SYMPLECTIC MULTIPLE FLAG VARIETIES OF FINITE TYPE

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Dedicated to Professor David Buchsbaum on his retirement.

In our paper [6], we examined the following question in the case of the group $G = \mathrm{GL}_m$:

Problem: Given a reductive algebraic group G, find all k-tuples of parabolic subgroups $(P_1, \dots, P_k)$ such that the product of flag varieties $G/P_1 \times \cdots \times G/P_k$ has finitely many orbits under the diagonal action of G. In this case we call $G/P_1 \times \cdots \times G/P_k$ a *multiple flag variety of finite type.*

In this paper, we solve this problem for the symplectic group $G = \mathrm{Sp}_{2n}$. We also give a complete enumeration of the orbits, and explicit representatives for them. The cases in our classification where one of the parabolics is a Borel subgroup, $P_1 = B$, are exactly those for which $G/P_2 \times \cdots \times G/P_k$ is a spherical variety under the diagonal action of $G = \mathrm{Sp}_{2n}$, and our results specialize to classify the B-orbits on these spherical varieties.

Our main tool is, as in [6], the algebraic theory of quiver representations. Rather unexpectedly, it turns out that two multiple symplectic flags lie in the same Sp_{2n}-orbit if and only if they lie in the same GL_{2n}-orbit (a consequence of a general result in Proposition 2.1 below). This allows us to reduce our problem for Sp_{2n} to one about GL_{2n}, which we solve using the results and the quiver techniques of [6].

All of our methods extend in an obvious way to the orthogonal groups $G = \mathrm{SO}_m$, but the combinatorics of orbits becomes much more complicated and we will not present it here. Our work intersects with that of Littelmann [5], who solved our problem for an arbitrary G, but with the restrictions that $k = 3$, $P_1 = B$, and P_2, P_3 are maximal parabolic subgroups.

Acknowledgments. Some crucial technical results in our paper are influenced by a work in progress [2]. We thank Harm Derksen for generously sharing his insights with us, and Michel Brion, Dominique Luna, and Dmitry Panyushev for their helpful interest in our work.

contents

1. Main results

Date: November 23, 2004.

1991 *Mathematics Subject Classification.* 14M15, 16G20, 14L30.

The research of Peter Magyar, Jerzy Weyman and Andrei Zelevinsky was supported in part by an NSF Postdoctoral Fellowship and NSF grants #DMS-9700884 and #DMS-9625511, respectively.

1.1. **Classification of finite types.** We recall the notation of [6]. A *composition* of a positive integer m is a sequence of non-negative integers $\mathbf{a} = (a_1, \dots, a_p)$ whose sum is equal to m. The components a_i are called *parts* of $\mathbf{a}$. Define the *opposite* of $\mathbf{a} = (a_1, \dots, a_p)$, by $\mathbf{a}^{\mathrm{op}} = (a_p, \dots, a_1)$, and say that $\mathbf{a}$ is *symmetric* if $\mathbf{a} = \mathbf{a}^{\mathrm{op}}$. Let (a^k) denote the composition with k parts all equal to a.

Thoughout this paper, all vector spaces are over a fixed algebraically closed field. For a vector space V of dimension m and a composition $\mathbf{a}$ of m, we denote by $\mathrm{Fl}_{\mathbf{a}}(V)$ the variety of flags $A = (0 = A_0 \subset A_1 \subset \cdots \subset A_p = V)$ of vector subspaces in V such that

$$\dim(A_i/A_{i-1}) = a_i \quad (i = 1, \dots, p) \ .$$

Now let V be a $2n$-dimensional symplectic vector space possessing a non-degenerate alternating bilinear form $\langle \, , \, \rangle$. The group of automorphisms of V preserving the form is $\mathrm{Sp}(V) = \mathrm{Sp}_{2n}$. A subspace $U \subset V$ is *isotropic* if $\langle U, U \rangle = 0$. For a symmetric composition $\mathbf{a}$ of $2n$, we denote by $\mathrm{SpFl}_{\mathbf{a}}(V)$ the variety of flags of dimension vector $\mathbf{a}$ in V that are formed by isotropic subspaces and their orthogonals:

$$\mathrm{SpFl}_{\mathbf{a}}(V) = \{A \in \mathrm{Fl}_{\mathbf{a}}(V) \mid \langle A_i, A_{p-i} \rangle = 0 \text{ for all } i\}.$$

This is a standard realization of a partial flag variety $\mathrm{Sp}(V)/P$. (See [3, §23.3].) The complete flag variety $\mathrm{Sp}(V)/B$ corresponds to the compositon $\mathbf{a} = (1^{2n})$.

A tuple of symmetric compositions $\mathbf{d} = (\mathbf{a}_1, \dots, \mathbf{a}_k)$ of the same number $2n$ is said to be *of symplectic finite type* (Sp-finite) if the group $\mathrm{Sp}(V)$, acting diagonally, has finitely many orbits in the *multiple flag variety* $\mathrm{SpFl}_{\mathbf{d}}(V) = \mathrm{SpFl}_{\mathbf{a}_1}(V) \times \cdots \times \mathrm{SpFl}_{\mathbf{a}_k}(V)$. We will classify all such tuples.

We say that a composition is *trivial* if it has only one part, $\mathbf{a} = (2n)$. Then the corresponding flag variety $\mathrm{SpFl}_{\mathbf{a}}$ consists of a single point, so adjoining any number of trivial compositions to a tuple gives essentially the same multiple flag variety and does not affect the finite-type property.

Theorem 1.1. *If a tuple of non-trivial symmetric compositions* $(\mathbf{a}_1, \dots, \mathbf{a}_k)$ *is of symplectic finite type then* $k \leq 3$.

Thus we only need to classify *triples* of symplectic finite type. We will write $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ instead of $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$.

The vanishing of some part $a_i = 0$ of $\mathbf{a}$ means that in any flag $A \in \mathrm{Fl}_{\mathbf{a}}$, the subspace A_i coincides with A_{i-1}. Thus, removing zero parts a_i and a_{p+1-i} from a symmetric composition $\mathbf{a}$ does not change $\mathrm{Fl}_{\mathbf{a}}$ or $\mathrm{SpFl}_{\mathbf{a}}$ up to isomorphism. Given $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, let p, q and r denote their respective numbers of non-zero parts. Assume without loss of generality that $p \leq q \leq r$.

Theorem 1.2. *A triple* $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ *of symmetric compositions of* $2n$ *is of symplectic finite type if and only if it belongs to one of the following classes:*

$(\mathrm{Sp}A_{q,r})$: $(p, q, r) = (1, q, r)$, $1 \leq q \leq r$.

$(\mathrm{Sp}D_{r+2})$: $(p, q, r) = (2, 2, r)$, $2 \leq r$.

$(\mathrm{Sp}E_6)$: $(p, q, r) = (2, 3, 3)$.

$(\mathrm{Sp}E_7)$: $(p, q, r) = (2, 3, 4)$.

$(\mathrm{Sp}E_8)$: $(p, q, r) = (2, 3, 5)$.

$(\mathrm{Sp}E^{(b)}_{r+3})$: $(p, q, r) = (2, 3, r)$, $3 \leq r$, $\mathbf{b}$ *has non-zero parts* $(1, 2n-2, 1)$.

$(\mathrm{Sp}Y_{r+4})$: $(p, q, r) = (3, 3, r)$, $3 \leq r$, $\mathbf{a}, \mathbf{b}$, *or* $\mathbf{c}$ *has non-zero parts* $(1, 2n-2, 1)$.

The labels are taken from [6], and refer to Dynkin graphs associated to the first five cases. Note that except for type (SpY), the dimension vectors $\mathbf{d}$ of all the above types also appear on our list of GL-finite dimensions in [6, Theorem 2.2]. That is, not only does Sp_{2n} have finitely many orbits on $\mathrm{SpFl}_{\mathbf{d}}$, but GL_{2n} has finitely many orbits on $\mathrm{Fl}_{\mathbf{d}}$.

As in [6], the type (SpA) covers all symplectic multiple flag varieties with only one or two non-trivial factors. Note that if $p = 2$ then $\mathbf{a} = (n, n)$, and the corresponding variety $\mathrm{SpFl}_{(n,n)}(V)$ is the variety of all Lagrangian subspaces in V. Thus the case (SpD) covers triple symplectic flag varieties in which two of the factors are Lagrangian Grassmannians, and the third factor is arbitrary.

In general, each flag in a symplectic flag variety is completely determined by its "lower half" consisting of isotropic subspaces. Thus, the cases (SpE_6), (SpE_7), and (SpE_8) correspond to triple flag varieties in which: the first flag contains a single Lagrangian subspace; the second flag contains a single isotropic subspace; and the third flag contains at most two isotropic subspaces. The cases (Sp$E^{(b)}_{r+3}$) and (SpY_{r+4}) correspond to triple flag varieties in which: the first flag contains a single isotropic subspace; the second flag contains a single line (automatically isotropic); and the third flag contains arbitrary isotropic subspaces.

