The E-Uniqueness Theorem

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April 7, 2002

1 Introduction

Let G be a finite group and p a prime dividing the order of G. We say that G has *characteristic* p if $\mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G)$ and we say that G has *local characteristic* p if all p-local subgroups of G have characteristic p. This paper is part of the project to classify all finite groups of local characteristic p. The classification is divided into two main part: The E-uniqueness case ($E!$) and the non E-uniqueness case ($\neg E!$). To explain these cases we need to introduce some notation:

Let G be a finite group of local characteristic p,
$S \in \mathrm{Syl}_p(G)$.
$Z = \Omega_1 \mathrm{Z}(S)$,
$\mathcal{L} = \{L \leq G \mid \mathrm{C}_G(\mathrm{O}_p(L)) \leq \mathrm{O}_p(L)\}$,
$\mathcal{M}$ is set of maximal elements of $\mathcal{L}$;

If $\mathcal{T}$ is a set of subgroups of G and $H \leq G$, then $\mathcal{T}(H) = \{T \in \mathcal{T} \mid H \leq T\}$ and $\mathcal{T}_H = \{T \in \mathcal{T} \mid T \leq H\}$.

We say that $T \in \mathcal{L}$ is a uniqueness subgroup of G if T is contained in a unique maximal p-local of G, that is if $|\mathcal{M}(T)| = 1$.

For $L \in \mathcal{L}$ let Y_L be the the largest p-reduced normal subgroup of G (see 4.1).

For H a finite group, $\mathrm{F}^*_p(H)$ is defined by $\mathrm{F}^*_p(H)/\mathrm{O}_p(H) = \mathrm{F}^*(H/\mathrm{O}_p(H))$.

$\tilde{C}$ is a maximal p-local containing $\mathrm{N}_G(Z)$ (in symbols: $\tilde{C} \in \mathcal{M}(N_G(Z))$).

$E = \mathrm{O}^p(\mathrm{F}^*_p(\mathrm{C}_{\tilde{C}}(Y_{\tilde{C}})))$.

$E!$ now means that E is a uniqueness subgroup and $\neg E!$ means that E is contained in at least two different maximal p-locals of G.

2 Some unnecessary comments on groups of parabolic characteristic p

Let G be a finite group and p a prime dividing the order of p. A subgroup P of G is called a *parabolic* if it contains a Sylow p- subgroup of G. A parabolic P is called a *local* *parabolic* if $O_p(P) \neq 1$. A parabolic is called regular, if it contains the normalizer of Sylow p-subgroup. G is of (regular) parabolic characteristic p if all (regular) local parabolics are of characteristic p. We eventually hope to extend the classification of groups of local characteristic p to the groups of regular parabolic characteristic p.

The Monster and the Baby Monster are example of groups which are of parabolic characteristic 2, but not of local characteritic 2. J_1 is a group which is of regular parabolic characteristic 2, but not of parabolic characteristic 2.

3 An unnecessary section on bricks

Definition 3.1 *Let G be a finite group. A* brick *of G is a perfect subnormal subgroup B of G such that B has a unique maximal normal subgroup M_B.* $\mathrm{Bri}(G)$ *denotes the sets of all bricks of G.*

Lemma 3.2 [**minimal subnormal supplement**] *Let G be a finite group and D a normal subgroup of G with G/D perfect.*

(a) *There exists a the unique minimal subnormal supplement $B = \mathrm{B}(G, D)$ to D in G.*

(b) *B is normal in G*

(c) *If G/D is simple, then B is the unique brick of G with $B \not\leq D$. Moreover $[B, D] \leq M_B = B \cap D$.*

(d) *If G is perfect, then $G = BD^\infty$.*

Proof: (a) Let B_1 and B_2 be minimal subnormal supplements to D in G. We need to show that $B_1 = B_2$. If $G = B_i$ for some i this is obvious. So we may assume that $B_i \leq M_i$ for a proper normal subgroup M_i of G. Then $G = M_iD$. Put $M = [M_1, M_2]$. Since G/D is perfect, $G = [G, G]D = [M_1D, M_2D]D = MD$. By induction theee exists a unique minimal supplement B to $M \cap D$ in M. Since $G = MD$ and $M \leq M_i$, $M_i = M(D \cap M_i$ and so $M_i = B(D \cap M_i$. By induction $B = \mathrm{B}(M_i, D \cap M_i) = B_i$ and thus $B_1 = B_2$.

