.2(a) and (e) imply that either $C_H(Q_H) \geq O^p(H)$ or $Q_H \in \mathrm{Syl}_p(C_H(Q_H))$ and $C_H(Q_H)$ is p-closed. In the former case, $O^p(H) \leq C_G(A) = L$ which is not the case. Thus, as $Z(S) \leq C_H(Q_H)$, we have

5.1.12. *$C_H(Q_H) \leq Q_H$ and $Z(S) \leq Q_H$.*

5.1.13. *$p = 2$ and $\Omega_1(Z(S)) = A$ has order 2.*

Since $A \leq Z(T)$ and H is p-minimal by (5.1.10), we have $C_H(\Omega_1(Z(T))) \leq H \cap L$. Also, since $T < S$, $N_S(J(T)) > T$ and so the maximal choice of H shows that $J(T)$ is not normal in H. It follows that $H > H \cap L \geq \langle N_H(J(T)), C_H(\Omega_1(Z(T)))\rangle$. Therefore, since $\Omega_1(Z(S)) \leq \Omega_1(Z(Q_H))$ and $H \cap L$ centralizes A, we infer from Corollary 3.9, (5.1.11) and (5.1.12) that $p = 2$ and $|A| \leq |\Omega_1(Z(S))| = 2$.

5.1.14. $Q_P = 1$.

Since $A \cap Q_P = 1$ by (5.1.5), (5.1.13) implies $Q_P \cap \Omega_1(Z(S)) = 1$. Hence also $Q_P = 1$.

5.1.15. *Q_H is a fours group, $T \cong \mathrm{Dih}(8)$, $H/C_H(Q_H) \cong \mathrm{Sym}(3)$, $S \cong \mathrm{SDih}(16)$ and $O^2(P) \cong 3^2$ or 3^{1+2}_+.*

From (5.1.4), (5.1.14) and Lemma 3.6, $|S| \le 2^4$. Note that $1 < A < Q_H < T < S$. Thus $|S| = 2^4$, $|Q_H| = 4$ and $|T| = 8$. Since $A \ntrianglelefteq H$, $Q_H \cong 2^2$ and since $Q_H \ntrianglelefteq S$, $T = \langle Q_H^S \rangle \cong \mathrm{Dih}(8)$. Together with Lemma 3.6 we infer that (5.1.15) holds.

5.1.16. *G has a unique conjugacy class of involutions.*

By (5.1.15) all involutions in S are conjugate in S to an involution in the fours group Q_H. Since H acts transitively by conjugation on $Q_H^\sharp$, (5.1.16) follows.

From (5.1.15) and (5.1.16) either G is simple or G' is simple and $T \in \mathrm{Syl}_2(G')$. Suppose first that G is simple. Then, since $S \cong \mathrm{SDih}(16)$, $G \cong \mathrm{Mat}_{11}$ or $G \cong \mathrm{PSL}_3^\epsilon(q)$ with $q + \epsilon \equiv 4 \pmod 8$ by [ABG]. In the first case there exists $S \le K \le G$ with $K \cong \mathrm{Mat}_{10}$. So $K \in \mathcal{P}_G(S)$, but neither $P = K$ nor $K \le L$, a contradiction. If $\epsilon = +$, then $L = C_G(A) \cong \mathrm{GL}_2(q)$ is not a maximal subgroup of G. So $\epsilon = -$. In this case, as the Sylow 3-subgroups of $\mathrm{PSL}_3^-(q)$ are abelian, we have $O^2(P) = 3^2$ and $P \cong 3^2 : \mathrm{SDih}(16)$. Since the Sylow 3-subgroups of G are not cyclic, we infer that 3 divides $q+1$ and that $|Z(\mathrm{SL}_3^-(q))| = 3$. It follows that the preimage J of $O^2(P)$ in $\mathrm{SL}_3^-(q)$ is extraspecial and that $N_{\mathrm{SL}_3^-(q)}(J)/J \cong \mathrm{Q}_8$. This contradicts $S \le N_G(O^2(P))$ and $|S| = 2^4$. Therefore $G \ncong \mathrm{PSL}_3^-(q)$. So G is not simple.

Hence G' is simple and by [GW], $G' \cong \mathrm{Alt}(7)$ or $G' \cong \mathrm{PSL}_2(q)$ with $q \cong 7, 9 \pmod{16}$. The former possibility is ruled out as a Sylow 2-subgroup of $\mathrm{Sym}(7)$ is not isomorphic to $\mathrm{SDih}(16)$. Hence $G' \cong \mathrm{PSL}_2(q)$. If $q \ne 3^a$, then G has cyclic Sylow 3-subgroups and this contradicts $O^2(P) \le G'$. Thus $q = 3^a$. In this case $P \cap G'$ normalizes a 3-subgroup and hence a Sylow 3-subgroup U of G'. But $N_{G'}(U)/U$ is cyclic, $T \le G'$ and $P \ge T \cong \mathrm{Dih}(8)$. This contradiction completes the proof of Theorem 5.1. □

Our final observation turns out to be very useful in [MPR].

Corollary 5.2. *Let p be a prime and G be a finite $\mathcal{CK}$-group with $Q_G = 1$. Suppose that $P \in \mathcal{P}_G(S)$ is an isolated subgroup of G with $O^p(P/Q_P) \in \mathcal{L}_1(p)$ and that $O^p(P)$ is not normal G. Then the following hold.*

(a) If $[Q_P, O^p(P)] = 1$, then $C_{Q_L}(O^p(L)) = 1$.
(b) $Q_L \cap O^p(G) \ne 1$.
(c) $L \cap O^p(G)$ is a p-local subgroup of $O^p(G)$.

Proof. Put $D := O_p(Z(L))$ where $L := L_G(P, S)$. By Theorem 5.1, $D \le Q_P$ and so $[D, O^p(P)] = 1$ and $D \trianglelefteq \langle O^p(P), L \rangle = G$. Thus $D = 1$ and, since $C_{C_{Q_L}(O^p(L))}(S) \le D$, (a) holds.

For (b), suppose that $Q_L \cap O^p(G) = 1$. Then

$$[Q_L, O_p(O^p(P))] \leq Q_L \cap O^p(P) \leq Q_L \cap O^p(G) = 1.$$

Therefore $[\langle Q_L^P \rangle, O_p(O^p(P))] = 1$. Since $O^p(P) \leq \langle Q_L^P \rangle$ by Lemma 4.4(a), we get $[O^p(P), O_p(O^p(P))] = 1$. Now

$$[Q_P, O^p(P)] \leq Q_P \cap O^p(P) = O_p(O^p(P)),$$

and so $[Q_P, O^p(P), O^p(P)] = 1$. Hence $[Q_P, O^p(P)] = 1$. Thus $C_{Q_L}(O^p(L)) = 1$ by part (a), Further

$$[Q_L, O^p(L)] \leq Q_L \cap O^p(L) \leq Q_L \cap O^p(G) = 1$$

and consequently $Q_L \leq C_{Q_L}(O^p(L)) = 1$, a contradiction. Hence $Q_L \cap O^p(G) \neq 1$ and (b) holds. □

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