Recall that the variety $\mathrm{SpFl}_{(\mathbf{a},\mathbf{b})}$ is *spherical* whenever $(\mathbf{a}, \mathbf{b}, (1^{2n}))$ is of Sp-finite type. For most of the above Sp-finite triples $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, the triple $(\mathbf{a}, \mathbf{b}, (1^{2n}))$ is also Sp-finite, and the variety $\mathrm{SpFl}_{(\mathbf{a},\mathbf{b})}$ is spherical. The exceptions are the cases (SpE_6), (SpE_7), and (SpE_8), provided $2n \geq 6$ and $(\mathbf{a}, \mathbf{b}, (1^{2n}))$ is not of type (Sp$E^{(b)}_{r+3}$). These latter cases go beyond the scope of Littelmann's classification [5].

1.2. **Flag categories.** For each Sp-finite triple $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$, we describe the $\mathrm{Sp}(V)$-orbits on the triple flag variety $\mathrm{SpFl}_{\mathbf{d}}(V)$. Remarkably, we can do so in the same categorical framework as in the case of $G = \mathrm{GL}_m$. Later, in Corollary 1.5, we give the parametrization of the orbits in purely combinatorial terms.

For a composition $\mathbf{a} = (a_1, \ldots , a_p)$, we write $|\mathbf{a}| = a_1 + \cdots + a_p$. The number p of parts of $\mathbf{a}$ will be denoted $\ell(\mathbf{a})$, called the *length* of $\mathbf{a}$. For any positive integers p, q, and r, we consider an additive semigroup of triples of compositions:

$$\Lambda_{pqr} = \{(\mathbf{a}, \mathbf{b}, \mathbf{c}) \mid (\ell(\mathbf{a}), \ell(\mathbf{b}), \ell(\mathbf{c})) = (p, q, r) \text{ and } |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|\}.$$

(Here, in contrast to the notation of Theorem 1.2, the numbers p, q, r include the zero parts of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.)

Introduce the additive category $\mathcal{F} = \mathcal{F}_{pqr}$ whose objects are families $(V; A, B, C)$, where V is any finite-dimensional vector space, and (A, B, C) is a triple of flags in V belonging to any $\mathrm{Fl}_{\mathbf{a}}(V) \times \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$ with $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda_{pqr}$. The triple $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ is called the *dimension vector* of $(V; A, B, C)$. A *morphism* in $\mathcal{F}$ from $(V; A, B, C)$ to $(V'; A', B', C')$ is a linear map $f : V \to V'$ such that $f(A_i) \subset A'_i$, $f(B_i) \subset B'_i$, and $f(C_i) \subset C'_i$ for all i. Direct sum of objects is taken componentwise on each member of each flag.

Each triple flag in $\mathrm{Fl}_{\mathbf{a}}(V) \times \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$ corresponds naturally to an object of $\mathcal{F}$, and $\mathrm{GL}(V)$-orbits in the triple flag variety are naturally identified with isomorphism classes of objects in $\mathcal{F}$ with dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. The advantage of translating the GL-orbit problem into the additive category is that each object splits *uniquely* into indecomposable objects, so that an isomorphism class is uniquely specified by the multiplicities of its indecomposable summands (cf. [6]). Thus to classify all isomorphism classes in a given dimension it is enough to find the *indecomposable* isomorphism classes in all smaller dimensions. But indecomposables are rather rare, and this becomes a tractable and familiar problem in the theory of quivers.

Next we translate the Sp-orbit problem into categorical terms. Let $\mathrm{Sp}\mathcal{F} = \mathrm{Sp}\mathcal{F}_{pqr}$ be the full subcategory of $\mathcal{F}_{pqr}$ consisting of the objects $(V; A, B, C)$ which have symmetric dimension vector and such that V admits a non-degenerate symplectic form $\langle\, , \rangle$ with $\langle A_{p-i}, A_i\rangle = \langle B_{q-i}, B_i\rangle = \langle C_{r-i}, C_i\rangle = 0$ for all i. Now fix a symplectic form on V. Clearly a triple flag in $\mathrm{SpFl}_{\mathbf{a}}(V) \times \mathrm{SpFl}_{\mathbf{b}}(V) \times \mathrm{SpFl}_{\mathbf{c}}(V)$ may be thought of as an object of $\mathrm{Sp}\mathcal{F}_{pqr}$, and if two such triple flags are in the same $\mathrm{Sp}(V)$-orbit, then they are equivalent as objects of $\mathrm{Sp}\mathcal{F}_{pqr}$. Also, since all symplectic forms on V are conjugate, every isomorphism class of $\mathrm{Sp}\mathcal{F}_{pqr}$ contains at least one $\mathrm{Sp}(V)$-orbit of our triple symplectic flag variety. Our first key technical result is that each $\mathrm{Sp}\mathcal{F}$-class contains *exactly* one $\mathrm{Sp}(V)$-orbit.

Our second key fact is a nice description of the indecomposable objects of $\mathrm{Sp}\mathcal{F}$ in terms of indecomposables in $\mathcal{F}$. To give this description, we define a contravariant duality functor $*$ on $\mathcal{F}_{pqr}$. For a single flag $A = (A_1 \subset \cdots \subset A_{p-1} \subset V)$, define $A^* = ((V/A_{p-1})^* \subset \cdots \subset (V/A_1)^* \subset V^*)$, where V^* denotes the dual vector space and $(V/A_i)^*$ the subspace of linear forms vanishing on A_i. Let $(V; A, B, C)^* = (V^*; A^*, B^*, C^*)$. If an object has dimension vector $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$, its dual has dimension vector $\mathbf{d}^{\mathrm{op}} = (\mathbf{a}^{\mathrm{op}}, \mathbf{b}^{\mathrm{op}}, \mathbf{c}^{\mathrm{op}})$.

Notice that an object in $\mathrm{Sp}\mathcal{F}$ must be isomorphic to its dual in $\mathcal{F}$, since the symplectic form identifies $V \xrightarrow{\sim} V^*$. Clearly, not all self-dual objects of $\mathcal{F}$ lie in $\mathrm{Sp}\mathcal{F}$ (for example, V might have odd dimension). But it is easy to see that $I \oplus I^*$ is in $\mathrm{Sp}\mathcal{F}$ for any object I of $\mathcal{F}$; indeed, if F is any flag in V then $F \oplus F^*$ is a symplectic flag in $V \oplus V^*$ with respect to the symplectic form having both V and V^* isotropic and inducing the natural pairing between V and V^*.

Theorem 1.3. *(i) Two triple symplectic flags in* $\mathrm{SpFl}_{\mathbf{d}}(V)$ *lie in the same* $\mathrm{Sp}(V)$-*orbit if and only if they are isomorphic as objects in* $\mathrm{Sp}\mathcal{F}_{pqr}$.
(ii) Let J be an object of $\mathcal{F}_{pqr}$. Then J is an indecomposable object of $\mathrm{Sp}\mathcal{F}_{pqr}$ *if and only if*

(a) J lies in $\mathrm{Sp}\mathcal{F}_{pqr}$ *and J is indecomposable in $\mathcal{F}_{pqr}$, or*
(b) $J \cong I \oplus I^*$ for some $\mathcal{F}_{pqr}$-indecomposable I not belonging to $\mathrm{Sp}\mathcal{F}_{pqr}$.

As a consequence of part (ii), any object of $\mathrm{Sp}\mathcal{F}$ can be *uniquely* written as a sum of indecomposable objects of $\mathrm{Sp}\mathcal{F}$. Thus, as in the GL case, to classify orbits on a multiple Sp-flag variety it is enough to find the $\mathrm{Sp}\mathcal{F}$-indecomposables which can appear as summands of an object with the given dimension vector.

1.3. **Classification of orbits.** For a dimension vector $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ with lengths (p, q, r), we have seen the Sp-orbits in $\mathrm{SpFl}_{\mathbf{d}}$ are naturally identified with direct sums $\bigoplus_J m_J J$, where J runs over all indecomposable isomorphism classes in $\mathrm{Sp}\mathcal{F}_{pqr}$, and the m_J are non-negative integers with $\sum_J m_J \dim(J) = \mathbf{d}$. We proceed to exhibit representatives for all J which can appear in such a decomposition for the dimensions $\mathbf{d}$ of Theorem 1.2. This will give a classification of orbits, as well as explicit triples of flags lying in each orbit.