(b) Let $g \in G$. The also B^g is a minimal subnormal supplement to D in G and so $B = B^g$ by the uniqueness of B.

(c) Let M be a normal subgroup of B. Suppose that $M \not\leq D$. The $MD/D \trianglelefteq BD/D = G/D$. Since G/D is simple, $G = MD$ and so the minimality of B implies $M = B$. Thus $B \cap D$ is the unique maximal normal subgroup of G and B is a brick. Let $\tilde{B}$ be any brick of G with $\tilde{B} \not\leq D$. Then $\tilde{B}D/D$ is a non-trivial subnormal subgroup of the simple G/D and so $\tilde{B}D = G$. Thus $B \leq \tilde{B}$. Moreover, $\tilde{B}/\tilde{B} \cap D$ is simple and so $\tilde{B} \cap D = M_{\tilde{B}}$. In particular $B \not\leq M_{\tilde{B}}$ and so $B = \tilde{B}$.

(d) Since G/B is perfect and $G/B = DB/B$ we get $G/D = D^\infty B/B$. □

Proposition 3.3 [**bricks and subnormal subgroups**] *Let B be a brick of the finite group G and $N \trianglelefteq\trianglelefteq G$. Then either $B \leq N$ or N normalizes B and $[B, N] \leq M_B$.*

Proof: If $N = G$, $B \leq N$. So we may assume that N is contained in a maximal normal subgroup D of G. If $B \leq D$ we are done by induction. So suppose that $B \not\leq D$. Then by 3.2 $D = \mathrm{B}(G, D)$ and $[B, N] \leq [B, D] \leq M_B$. □

Lemma 3.4 [**products of bricks**] *Let B_1 and B_2 be bricks of the finite group G. Then $\langle B_1, B_2\rangle = B_1B_2$ and exactly one of the following holds.*

1. $B_1 = B_2$
2. $B_1 \leq M_{B_2}$,
3. $B_2 \leq M_{B_1}$.
4. $[B_1, B_2] \leq M_{B_1} \cap M_{B_2}$.

Proof: If $B_1 \not\leq B_2$ and $B_2 \not\leq B_1$ then by 3.3 $[B_1, B_2] \leq M_{B_1} \cap M_{B_2}$. So we may assume $B_1 \leq B_2$. But then $B_1 = B_2$ or $B_1 \leq M_{B_2}$. So one of (1)-(4) holds. Since B_i is perfect its easy to see that at most one of (1)-(4) can hold. Moreover in all four cases, $\langle B_1, B_2\rangle = B_1B_2$. □

Lemma 3.5 [**Ginfty**] $\mathrm{Bri}(G) = \mathrm{Bri}(G^\infty)$ *and* $G^\infty = \prod_{B\in\mathrm{Bri}(G)} B$.

Proof: Note that a brick of G^∞ is a brick of G and all bricks of G are contained in G^∞. Thus $\mathrm{Bri}(G) = \mathrm{Bri}(G^\infty)$. Let D be a maximal normal subgroup of G^∞ Then by 3.2 there exists a brick B with $G^\infty = BD^\infty$. By induction D^∞ is the products of its bricks. So also G^∞ is the products of its bricks. □

4 The Largest p-reduced normal subgroup

Let L be a finite group of characteritic p. An elementary abelian normal subgroup V of L is called p-reduced if any normal subgroup of G which acts unipotently on V has to act trivially. Note that this is equivalent to $\mathrm{O}_p(L/\mathrm{C}_L(V)) = 1$.