We say a composition is *compressed* if it has no zero parts, and a flag with no repeated subspaces is also called compressed. Let $\mathbf{a}_{\mathrm{cpr}}$ denote the composition obtained from $\mathbf{a}$ by removing all zero parts; let A_{cpr} denote the flag obtained from A by removing all repetitions of subspaces; and let similar notation hold for tuples of compositions and multiple flags. Even if an object in $\mathrm{Sp}\mathcal{F}_{pqr}$ is compressed, its direct summand J might not be compressed. However, if we have a representative for J_{cpr} and we know $\dim(J)$, then we can immediately construct a representative for J. Thus in our list we need only produce representatives for compressed $\mathrm{Sp}\mathcal{F}$-indecomposables $J = J_{\mathrm{cpr}}$.

As a further normalization, if an object $J = (V; A, B, C)$ has dimension $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ where $p \leq q \leq r$ does *not* hold, we switch A, B, and C to get an object $J^{<}$ of dimension $\mathbf{d}^{<}$ for which the lengths of the flags *are* non-decreasing. (If there is more than one way to do such switching, choose an arbitrary one.)

We adopt the following notation for $\mathrm{Sp}\mathcal{F}$-indecomposables J:
(1) Suppose that in dimension $\mathbf{d}$ there are exactly k isomorphism classes of $\mathrm{Sp}\mathcal{F}$-indecomposables which are also indecomposable in $\mathcal{F}$. Then we denote these indecomposables by $J = I^1_{\mathbf{d}}, \dots, I^k_{\mathbf{d}}$ (or simply $J = I_{\mathbf{d}}$ if $k = 1$).
(2) Suppose $\mathbf{d} = \mathbf{e} + \mathbf{e}^{\mathrm{op}}$, and in dimension $\mathbf{e}$ there is a unique isomorphism class of $\mathcal{F}$-indecomposables $I_{\mathbf{e}}$ which is not in $\mathrm{Sp}\mathcal{F}$. Then $J = I_{\mathbf{e}} \oplus I^*_{\mathbf{e}}$ is a $\mathrm{Sp}\mathcal{F}$-indecomposable of dimension $\mathbf{d}$, which we denote $J = I^{\mathrm{sym}}_{\mathbf{e}}$.

Theorem 1.4. *An object J is a $\mathrm{Sp}\mathcal{F}$-indecomposable summand of a $\mathrm{Sp}\mathcal{F}$ object with symplectic finite dimension if and only if $J^{<}_{\mathrm{cpr}}$ is isomorphic to one of the objects in the table below. (In the right-hand column of the table, we abbreviate $I^j_{\mathbf{d}}$ to $\mathbf{d}^j$ and $I^{\mathrm{sym}}_{\mathbf{e}}$ to $\mathbf{e}^{\mathrm{sym}}$, and we omit commas.)*

dimensions $\mathbf{d}$	Sp-indecomposables J
$((2)(2)(2))$	$((1)(1)(1))^{\mathrm{sym}}$
$((2)(2)(11))$	$((1)(1)(10))^{\mathrm{sym}}$
$((2)(11)(11))$	$((1)(10)(10))^{\mathrm{sym}},\ ((1)(10)(01))^{\mathrm{sym}}$
$((11)(11)(11))$	$((11)(11)(11)),$ $((10)(10)(10))^{\mathrm{sym}},\ ((10)(10)(01))^{\mathrm{sym}},$ $((10)(01)(10))^{\mathrm{sym}},\ ((10)(01)(01))^{\mathrm{sym}}$
$((22)(22)(121))$	$((11)(11)(110))^{\mathrm{sym}}$
$((22)(22)(1^4))$	$((11)(11)(1100))^{\mathrm{sym}},\ ((11)(11)(1010))^{\mathrm{sym}}$
$((22)(121)(121))$	$((11)(110)(110))^{\mathrm{sym}},\ ((11)(110)(011))^{\mathrm{sym}}$
$((22)(121)(1^4))$	$((22)(121)(1^4)),$ $((11)(110)(1100))^{\mathrm{sym}},\ ((11)(110)(1010))^{\mathrm{sym}},$ $((11)(110)(0101))^{\mathrm{sym}},\ ((11)(110)(0011))^{\mathrm{sym}}$
$((121)(121)(121))$	$((121)(121)(121)),$ $((110)(110)(110))^{\mathrm{sym}},\ ((110)(110)(011))^{\mathrm{sym}},$ $((110)(011)(110))^{\mathrm{sym}},\ ((110)(011)(011))^{\mathrm{sym}}$
$((121)(121)(1^4))$	$((121)(121)(1^4))^1,\ ((121)(121)(1^4))^2,$ $((110)(110)(1100))^{\mathrm{sym}},\ ((110)(110)(1010))^{\mathrm{sym}},$ $((110)(110)(0101))^{\mathrm{sym}},\ ((110)(110)(0011))^{\mathrm{sym}},$ $((110)(011)(1100))^{\mathrm{sym}},\ ((110)(011)(1010))^{\mathrm{sym}},$ $((110)(011)(0101))^{\mathrm{sym}},\ ((110)(011)(0011))^{\mathrm{sym}}$

$((33)(2^3)(2^3))$	$((21)(1^3)(1^3))^{\rm sym}$
$((33)(2^3)(2112))$	$((21)(1^3)(1101))^{\rm sym},\ ((21)(1^3)(1011))^{\rm sym}$
$((33)(2^3)(1221))$	$((21)(1^3)(1110))^{\rm sym},\ ((21)(1^3)(0111))^{\rm sym}$
$((33)(2^3)(11211))$	$((33)(2^3)(11211)),$
	$((21)(1^3)(11100))^{\rm sym},\ ((21)(1^3)(10110))^{\rm sym},$
	$((21)(1^3)(01101))^{\rm sym},\ ((21)(1^3)(00111))^{\rm sym}$
$((141)(2^3)(2^3))$	$((120)(1^3)(1^3))^{\rm sym}$
$((141)(2^3)(2112))$	$((120)(1^3)(1101))^{\rm sym}\ ((120)(1^3)(1011))^{\rm sym}$
$((141)(2^3)(1221))$	$((120)(1^3)(1110))^{\rm sym}\ ((120)(1^3)(0111))^{\rm sym}$
$((141)(2^3)(11211))$	$((141)(2^3)(11211)),$
	$((120)(1^3)(11100))^{\rm sym},\ ((120)(1^3)(10110))^{\rm sym},$
	$((120)(1^3)(01101))^{\rm sym},\ ((120)(1^3)(00111))^{\rm sym}$
$((141)(2^3)(1^6))$	$((141)(2^3)(1^6))^1,\ ((141)(2^3)(1^6))^2,$
	$((120)(1^3)(111000))^{\rm sym},\ ((120)(1^3)(110100))^{\rm sym},$
	$((120)(1^3)(101010))^{\rm sym},\ ((120)(1^3)(100110))^{\rm sym},$
	$((120)(1^3)(011001))^{\rm sym},\ ((120)(1^3)(010101))^{\rm sym},$
	$((120)(1^3)(001011))^{\rm sym},\ ((120)(1^3)(000111))^{\rm sym}$
$((44)(323)(2^4))$	$((22)(211)(1^4))^{\rm sym}$
$((44)(323)(21212))$	$((22)(211)(11101))^{\rm sym},\ ((22)(211)(10111))^{\rm sym}$
$((44)(323)(12221))$	$((22)(211)(11110))^{\rm sym},\ ((22)(211)(01111))^{\rm sym}$
$((44)(242)(21212))$	$((22)(121)(11101))^{\rm sym}$
$((44)(242)(12221))$	$((22)(121)(11110))^{\rm sym}$
$((55)(424)(2^5))$	$((32)(212)(1^5))^{\rm sym}$
$((55)(343)(2^5))$	$((32)(221)(1^5))^{\rm sym},\ ((32)(122)(1^5))^{\rm sym}$
$((66)(4^3)(32223))$	$((3^2)(2^3)(21111))^{\rm sym}$
$((66)(4^3)(23232)$	$((3^2)(2^3)(12111))^{\rm sym}$

Note that for all the above Sp$\mathcal{F}$-indecomposables $J = (V; A, B, C)$, the dimension of the ambient space V is at most 12. This is in contrast to the $\mathcal{F}$-indecomposables of GL-finite type [6, Theorem 2.9], which occur in all dimensions.

In all the above cases with $J = I_{\mathbf{e}}^{\rm sym}$, the dimension vector $\mathbf{e}$ is GL-finite (i.e. there are only finitely many GL-orbits in $\mathrm{Fl}_{\mathbf{e}}$). Thus we may read off an explicit representative for $I_{\mathbf{e}}$ from the list of $\mathcal{F}$-indecomposables of GL-finite type in [6, Theorem 2.9]. For the remaining indecomposable classes $J = I_{\mathbf{d}}^i$ we give symplectic representatives below.

We present each indecomposable $J = (V; A, B, C)$ with dimension vector $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ as follows. The space $V = V_{2n}$ has basis $e_1, \dots, e_{2n}$ (where $2n = |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$), with the standard symplectic form $\langle e_i, e_{2n+1-i}\rangle = -\langle e_{2n+1-i}, e_i\rangle = 1$ for $i = 1, \dots, n$ and $\langle e_i, e_j\rangle = 0$ otherwise. We list explicit bases for the isotropic spaces in A, B, and C (the rest of the spaces being the orthogonals of those given).

$$
\begin{aligned}
I_{((11)(11)(11))} &= (V_2;\,(e_1+e_2),\ (e_1),\ (e_2))\\
I_{((22)(121)(1^4))} &= (V_4;\,(e_1+e_3+e_4, e_2+e_4),\ (e_1),\ (e_4)\subset(e_3,e_4))\\
I_{((121)(121)(121))} &= (V_4;\,(e_1+e_2+e_4),\ (e_1),\ (e_4))\\
I^1_{((121)(121)(1^4))} &= (V_4;\,(e_1+e_2+e_4),\ (e_1),\ (e_4)\subset(e_2,e_4))\\
I^2_{((121)(121)(1^4))} &= (V_4;\,(e_1+e_2+e_4),(e_1),\ (e_4)\subset(e_3,e_4))\\
I_{((33)(2^3)(11211))} &= (V_6;\,(e_1+e_5+e_6, e_2+e_4+e_6, e_3+e_5),\ (e_1,e_2),\ (e_6)\subset(e_5,e_6))\\
I_{((141)(2^3)(11211))} &= (V_6;\,(e_1+e_3+e_5+e_6),\ (e_1,e_2),\ (e_6)\subset(e_5,e_6))\\
I^1_{((141)(2^3)(1^6))} &= (V_6;\,(e_1+e_3+e_5+e_6),\ (e_1,e_2),\ (e_6)\subset(e_5,e_6)\subset(e_3,e_5,e_6))\\
I^2_{((141)(2^3)(1^6))} &= (V_6;\,(e_1+e_3+e_5+e_6),\ (e_1,e_2),\ (e_6)\subset(e_5,e_6)\subset(e_4,e_5,e_6))\ .
\end{aligned}
$$

1.4. **Enumeration of orbits.** From Theorem 1.4 we may deduce the following enumeration of orbits on multiple symplectic flag varieties, similar to the Kostant partition function.

Let $\mathrm{Sp}\Pi_{pqr}$ be the set of the symmetric triples $\mathbf{d}\in\Lambda_{pqr}$ such that $\mathbf{d}^<_{\mathrm{cpr}}$ is one of the dimension vectors in the left-hand column of the table in Theorem 1.4. For each $\mathbf{d}\in\mathrm{Sp}\Pi_{pqr}$, let $\mu_{\mathbf{d}}$ be the number of Sp-indecomposables of dimension $\mathbf{d}^<_{\mathrm{cpr}}$ listed in the right-hand column next to $\mathbf{d}^<_{\mathrm{cpr}}$. Define $\widetilde{\mathrm{Sp}\Pi}_{pqr}\subset\mathrm{Sp}\Pi_{pqr}\times\mathbf{Z}_+$ by

$$\widetilde{\mathrm{Sp}\Pi}_{pqr}=\bigcup_{\mathbf{d}\in\mathrm{Sp}\Pi_{pqr}}\{\mathbf{d}\}\times\{1,2,\ldots,\mu_{\mathbf{d}}\}\ .$$

Corollary 1.5. *Let $\mathbf{d}=(\mathbf{a},\mathbf{b},\mathbf{c})\in\Lambda_{pqr}$ be a triple of symmetric compositions.*
(i) The Sp_{2n}-orbits on $\mathrm{SpFl}_{\mathbf{d}}$ are in natural bijection with families of non-negative integers $(m_{\mathbf{e},i})$ indexed by $(\mathbf{e},i)\in\widetilde{\mathrm{Sp}\Pi}_{pqr}$ such that

$$\sum_{(\mathbf{e},i)\in\widetilde{\mathrm{Sp}\Pi}} m_{\mathbf{e},i}\ \mathbf{e}=\mathbf{d}\ .$$

(ii) The number of Sp_{2n}-orbits on $\mathrm{SpFl}_{\mathbf{d}}$ is:

$$\sum_{(m_{\mathbf{e}})}\prod_{\mathbf{e}\in\mathrm{Sp}\Pi}\binom{\mu_{\mathbf{e}}+m_{\mathbf{e}}-1}{\mu_{\mathbf{e}}-1},$$

where the sum runs over all families of non-negative integers $(m_{\mathbf{e}})$ indexed by $\mathbf{e}\in\mathrm{Sp}\Pi_{pqr}$ such that

$$\sum_{\mathbf{e}\in\mathrm{Sp}\Pi} m_{\mathbf{e}}\,\mathbf{e}=\mathbf{d}\ .$$

Part (i) is a consequence of Theorem 1.4. Part (ii) follows from (i), since the binomial coefficient $\binom{\mu+m-1}{\mu-1}$ is the number of nonnegative integer solutions of the equation $m_1+\cdots+m_\mu=m$. In most examples of interest, only a few types of compressed indecomposable dimension vectors can contribute to a decomposition of $\mathbf{d}$, and we can obtain compact expressions for the number of orbits.

Example. The dimension vector $((n,n),(n,n),(1^{2n}))$, spherical of type $(\mathrm{Sp}D_{2n+2})$. The multiple flag variety consists of triples containing two Lagrangian subspaces and one complete isotropic flag. Let c_n be the number of Sp_{2n}-orbits on the flag variety $\mathrm{SpFl}_{((n,n),(n,n),(1^{2n}))}$.

For $n = 2$, there are only two families $(m_{\mathbf{e}})$ with $\sum_{\mathbf{e}} m_{\mathbf{e}}\,\mathbf{e} = ((22)(22)(1^4))$:
(1) $m_{((22)(22)(1^4))} = 1$, all other $m_{\mathbf{e}} = 0$;
(2) $m_{((11)(11)(1001))} = m_{((11)(11)(0110))} = 1$, all other $m_{\mathbf{e}} = 0$.
(Recall that a summand $\mathbf{e}$ must be a triple of *symmetric* compositions.) Since $\mu_{((22)(22)(1^4))} = 2$ and $\mu_{((11)(11)(1001))} = \mu_{((11)(11)(0110))} = \mu_{((11)(11)(11))} = 5$, we get

$$c_2 = \binom{2+1-1}{2-1} + \binom{5+1-1}{5-1}^2 = 27\ .$$

For general n, any symplectic-indecomposable summand of $((nn)(nn)(1^{2n}))$ must have compressed form: $((11)(11)(11))$ with $\mu = 5$; or $((22)(22)(1^4))$ with $\mu = 2$. A family $(m_{\mathbf{e}})$ with $\sum_{\mathbf{e}} m_{\mathbf{e}}\,\mathbf{e} = ((nn)(nn)(1^{2n}))$ has $m_{\mathbf{e}} \le 1$ and is equivalent to a partition of the set $\{1, 2, \ldots, n\}$ into subsets of sizes 1 and 2. An easy computation gives the exponential generating function

$$\sum_{n=0}^{\infty} \frac{c_n x^n}{n!} = 1 + 5x + \frac{27}{2!}x^2 + \cdots = e^{x^2+5x}\ .$$

Example. The dimension vector $((n, n), (1, 2n-2, 1), (1^{2n}))$, spherical of type $(\mathrm{Sp}E^{(b)}_{2n+3})$. The multiple flag variety consists of triples containing a Lagrangian subspace, a line, and a complete isotropic flag.

There are 3 compressed vectors that can serve as symplectic-indecomposable summands: $((11)(2)(11))$ with $\mu = 2$; $((11)(11)(11))$ with $\mu = 5$; and $((22)(121)(1^4))$ with $\mu = 5$. Furthermore, all the summands but one must be of the first kind. The families $m_{\mathbf{e}}$ are equivalent to choosing either one or two elements in $\{1, 2, \ldots, n\}$. Thus the number of orbits is equal to

$$n \cdot \binom{5+1-1}{5-1}\binom{2+1-1}{2-1}^{n-1} + \binom{n}{2} \cdot \binom{5+1-1}{5-1}\binom{2+1-1}{2-1}^{n-2} = 5 \cdot 2^{n-3} n(n+3)\ .$$

2. Proof of Theorems 1.1 – 1.4

2.1. **Proof of Theorem 1.3(i).** Although we discuss only alternating bilinear forms and the symplectic group, the arguments in the proof of Theorem 1.3 hold almost verbatim for symmetric forms and the orthogonal group.