Lemma 4.1 [**YL**] *Let L be a finite group of characteritic p and $S \in \mathrm{Syl}_p(L)$*

(a) *There exists a unique maximal p-reduced normal subgroup Y_L of L.*

(b) *Let $R \leq L$ and X a p-reduced normal subgroup of R. Then $\langle X^L\rangle$ is a p-reduced normal subgroup of L. In particular, $Y_R \leq Y_L$.*

(c) *Let $S_L = C_S(Y_L)$ and $L^f = N_G(S_L)$. Then $L = L_f\,\mathrm{C}_L(Y_L)$, $S_L = O_p(L^f)$ and $Y_L = \Omega_1 Z(S_L)$.*

(d) $Y_S = \Omega_1 \mathrm{Z}(S)$, $Z_L := \langle \Omega_1 \mathrm{Z}(S)^L \rangle$ *is p-reduced for L and* $\Omega_1 \mathrm{Z}(S) \leq Z_L \leq Y_L$.

Proof: (a) Let Y_L be the subgroup generated by the p-reduced normal subgroups of L. Let N be a normal subgroup acting unipotently on Y_L. Then N also acts unipotently on all the generators of Y_L. Hence N centralizes all the generators of Y_L and so Y_L. Thus Y_L is p-reduced.

(c) Let $Y = \langle X^L \rangle$ and $C = \mathrm{C}_L(Y)$. Let $N/C = \mathrm{O}_p(L/C)$. Then $N = (N \cap S)C$ and in particular, $N = (N \cap L)C$. As X is p reduced, $N \cap L$ centralizes X. The same is true for C and so also for N. Since N is normal in L and $Y = \langle X^L \rangle$, N centralizes Y. Thus $N = C$ and Y is p-reduced.

(b) Put $C = \mathrm{C}_L(Y_L)$. By Frattini, $L = L^f C$. Since $O_p(L/C) = 1$ we conclude $O_p(L_f) \leq C$ Hence $O_p(L_f) \leq C \cap S = S_L$ and so $O_p(L_f) = S_L$). Let $X = \Omega_1(Z(S_L))$. Then clearly $Y_L \leq X$ and L_f normalizes Y. Put $Y = \langle Y^L \rangle = \langle Y^C \rangle$. Clearly X is p-reduced for S_L and so by (b) applied to C, Y is p-reduced for C. Let N be a normal subgroup of L acting unipotently on Y. Since $Y_L \leq Y$ and Y_L is p-reduced for L, $N \leq C$. As Y is p-reduced for C, N centralizes C and so Y is p-reduced for L. By maximality of Y_L we get $Y \leq Y_L$. But $Y_L \leq X \leq Y$ and so $Y_L = X = Y$.

(d) Clearly S centralizes Y_S and so $Y_S \leq \Omega_1 \mathrm{Z}(S)$. Also $\Omega_1 \mathrm{Z}(S)$ is p-reduced for S and so $\Omega_1 \mathrm{Z}(S) \leq Y_S$. Thus $\Omega_1 \mathrm{Z}(S) = Y_L$. The remaining parts now follow from (b). □

Lemma 4.2 [**YL and subnormal subgroups**] *Let L be of characteristic p and K a subnormal subgroup of L.*

(a) $Y_L \cap K$ *and* $[Y_L, \mathrm{O}^p(K)]$ *are p-reduced for K*

(b) $[Y_L, \mathrm{O}^p(K)] \leq Y_L \cap K = Y_L \cap Y_K \leq Y_K$.

(c) $\mathrm{C}_K(Y_K) = \mathrm{C}_K([Y_L, \mathrm{O}^p(K)]) = \mathrm{C}_K(Y_L \cap Y_K) = \mathrm{C}_K(Y_L)$.