For a k-tuple of positive integers $(p_1, \ldots, p_k)$, we define the semigroup $\Lambda_{p_1,\ldots,p_k}$ and the categories $\mathcal{F}_{p_1,\ldots,p_k}$ and $\mathrm{Sp}\mathcal{F}_{p_1,\ldots,p_k}$ analogously to their counterparts for $k = 3$. When there is no risk of ambiguity, we drop the subscripts $(p_1, \ldots, p_k)$ and write Λ, $\mathcal{F}$, $\mathrm{Sp}\mathcal{F}$, etc.

Our first task is to show that each isomorphism class in $\mathrm{Sp}\mathcal{F}$ corresponds to a unique Sp-orbit in a multiple symplectic flag variety. This is a consequence of the following general fact, which is analogous to (but sharper than) a lemma of Richardson [7], and which generalizes a result of Derksen and Weyman [2].

Consider a group G acting on a set X, and suppose we have involutions $g \mapsto g^\sigma$ on G and $x \mapsto \sigma x$ on X such that for all g and x,

$$\sigma(g(\sigma x)) = (g^\sigma)(x).$$

Let G^σ and X^σ denote the fixed point sets of σ. Suppose further that:
(1) The group G is a subgroup in $M^\times$, the invertible elements of some finite-dimensional associative algebra M over an algebraically closed field $\mathbf{k}$.
(2) The anti-involution $g \mapsto g^* := (g^\sigma)^{-1}$ on G extends to a $\mathbf{k}$-linear anti-involution $m \mapsto m^*$ of the algebra M.

(3) For any $x \in X^\sigma$, the stabilizer $H = \mathrm{Stab}_G(x)$ is the group of invertible elements of its linear span $\mathrm{Span}_{\mathbf{k}}(H)$ in M.

Proposition 2.1. *Let G and X be as above. If two points in X^σ are G-conjugate then they are G^σ-conjugate.*

Proof. As a preliminary, let us show that if $H = \mathrm{Span}_{\mathbf{k}}(H)^\times$ then any element $h \in H$ with $h^* = h$ can be written as $h = k^2 = k^*k$ for some $k = k^* \in H$. Since M is finite-dimensional, the subalgebra $\mathbf{k}[h] \subset M$ is isomorphic to a quotient of a polynomial ring, $\mathbf{k}[h] \cong \mathbf{k}[t]/(p(t))$, where $p(t)$ is the minimal polynomial of h. Furthermore, since h is invertible, $p(0) \neq 0$. Now, by the usual theory of finitely-generated $\mathbf{k}[t]$-modules, h has a square root in $\mathbf{k}[h]$: that is, $h = k^2$ for some $k = q(h) \in \mathbf{k}[h]$. Clearly $k^* = k$, and also $k \in \mathrm{Span}_{\mathbf{k}}(H)^\times = H$, as desired.

Now, let $x \in X^\sigma$, and let H be the stabilizer of x in G. Consider any point $gx \in X^\sigma$ for $g \in G$. We have

$$gx = \sigma(gx) = \sigma g(\sigma x) = g^\sigma x,$$

and hence $g^*gx = x$. Thus the element $h := g^*g$ lies in H and satisfies $h^* = h$.

By the above discussion, we can write $h = k^*k$ for some $k \in H$. Therefore we have $(g^\sigma)^{-1}g = (k^\sigma)^{-1}k$, and $gk^{-1} = (gk^{-1})^\sigma \in G^\sigma$. But $k \in H = \mathrm{Stab}_G(x)$, so

$$gx = gk^{-1}x \in G^\sigma x,$$

and we are done. □

The proof of Theorem 1.3(i) is now completed as follows. Let $G = \mathrm{GL}(V)$, and X be a multiple GL-flag variety $\mathrm{Fl}_{\mathbf{d}}(V)$ with symmetric dimension vector $\mathbf{d}$. Choose a symplectic form on V, and let $m \mapsto m^*$ be the corresponding adjoint map on $M = \mathrm{End}(V)$. Take $\sigma : G \to G$ to be $\sigma(g) = (g^*)^{-1}$. For $x \in X$, let $\sigma(x)$ be the multiple flag formed by all orthogonals of subspaces in x. Then $G^\sigma = \mathrm{Sp}(V)$, and $X^\sigma = \mathrm{SpFl}_{\mathbf{d}}$. Conditions (1)-(3) of Proposition 2.1 are clearly satisfied. (In fact, (3) is valid for any multiple flag $x \in \mathrm{Fl}_{\mathbf{d}}$, not necessarily σ-fixed, since linear maps that preserve x form a subalgebra in $M = \mathrm{End}(V)$.) By Proposition 2.1, two elements of $\mathrm{SpFl}_{\mathbf{d}}(V)$ lie in the same $\mathrm{GL}(V)$-orbit if and only if they lie in the same $\mathrm{Sp}(V)$-orbit. This is precisely what was to be shown. □

2.2. **Proof of Theorem 1.3(ii).** We begin by recalling some well-known facts about $\mathcal{F}$. (See for example [6, §3.1].) Although $\mathcal{F}$ is not an abelian category, it has a full faithful embedding into the abelian category of *quiver representations*, and an indecomposable of $\mathcal{F}$ is also indecomposable in the quiver category. Therefore we can apply the Krull-Schmidt theorem for abelian categories [1] to conclude that an object in $\mathcal{F}$ has a unique expression as a direct sum of indecomposables. Furthermore, the endomorphism ring of an indecomposable object in $\mathcal{F}$ is local: that is, every endomorphism is either invertible or nilpotent.

In what follows, we fix $J = (V; F)$, a multiple symplectic flag in $\mathrm{Sp}\mathcal{F}$. Here V is a symplectic vector space and F is a tuple of symplectic flags. We shall examine how $(V; F)$ decomposes in the larger category $\mathcal{F}$.

Any subspace $U \subset V$ induces a subobject $I = (U; F \cap U)$ in $\mathcal{F}$. A splitting of the vector space $V = U \oplus W$ induces a splitting of $(V; F)$ in $\mathcal{F}$ if and only if $F = (F \cap U) \oplus (F \cap W)$; that is, for any space $A \subset V$ which is a member of any flag of F, we have $A = (A \cap U) \oplus (A \cap W)$.

We shall repeatedly use an elementary linear algebra fact. For any vector space V and subspaces $X, Y, Z \subset V$, the following conditions are equivalent:

$$(*) \quad \begin{aligned} X \cap (Y+Z) = (X\cap Y) + (X\cap Z) &\iff X + (Y\cap Z) = (X+Y) \cap (X+Z) \\ &\iff Z \cap (X+Y) = (Z\cap X) + (Z\cap Y)\ . \end{aligned}$$

If any of these conditions holds, the subspaces X, Y, and Z generate a distributive lattice.

Lemma 2.2. *Let $(V;F)$ be a symplectic multiple flag and $U \subset V$ a subspace such that $U\cap U^\perp = 0$ and $(U;F\cap U)$ has symmetric dimension vector. Then both multiple flags $(U;F\cap U)$ and $(U^\perp;F\cap U^\perp)$ are symplectic, and we have the splitting in $\mathrm{Sp}\mathcal{F}$: $(V;F) = (U;F\cap U) \oplus (U^\perp;F\cap U^\perp)$.*

Proof. The condition $U \cap U^\perp = 0$ means that the symplectic form on V is non-degenerate on U and on $U^\perp$. Let $A, A^\perp \subset V$ be two orthogonal members of a flag in F. Now, the orthogonal of $A^\perp \cap U$ in U is

$$(A^\perp \cap U)^\perp \cap U = (A + U^\perp) \cap U \supset A \cap U\ .$$

Since the form is non-degenerate on U, we have $\dim((A^\perp \cap U)^\perp \cap U) = \dim U - \dim(A^\perp \cap U)$; but since F has symmetric dimension vector, $\dim(A\cap U) = \dim U - \dim(A^\perp\cap U)$. Thus the above containment is an equality: $(A+U^\perp)\cap U = (A\cap U) = (A \cap U) + (U^\perp \cap U)$.

Hence by $(*)$,

$$A = A \cap (U \oplus U^\perp) = (A \cap U) \oplus (A \cap U^\perp).$$

This implies the splitting of $(V;F)$ in $\mathcal{F}$. In particular, $(U^\perp;F \cap U^\perp)$ also has symmetric dimension vector. Thus the summands are symplectic. □

Lemma 2.3. *Let $(V;F)$ be a symplectic multiple flag which splits in $\mathcal{F}$ as $(V;F) = (U;F \cap U) \oplus (W;F \cap W)$. Then:*

(i) We have the adjoint splitting $(V;F) = (U^\perp;F \cap U^\perp) \oplus (W^\perp;F \cap W^\perp)$.

(ii) The multiple flag $(W^\perp;F \cap W^\perp)$ is isomorphic to the dual $(U;F \cap U)^*$. In particular, $(W^\perp;F \cap W^\perp)$ has dimension vector opposite to that of $(U;F \cap U)$.

(iii) The projections $\alpha : V \cong U \oplus W \to U$ and $\beta : V \cong U^\perp \oplus W^\perp \to W^\perp$ induce morphisms

$$\alpha : (W^\perp;F \cap W^\perp) \to (U;F \cap U). \quad \text{and} \quad \beta : (U;F \cap U) \to (W^\perp;F \cap W^\perp)$$

Proof. (i) The space V splits as $V = U^\perp \oplus W^\perp$ because the symplectic form is non-degenerate. Now, for orthogonal members $A, A^\perp$ of F, we have $A^\perp = (A^\perp \cap U) + (A^\perp \cap W)$, which is equivalent to $A = (A + U^\perp) \cap (A + W^\perp)$. But this is equivalent to $A = (A \cap U^\perp) + (A \cap W^\perp)$ by $(*)$.
(ii) The symplectic form on V induces a pairing between U and $W^\perp$, which is non-degenerate since $W^\perp \cap U^\perp = 0$. The elements of $W^\perp$ which pair to zero with $A\cap U$ are exactly

$$\begin{aligned} (A \cap U)^\perp \cap W^\perp &= (A^\perp + U^\perp) \cap W^\perp \\ &= (A^\perp \cap W^\perp) + (U^\perp \cap W^\perp) = A^\perp \cap W^\perp\ . \end{aligned}$$

(Here we used (i) and $(*)$ for the second equality.) Thus $W^\perp$ is isomorphic to U^*; and $A^\perp \cap W^\perp$, a member of $F \cap W^\perp$, is isomorphic to $(U/(A \cap U))^*$.

(iii) For A a member of F, we have

$$\alpha(A \cap W^{\perp}) = ((A \cap W^{\perp}) + W) \cap U \subset (A + W) \cap U = A \cap U \ .$$

(Here we used $(*)$ for the last equality.) Thus $\alpha(F \cap W^{\perp}) \subset F \cap U$, and similarly $\beta(F \cap U) \subset F \cap W^{\perp}$. □

Now let $J = (V; F)$ be a symplectic multiple flag which is indecomposable in $\mathrm{Sp}\mathcal{F}$, but which splits in $\mathcal{F}$ as $(V; F) = (U; F \cap U) \oplus (W; F \cap W)$, where $I = (U; F \cap U)$ is indecomposable in $\mathcal{F}$. We know that the endomorphism $\alpha\beta : (U; F \cap U) \to (U; F \cap U)$ defined in Lemma 2.3 must be either invertible or nilpotent. That is, at least one of $\alpha\beta$ and $\mathrm{id}_U - \alpha\beta$ is an isomorphism.

Case 1. The map $\alpha\beta$ is an isomorphism. Since $\dim(U) = \dim(W^{\perp})$, the maps α and β are invertible and give isomorphisms between $(U; F \cap U)$ and $(W^{\perp}; F \cap W^{\perp}) \cong (U; F \cap U)^*$. In particular, $(U; F \cap U)$ has symmetric dimension vector. We also have $U \cap U^{\perp} = \mathrm{Ker}(\beta) = 0$. Hence by Lemma 2.2, $(V; F)$ splits in $\mathrm{Sp}\mathcal{F}$ as $(V; F) = (U; F \cap U) \oplus (U^{\perp}; F \cap U^{\perp})$. But since $(V; F)$ is indecomposable in $\mathrm{Sp}\mathcal{F}$, we must have $(V; F) = (U; F \cap U)$: that is, $(V; F) = J = I$ is $\mathcal{F}$-indecomposable.

Case 2. The map $\mathrm{id}_U - \alpha\beta$ is an isomorphism. It follows from the definitions that

$$\mathrm{Ker}(\mathrm{id}_U - \alpha\beta) = U \cap (W^{\perp} + (U^{\perp} \cap W)) \ .$$

Hence in our case, we have

$$U \cap W^{\perp} \subset U \cap (W^{\perp} + (U^{\perp} \cap W)) = 0 \ ,$$

so that the sum $U \oplus W^{\perp}$ is direct.

Furthermore, consider the projections $\gamma : U \oplus W^{\perp} \to U$ and $\delta : U^{\perp} \oplus W^{\perp} \to U^{\perp}$. We may easily check that γ restricts to the following isomorphism (with inverse δ):

$$\gamma : (U + W^{\perp}) \cap (U + W^{\perp})^{\perp} = (U + W^{\perp}) \cap U^{\perp} \cap W \xrightarrow{\sim} U \cap (W^{\perp} + (U^{\perp} \cap W)) = 0.$$

Thus the symplectic form on V is non-degenerate on $U + W^{\perp}$.

Finally, note that for any member A of F, we have

$$\begin{aligned} \dim(A \cap (U + W^{\perp})) &\leq \dim(U + W^{\perp}) - \dim(A^{\perp} \cap (U + W^{\perp})) \\ &\leq \dim U + \dim W^{\perp} - \dim(A^{\perp} \cap U) - \dim(A^{\perp} \cap W^{\perp}) \\ &= \dim(A \cap U) + \dim(A \cap W^{\perp}) \end{aligned}$$

where the last equality is because $(W^{\perp}; F \cap W^{\perp}) \cong (U; F \cap U)^*$ and $(U; F \cap U) \cong (W^{\perp}; F \cap W^{\perp})^*$ by Lemma 2.3. Thus $A \cap (U + W^{\perp}) = (A \cap U) + (A \cap W^{\perp})$, and we have

$$(U + W^{\perp}; F \cap (U + W^{\perp})) \ \cong \ (U; F \cap U) \oplus (W^{\perp}; F \cap W^{\perp}),$$

with symmetric dimension vector.

As before, we may now apply Lemma 2.2 to conclude that $U \oplus W^{\perp}$ induces a symplectic summand of $(V; F)$, so that $V = U \oplus W^{\perp}$ and $(V; F) \cong (U; F \cap U) \oplus (U; F \cap U)^*$. That is, $J \cong I \oplus I^*$.

This concludes the proof of Theorem 1.3. □

2.3. Proofs of Theorems 1.1 and 1.2.

Given dimension vectors $\mathbf{d}, \mathbf{d}' \in \Lambda_{p_1,\dots,p_k}$, we say $\mathbf{d}'$ is a *summand* of $\mathbf{d}$ if $\mathbf{d} - \mathbf{d}' \in \Lambda_{p_1,\dots,p_k}$.

Proposition 2.4. *Let $\mathbf{d} \in \Lambda_{p_1,\dots,p_k}$ be a symmetric dimension vector.*
(i) If some symmetric summand of $\mathbf{d}$ is not Sp-finite, then $\mathbf{d}$ is not Sp-finite.
(ii) If there exist only finitely many $\mathrm{Sp}\mathcal{F}$-indecomposable classes whose dimension is a summand of $\mathbf{d}$, then $\mathbf{d}$ is Sp-finite.

Proof. (i) If a summand $\mathbf{d}'$ is Sp-infinite, there are infinitely many distinct Sp$\mathcal{F}$-classes of dimension $\mathbf{d}'$. Taking a direct sum with any Sp$\mathcal{F}$-class of dimension $\mathbf{d}-\mathbf{d}'$ we obtain (by the unique decomposition in Sp$\mathcal{F}$) an infinite family of distinct Sp$\mathcal{F}$-classes of dimension $\mathbf{d}$, and thus infinitely many Sp-orbits on $\mathrm{SpFl}_{\mathbf{d}}$.
(ii) The Sp-orbits on $\mathrm{SpFl}_{\mathbf{d}}$ are in bijection with families of integers $(m_I)_I$ with $\sum_I m_I \dim(I) = \mathbf{d}$, where I runs over all Sp$\mathcal{F}$-indecomposables. The hypothesis ensures that there are only finitely many such families. □

Proof of Theorem 1.1. If $\mathbf{d} = (\mathbf{a}_1 \ldots \mathbf{a}_k)$ with $k \geq 4$ were Sp-finite, then $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ would be as well. Thus by Proposition 2.4(i), it is enough to show that the summand $\mathbf{f}_0 = ((1^2), (1^2), (1^2), (1^2))$ is *not* Sp-finite. But in this case we have $\mathrm{SpFl}_{\mathbf{f}_0} = \mathrm{Fl}_{\mathbf{f}_0}$ since every vector in a symplectic vector space is isotropic. So our statement follows from Theorem 1.3 and the well-known fact that $\mathbf{f}_0$ is of infinite GL-type. □

Proof of Theorem 1.2. First let us prove that all the cases in the theorem are indeed Sp-finite. By Theorem 1.3, we must show there are finitely many Sp$\mathcal{F}$-isomorphism classes in dimension $\mathbf{d}$ of type (SpA)—(SpY). For all the types except the last, this follows from the GL-classification in [6, Theorem 2.2]. That is, the $\mathbf{d}$ of types (SpA)—(Sp$E^{(b)}$) are all GL-finite, meaning there are only finitely many $\mathcal{F}$-isomorphism classes in dimension $\mathbf{d}$; but there must be even fewer Sp$\mathcal{F}$-isomorphism classes.