Proof:

Note that $\mathrm{O}^p(K) = \mathrm{O}^p(KY_L)$ and so $[Y_L, \mathrm{O}^p(K)] \leq Y_L \cap \mathrm{O}^p(K) \leq Y_L \cap K$. Let D be the largest normal subgroup of K acting unipotently on $[Y_L, \mathrm{O}^p(K)]$. Since K acts acts unipotently on $Y_L/[Y_L, \mathrm{O}^p(K)$ we get that D is unipotent on Y_L. Since D is subnormal in L and Y_L is p-reduced, D centralizes Y_L, $Y_L \cap K$ and $[Y_L, \mathrm{O}^p(K)]$. Thus $Y_L \cap K$ and $[Y_L, \mathrm{O}^p(K)]$ are p-reduced, $D = C_K([Y_L, \mathrm{O}^p(K)$. Thus (a) and (b) hold and for (c) it remains to show that $C_K(Y_L) \leq C_K(Y_K)$.

For this we may assume by induction on the subnormal length that K is normal in L. Then also Y_K is normal in L. Let V be a normal subgroup of L contained in Y_K which is minimal with respect to $C_K(V) = C_K(Y_L)$. Then $\mathrm{O}_p(K/C_K(V) = 1$ and V is p-reduced for K. Let D be the largest normal subgroup of L acting unipotently on V. Then $[K, D] \leq K \cap D \leq C_L(V)$. Put $W = C_V(D)$. Since $D/\mathrm{O}_p(D)$ is a p-group, the $P \times Q$-Lemma implies $C_K(W)/C_K(V)$ is a p-group. Hence $C_K(W) = C_K(V) = C_K(Y_K)$.

The minimality of V yields $V = W$. So D centralizes V and V is p-reduced for L. Thus $V \leq Y_L$ and

$$C_K(Y_L) \leq C_K(V) = V_K(Y_K).$$

□

5 The Kieler Lemma and Point Stabilizers

Lemma 5.1 [**Kieler Lemma for modules**] *Let G be a finite group L a subnormal subgroup of G, p a prime and $S \in \mathrm{Syl}_p(G)$. Let V be a $GF(p)G$ module. Then*

$$\mathrm{C}_L(\mathrm{C}_V(S)) = \mathrm{C}_L(\mathrm{C}_V(S \cap L)).$$

Proof: Without loss L is normal in G and $G = LS$. Also $\mathrm{C}_V(S) \leq \mathrm{C}_V(S \cap L)$ and replacing G by $\mathrm{C}_G(\mathrm{C}_V(S)$ we may assume that $\mathrm{C}_V(S) \leq \mathrm{C}_V(G)$. For $T \in L/L \cap S$ and $v \in \mathrm{C}_V(L \cap S)$ define $v^T = v^t$ for any $t \in T$. Note that this is independ from the choice of $t \in T$ (Also we slightly are abusing notation as v^T usually is define as $\{v^t \mid t \in T\}$. Define

$$\pi : \mathrm{C}_V(S \cap L) \to V, v \to \sum_{T \in L/S \cap L} v^T$$

Let $v \in \mathrm{C}_V(S \cap L)$ and $l \in L$. Then

$$\pi(v^l) = \sum_{T \in L/S \cap L} v^{Tl}$$

Since $T \to Tl$ is a bijection of $S/S \cap L$ we conclude $\pi(v^l) = \pi(v)$ and so $\mathrm{Im}\,\pi \leq \mathrm{C}_V(L)$. Also if $v \in \mathrm{C}_V(L)$ then $\pi(v) = mv$ where $m = |L/L \cap S|$. $L \cap S$ is a Sylow p-subgroup of L. Thus p does not devide m and $\pi|_{\mathrm{C}_V(L)}$ is one to one. We conclude that

$$\mathrm{C}_V(S \cap L) = \ker \pi \oplus \mathrm{C}_V(L).$$

Let $s \in S$ the map $T \to T^s$ is a bijection of $L/S \cap L$ and thus

$$\pi(v)^s = \sum_{T \in L/S \cap L} v^{Ts} = \sum_{T \in L/S \cap L} = (v^s)^{T^s} = \pi(v^s)$$