It remains to show that every symmetric dimension vector of type (SpY) (that is, of the form $\mathbf{d} = ((1, 2n-2, 1), \mathbf{b}, \mathbf{c})$ with $\ell(\mathbf{b}) = 3$) is Sp-finite. We will use the criterion of Proposition 2.4(ii). First, notice that any symmetric summand of $\mathbf{d}$ with even total dimension is of the same type (SpY) (or the type (SpA) already dealt with). Thus any indecomposable summand of the form $J = I \oplus I^*$ must have $\dim(I) = \mathbf{e}$ with $\mathbf{e}+\mathbf{e}^{\mathrm{op}}$ of type SpY. But then it is easily seen from the classification of [6] that $\mathbf{e}$ is GL-finite thus producing only finitely many Sp$\mathcal{F}$-indecomposables in dimension $\mathbf{e} + \mathbf{e}^{\mathrm{op}}$.

Thus it only remains to limit the Sp$\mathcal{F}$-indecomposables which are also indecomposable in $\mathcal{F}$. For this purpose, let $\|\mathbf{a}\|^2 = a_1^2 + \cdots + a_p^2$, and for $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda$ with $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 2n$, define the *Tits quadratic form* Q by:

$$\begin{aligned} Q(\mathbf{d}) &= \dim \mathrm{GL}_{2n} - \dim \mathrm{Fl}_{\mathbf{d}} \\ &= \tfrac{1}{2}(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 - (2n)^2). \end{aligned}$$

The fundamental result of V. Kac [4] states that if $Q(\mathbf{d}) > 1$, then there is no indecomposable isomorphism class of $\mathcal{F}$ in dimension $\mathbf{d}$; and if $Q(\mathbf{d}) = 1$, then there is at most one indecomposable.

Thus, it only remains to check the symplectic finite type property for those $\mathbf{d}$ of type (SpY) with $Q(\mathbf{d}) \leq 0$. The following lemma is an easy calculation with the Tits form.

Lemma 2.5. *Consider a compressed symmetric dimension vector of the form* $\mathbf{d} = ((1, 2n-2, 1), \mathbf{b}, \mathbf{c})$ *with* $\ell(\mathbf{b}) = 3$.
(i) If $Q(\mathbf{d}) \leq 0$*, then* $\mathbf{d}$ *must be one of the following:*

$$\mathbf{d}_1 = ((1,2,1),(1,2,1),(1^4)), \quad \mathbf{d}_2 = ((1,4,1),(2^3),(1^6)) \ ;$$

in fact, $Q(\mathbf{d}_1) = Q(\mathbf{d}_2) = 0$.
(ii) If $Q(\mathbf{d}) = 1$*, then* $\mathbf{d}$ *must be one of the following:*

$$\mathbf{d}_1^+ = ((1,2,1),(1,2,1),(1,2,1)), \quad \mathbf{d}_2^+ = ((1,4,1),(2^3),(1,1,2,1,1)) \ .$$

We finish off the type (SpY) with the following statement, whose proof we postpone until §2.4.

Lemma 2.6. *For* $\mathbf{d} = \mathbf{d}_1$ *or* $\mathbf{d}_2$ *as above, there are exactly two* $\mathcal{F}$*-indecomposable classes of dimension* $\mathbf{d}$ *which lie in* Sp$\mathcal{F}$*.*

Next we show that any $\mathbf{d}$ *not* on the list of Theorem 1.2 is Sp-infinite. First we observe the following analogue of [6, Lemma 3.5]. (We wish to thank the vigilant referee who pointed out omissions in an earlier version of this Lemma.)

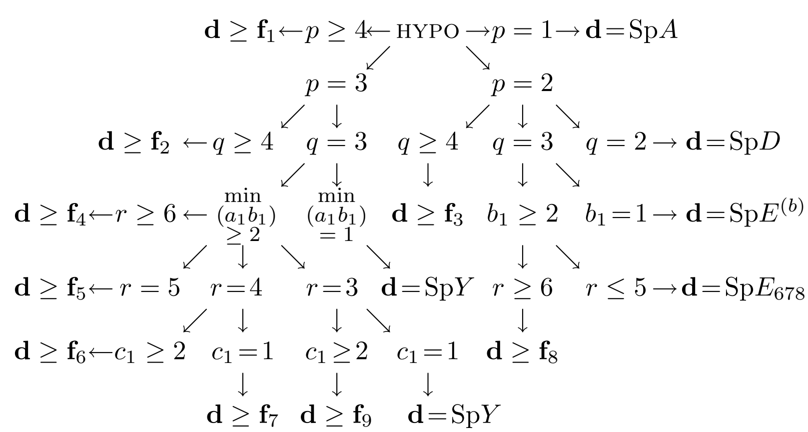
Lemma 2.7. *Let* $\mathbf{d}$ *be a compressed triple of symmetric compositions of the same even number. Then exactly one of the following holds:*
(i) $\mathbf{d}$ *belongs to one of the types* (SpA)—(SpY) *in Theorem 1.2.*
(ii) $\mathbf{d}$ *has a summand whose compressed form is one of the following nine dimension vectors:*

$$\mathbf{f}_1 = ((1^4)(1^4)(1^4)),\ \mathbf{f}_2 = ((1,2,1),(1^4),(1^4)),\ \mathbf{f}_3 = ((2,2),(1^4),(1^4))\ ,$$

$$\mathbf{f}_4 = ((2^3),(2^3),(1^6)),\ \mathbf{f}_5 = ((2^3),(2^3),(1,1,2,1,1)),\ \mathbf{f}_6 = ((2^3),(2^3),(2,1,1,2)),$$

$$\mathbf{f}_7 = ((2^3),(2^3),(1,2,2,1)),\ \mathbf{f}_8 = ((3,3),(2^3),(1^6)),\ \mathbf{f}_9 = ((2^3),(2^3),(2^3))\ ,$$

Proof. Consider the tree of implications:



The root of the tree is our

HYPOTHESIS: $\mathbf{d} \in \Lambda_{p,q,r}$ is a compressed triple of symmetric compositions of an even number, and $1 \le p \le q \le r$.

The arrows coming from a statement point to all possible cases resulting from the statement. We employ the abuse of notation $\mathbf{d} = \mathrm{Sp}A$, $\mathbf{d} = \mathrm{Sp}D$, etc to indicate that $\mathbf{d}$ belongs to the corresponding type in Theorem 1.2. We also write $\mathbf{d} \ge \mathbf{f}_i$ to indicate that $\mathbf{f}_i$ is a summand of $\mathbf{d}$. The lemma follows because every case ends in (i) or (ii), and these conditions are clearly disjoint. □

Now we can apply Proposition 2.4(i), provided we show that each of the dimension vectors $\mathbf{f}_1, \dots, \mathbf{f}_9$ is Sp-infinite.

For the first eight of these cases, this is done by a simple dimension count. The symplectic group and its isotropic Grassmannians have the following dimensions:

$$\dim \mathrm{Sp}_{2n} = n(2n+1), \quad \dim \mathrm{SpFl}_{(k,2n-2k,k)} = k(4n+1-3k)/2\ .$$

In particular, we have:

$$\dim \mathrm{Sp}_4 = 10, \quad \dim \mathrm{SpFl}_{(1^4)} = 4$$
$$\dim \mathrm{SpFl}_{(1,2,1)} = \dim \mathrm{SpFl}_{(2,2)} = 3,$$

$$\dim \mathrm{Sp}_6 = 21, \quad \dim \mathrm{SpFl}_{(1^6)} = 9,$$
$$\dim \mathrm{SpFl}_{(1,1,2,1,1)} = \dim \mathrm{SpFl}_{(1,2,2,1)} = \dim \mathrm{SpFl}_{(2,1,1,2)} = 8,$$
$$\dim \mathrm{SpFl}_{(2^3)} = 7, \quad \dim \mathrm{SpFl}_{(3,3)} = 6 \ .$$

We conclude that

$$\dim \mathrm{SpFl}_{\mathbf{f}_1} = 12 > \dim \mathrm{SpFl}_{\mathbf{f}_2} = \dim \mathrm{SpFl}_{\mathbf{f}_3} = 11 > \dim \mathrm{Sp}_4 = 10 \ ,$$
$$\dim \mathrm{SpFl}_{\mathbf{f}_4} = 23 > \dim \mathrm{SpFl}_{\mathbf{f}_5} = \dim \mathrm{SpFl}_{\mathbf{f}_6} = \dim \mathrm{SpFl}_{\mathbf{f}_7} = \dim \mathrm{SpFl}_{\mathbf{f}_8} = 22$$
$$> \dim \mathrm{Sp}_6 = 21 \ .$$

Thus these eight cases are of infinite symplectic type.