and we conclude that $\ker \pi$ is S-invariant. Suppose that $\ker \pi \neq 0$, then also $\mathrm{C}_{\ker \pi}(S) \neq 0$, but thus contradicts $\mathrm{C}_V(S) \leq \mathrm{C}_V(L)$ and $C_V(L) \cap \ker pi = 0$. Hence $\ker \pi = 0$ and so $\mathrm{C}_V(S \cap L) = \mathrm{C}_V(L)$. Thus

$$\mathrm{C}_L(\mathrm{C}_V(S \cap L) = L = \mathrm{C}_L(\mathrm{C}_V(S))$$

and the lemma is proved. □

Proposition 5.2 (Kieler Lemma) [**kieler lemma**] *Let G be a group of local characteritic p, L a subnormal subgroup of G and $S \in \mathrm{Syl}_p(G)$. Then*

$$\mathrm{C}_L(\Omega_1 \mathrm{Z}(S)) = \mathrm{C}_L(\Omega_1 \mathrm{Z}(S \cap L))$$

Proof: By induction G/L we may assume that L is normal subgroup of G and $G = LS$. Put $Z = \Omega_1\mathrm{Z}(S)$ and $Y = \Omega_1\mathrm{Z}(S \cap L)$. Since S normalizes $\mathrm{O}(L)$, $L \cap Z \neq 1$. Note that $L \cap Z \leq \Omega_1\mathrm{Z}(S \cap L)$. So S, $\mathrm{C}_L(Z)$ and $\mathrm{C}_L(Y)$ are all contained in $\mathrm{C}_G(L \cap Z)$ we may assume that $G = \mathrm{C}_G(L \cap Z)$. Since G is of local characteristic p, we now get that G is of characteristic p. Let $V = \Omega_1\mathrm{Z}\,\mathrm{O}(L)Z$. Since $\mathrm{O}_p(L) \leq S$, $Z \leq C_G(\mathrm{O}_p(L))$ and so $[Z, L] \leq C_G(\mathrm{O}_p(L)) \cap L$. Thus $[V, L] \leq \Omega_1\mathrm{Z}\,\mathrm{O}(L))$ and V is an elementary abelian normal p-subgroup of G. Note that $V = \Omega_1\mathrm{Z}\,\mathrm{O}_p(L)) \oplus X$ for some $X \leq \Omega_1\mathrm{Z}$ and so by a theorem of Gaschütz, $V = \Omega_1\mathrm{Z}\,\mathrm{O}_p(L) \oplus A$ for some normal subgroup A of G. But then $[A, G] = 1$, $A \leq \Omega_1\mathrm{Z}(G)$ and so $Z = (V \cap Z)A = (Z \cap L)A$. By assumption $Z \cap L \leq Z(G)$ and thus $Z = \Omega_1\mathrm{Z}(G) = C_V(G)$. Also $\mathrm{C}_V(S \cap L) = YA$ and so $\mathrm{C}_L(Y) = \mathrm{C}_L(C_V(S \cap L))$. The proof is now completed by 5.1. □

Definition 5.3 *Let G be a finite group, p a prime and $S \in \mathrm{Syl}_p(G)$. Then*

$$\mathrm{P}_G(S) := O^{p'}(\mathrm{C}_G(\Omega_1\mathrm{Z}(S))).$$

and

$$\mathrm{Pst}_p(G) = \{\ P_G(S) \mid S \in \mathrm{Syl}_p(G)\}.$$

The group $\mathrm{P}_G(S)$ is called a point stabilizer of G.

Lemma 5.4 [**alternative definition of PG(S)**] *Let G be a finite group, p a prime, $S \in \mathrm{Syl}_p(G)$ Then*

$$\mathrm{P}_G(S) = \langle T \in \mathrm{Syl}_p(G) \mid \Omega_1\mathrm{Z}(T) = \Omega_1\mathrm{Z}(S)\rangle.$$

Proof: Let $T \in \mathrm{Syl}_p(G)$ with $\Omega_1\mathrm{Z}(T) = \Omega_1\mathrm{Z}(S)$. Then clearly $T \leq \mathrm{P}_G(S)$. Conversely if $T \in \mathrm{Syl}_p(\mathrm{C}_G(\Omega_1\mathrm{Z}(S)$, then $[\Omega_1\mathrm{Z}(S), T] = 1$, $\Omega_1\mathrm{Z}(S)T$ is a p-group and so $\Omega_1\mathrm{Z}(S) \leq \Omega_1\mathrm{Z}(T)$ and so $\Omega_1\mathrm{Z}(T) = \Omega_1\mathrm{Z}(S)$. Since $\mathrm{P}_G(S)$ is just the group generated by the Sylow p-subgroups of $\mathrm{C}_G(\Omega_1\mathrm{Z}(S))$, the lemma is proved. □

Lemma 5.5 [**sylow subgroups and subnormal subgroups**] *Let G be a finite group, A_1 and A_2 subnormal subgroups of G and p a prime and $S \in \mathrm{Syl}_p(G)$.*

(a) *$A_i \cap S$ is a Sylow p-subgroup of A_i.*

(b) *For $i = 1, 2$ let S_i be a Sylow p-subgroup of A_i. Then $\langle S_1, S_2\rangle$ contains a Sylow p-subgroup of $\langle A_1, A_2\rangle$.*

(c) $\langle A_1, A_2\rangle \cap S = \langle A_1 \cap S, A_2 \cap S\rangle$.

Proof: (a) is well known [**?**].

(b) By induction on $|G/A_1||G/A_2|$. In particular we may assume that $G = \langle A_1, A_2\rangle$. If $A_i = G$ for some i, (b) holds. So we may assume that A_i lies in a maximal normal subgroup M_i of G. Let $\{1,2\} = \{i,j\}$ and $B_i = \langle A_i, A_j \cap M_i$. Then $B_i \leq M_i$ and by induction $\langle S_i, S_j \cap M_j$ contains a Sylow p-subgroup T_i of B_i. If $A_1 \neq B_1$ or $A_2 \neq B_2$ the by induction $\langle T_1, T_2\rangle$ contains a Sylow p-subgroup of G. But

$$\langle T_1, T_2\rangle \leq \langle\langle S_1, S_2 \cap M_1\rangle, \langle S_2, S_1 \cap M_2\rangle\rangle \leq \langle S_1, S_2\rangle$$

and (b) holds. So suppose that $A_1 = B_1$ and $A_2 = B_2$. Then $A_2 \cap M_1 \leq A_1$ and $A_1 \cap M_2 \leq A_1$. Thus

$$A_1 \cap M_2 = A_1 \cap A_2 = A_2 \cap M_1$$

.

Snce $A_i \cap M_j$ is normal in A_i we conclude that $A_1 \cap A_2$ is normal in $G\langle A_1 \cap A_2$. Replacing G by $G/A_1 \cap A_2$ we may assume that $A_1 \cap A_2 = 1$.

Since $G = \langle A_1, A_2\rangle = A_i M_j$ we have $A_i \cong A_i/A_1 \cap A_2 = A_i/A_i \cap M_j \cong G/M_j$ and so A_i is simple.

Suppose that A_1 is perfect. Then A_1 is a component and since $A_1 \not\leq A_2$, we get (see for example 3.3) $[A_1, A_2] = 1$. Clearly (b) holds in this case. So we may assume that for $i = 1$ and $i = 2$, A_i is an r_i group for some prime p. Hence $A_i \leq \mathrm{O}_{r_i}(G)$. If $r_1 \neq r_2$ we get again get $[A_1, A_2] = 1$. If $r_1 = r_2 \neq p$, then G is a p' group and (b) holds. Suppose finally that $r_1 = r_2 = p$. Then $A_i = S_i$ and $G = \langle S_1, \rangle S_2$. So (b) holds in all cases.

(c) By (a) and (b) both $\langle S \cap A_1, S \cap A_2\rangle$ and $\langle A_1, A_2\rangle \cap S$ are Sylow p-subgroups of $\langle A_1, A_2\rangle$. Since the first of these groups is contained in the second, they are equal. □

Lemma 5.6 [**point stabilizers and subnormal subgroups**] *Let G be a finite group of local characteristic p, A_1 and A_2 subnormal subgroups of G, $A = \langle A_1, A_2\rangle$, and $S \in \mathrm{Syl}_p(G)$.*

(a) $\mathrm{O}^{p'}(A_i \cap \mathrm{P}_G(S) = \mathrm{P}_{A_i}(S \cap A_i)$.

(b) *Let $S \in \mathrm{Syl}_p(G)$. Then*

$$\mathrm{P}_A(S \cap A) = \langle \mathrm{P}_{A_1}(S \cap A_1), \mathrm{P}_{A_2}(S \cap A_2)\rangle.$$

(c) *For $i = 1, 2$ let P_i be a point stabilizer of A_i. Then $\langle P_1, P_2\rangle$ contains a point stabilizer of A.*

Proof:

(a) Let $L = A_i$. By the Kieler Lemma 5.2, $C_L(\Omega_1 \mathrm{Z}(S)) = C_L(\Omega_1 \mathrm{Z}(S \cap L)$. Thus

$$\mathrm{P}_L(S \cap L) = \mathrm{O}^{p'}(C_L(\Omega_1 \mathrm{Z}(S))) = \mathrm{O}^{p'}(L \cap \mathrm{P}_G(S).$$

(b) Let T be a Sylow p-subgroup of $\mathrm{P}_A(S \cap A)$. By (a) $T \cap A_i \leq P_{A_i}(S \cap A_i)$. Hence by 5.5

$$T = \langle T \cap A_1, T \cap A_2 \rangle \leq \langle \mathrm{P}_{A_1}(S \cap A_1), \mathrm{P}_{A_2}(S \cap A_2) \rangle.$$

So (b) holds.

(c) Let $H = \langle P_1, P_2 \rangle$. Note that P_i contains a Sylow p-subgroup S_i of A_i and so by 5.5, H contains a Sylow p-subgroup T of A. Then $T \cap A_i = S_i^{g_i}$ for some $g_i \in H \cap A_i$ and so $\mathrm{P}_{A_i}(T \cap A_i)) = \mathrm{P}_i^{g_i} \leq H$. Hence by (b)

$$\mathrm{P}_A(T) = \langle \mathrm{P}_{A_1}(T \cap A_1), \mathrm{P}_{A_2}(T \cap A_2) \leq H.$$

So $P_A(T)$ is a point stabilizer of A contained in H. □

The following example show that under the assumtions of the previous lemma one might have:

$$\mathrm{C}_A(\Omega_1 \mathrm{Z}(S \cap A) \neq \langle \mathrm{C}_{A_1}(\Omega_1 \mathrm{Z}(S \cap A_1)), \mathrm{C}_{A_2}(\Omega_1 \mathrm{Z}(S \cap A_2)) \rangle$$

Indeed let q be a power of p with $q > 2$, $D = \Omega_4^+(q)$ and V the natural module for D. Then $D = D_1 \circ D_2$ with $D_i \cong \mathrm{Sl}_2(q)$. Let $G = V \rtimes D$ and $A_i = VD_i$. The A_i is normal in G and $G = A = A_1 A_2$. Let $S \in \mathrm{Syl}_p(G)$. Then it is easy to verify that $\mathrm{C}_{A_i}(\Omega_1 \mathrm{Z}(S \cap A_i)) = S \cap A_i$ and so

$$\mathrm{C}_{A_1}(\Omega_1 \mathrm{Z}(S \cap A_1)), \mathrm{C}_{A_2}(\Omega_1 \mathrm{Z}(S \cap A_2)) \rangle = S$$

On the otherhand $\mathrm{C}_G(\Omega_1 \mathrm{Z}(S))$ is cylic of order $q - 1$.

References

[BBSM] The Big Book of Small Modules