Now, $\dim \mathrm{SpFl}_{\mathbf{f}_9} = 21 = \dim \mathrm{Sp}_6$, but we can easily see that $\mathbf{f}_9$ is also Sp-infinite by using Theorem 1.3. It is known (from counting dimensions) that there are infinitely many non-isomorphic classes of $\mathcal{F}$ in dimension $\mathbf{f}_9/2 = ((1^3),(1^3),(1^3))$. Taking the direct sum of each such class with its dual, we obtain infinitely many non-isomorphic classes of $\mathrm{Sp}\mathcal{F}$ in dimension $\mathbf{f}_9$, meaning infinitely many Sp_6-orbits in $\mathrm{SpFl}_{\mathbf{f}_9}$.

This concludes the proof of Theorem 1.2. □

2.4. **Proof of Theorem 1.4.** To produce the list of finite-type symplectic indecomposables in Theorem 1.4, we must consider all possible Sp-indecomposable summands for each type (SpA)–(SpY) in Theorem 1.4. That is, we must refine our proof of Theorem 1.2, which showed that there are only finitely many such indecomposables. By Theorem 1.3, each Sp-indecomposable J in $\mathrm{Sp}\mathcal{F}_{pqr}$ is of the form $J = I$ or $J = I \oplus I^*$ for some GL-indecomposable I in $\mathcal{F}_{pqr}$.

For all our types except (SpY), the corresponding dimension vectors are not only Sp-finite, but GL-finite as well. Thus we may use our list of GL-indecomposables $I_{\mathbf{d}}$ of finite type from [6, Thm 2.9]. Each $I_{\mathbf{d}}$ is the unique GL-indecomposable in its dimension. (Notice that this implies $I^*_{\mathbf{d}} \cong I_{\mathbf{d}^{\mathrm{op}}}$.)

Scanning the list of [6] we find that for every symmetric $\mathbf{d}$ of type (SpA)–(Sp$E^{(b)}$), the GL-indecomposable $I_{\mathbf{d}}$ lies in $\mathrm{Sp}\mathcal{F}$ (as demonstrated by the symplectic representatives given in Theorem 1.4). Therefore we can obtain all Sp-indecomposables for these types either as:
(1) $J = I_{\mathbf{d}}$, where $\mathbf{d}$ is symmetric of type (SpA)–(Sp$E^{(b)}$) and $(I_{\mathbf{d}})_{\mathrm{cpr}}$ is one of the compressed GL-indecomposables from [6]; or
(2) $J = I_{\mathbf{e}} \oplus I_{\mathbf{e}^{\mathrm{op}}}$ where $\mathbf{e} + \mathbf{e}^{\mathrm{op}}$ is of type (SpA)–(Sp$E^{(b)}$), $\mathbf{e}$ is *not* symmetric, and $(I_{\mathbf{e}})_{\mathrm{cpr}}$, $(I_{\mathbf{e}^{\mathrm{op}}})_{\mathrm{cpr}}$ are among the compressed GL-indecomposables from [6].

Most of Theorem 1.4 consists of a systematic listing of these J in compressed form. (In compiling this list, one must remember that even if $\mathbf{d} = \mathbf{e} + \mathbf{e}^{\mathrm{op}}$ is compressed, the summand $\mathbf{e}$ might not be compressed.)

To complete our list, we must find the (compressed) Sp-indecomposables of type (SpY). Now, if $\mathbf{d}$ is any dimension vector of type (SpY) with summand $\mathbf{e} + \mathbf{e}^{\mathrm{op}}$, then $\mathbf{e}_{\mathrm{cpr}}$ belongs to one of the previous types (SpA)–(Sp$E^{(b)}$), and we can repeat the above procedure.

It remains to consider GL-indecomposables I of type (SpY) which are symplectic. By Kac's Theorem on indecomposables and Lemma 2.5, such I can occur only in the dimensions

$$\mathbf{d}_1 = ((1,2,1),(1,2,1),(1^4)), \quad \mathbf{d}_2 = ((1,4,1),(2^3),(1^6)) \ ,$$

$$\mathbf{d}_1^+ = ((1,2,1),(1,2,1),(1,2,1)), \quad \mathbf{d}_2^+ = ((1,4,1),(2^3),(1,1,2,1,1)).$$

The bottom two dimensions have at most one indecomposable each. We may easily check that the representatives $I_1^+ := I_{\mathbf{d}_1^+}$ and $I_2^+ := I_{\mathbf{d}_2^+}$ given in Theorem 1.4 are GL-indecomposable. Indeed, the automorphisms of I_1^+ in GL_4 are exactly the matrices of the form $M(a,b) = \left(\begin{smallmatrix} a&0&0&0\\ 0&a&b&0\\ 0&0&a&0\\ 0&0&0&a \end{smallmatrix}\right)$ with $a \neq 0$. Since this automorphism group contains no semi-simple elements other than scalars, I_1^+ is indecomposable (cf. [4, Lemma 2.3]). Similarly for I_2^+. It is clear that I_1^+, I_2^+ lie in Sp$\mathcal{F}$.

Finally we must find all the symplectic objects among the (infinitely many) GL-indecomposables in dimensions $\mathbf{d}_1$ and $\mathbf{d}_2$. This is the content of the lemma left unproved in §2.3:

Proof of Lemma 2.6. Our technique is to fiber $\mathrm{Fl}_{\mathbf{d}_i}$ over $\mathrm{Fl}_{\mathbf{d}_i^+}$ by dropping one subspace from the complete flag. In other words, consider the *contraction functor* $\pi_i : \mathcal{F}_{p,q,r} \to \mathcal{F}_{p,q,r-1}$ which sends an object $(V; A, B, C)$ to $(V; A, B, C')$, where C' is obtained from C by dropping the ith subspace C_i.

Lemma 2.8. *Suppose $(V; A, B, C)$ is an object of $\mathcal{F}_{p,q,r}$ whose contraction splits in $\mathcal{F}_{p,q,r-1}$ as*

$$\pi_i(V; A, B, C) = \bigoplus_{j=1}^{c+1} (V_j; A^j, B^j, C^j),$$

where $c = c_i + c_{i+1} = \dim(C_{i+1}/C_{i-1})$ and $V_1, \ldots, V_{c+1} \subset V$.

Then $(V; A, B, C)$ itself splits in $\mathcal{F}_{p,q,r}$ as

$$(V; A, B, C) \ = \ (V_k; A^k, B^k, C^k) \oplus (\tilde{V}_k; \tilde{A}^k, \tilde{B}^k, \tilde{C}^k)$$

for some k, where $\tilde{V}_k = \bigoplus_{j \neq k} V_j$.

Proof. It suffices to show that for any subspaces $C' \subset C \subset C'' \subset V$ with $\dim(C''/C') = c$, if C' and C'' split as

$$C' = \bigoplus_{j=1}^{c+1} (C' \cap V_j) \qquad C'' = \bigoplus_{j=1}^{c+1} (C'' \cap V_j),$$

then C splits as $C = (C \cap V_k) \oplus (C \cap \tilde{V}_k)$ for some k.

Since

$$c = \dim(C''/C') = \sum_{j=1}^{c+1} \dim((C'' \cap V_j) \, / \, (C' \cap V_j)) \ ,$$

there exists an index k with $C'' \cap V_k = C \cap V_k = C' \cap V_k$. Now, any $v \in C \subset C''$ has a decomposition $v = v_k + \tilde{v}$ for $v_k \in C'' \cap V_k$, $\tilde{v} \in C'' \cap \tilde{V}_k$; but then $v_k \in C \cap V_k$ and $\tilde{v} = v - v_k \in C \cap \tilde{V}_k$. This implies the desired decomposition. ◻

By the above lemma, any GL-indecomposable I of dimension $\mathbf{d}_i$ can split into at most two summands when contracted to $\mathbf{d}_i^+$; and if I is symplectic, then the contraction I^+ must be as well. This leaves only a few possibilities for I^+. We choose a representative for each possible I^+, and insert an extra middle-dimensional subspace to lift it to dimension $\mathbf{d}_1$. (This middle-dimensional space is automatically Lagrangian.) In geometric terms, we consider the fibration $\pi : \mathrm{Fl}_{\mathbf{d}_1} \to \mathrm{Fl}_{\mathbf{d}_1^+}$. The automorphism group of I^+ acts on the fiber, which is a projective line $\mathbf{P}^1$, and its orbits are the Sp$\mathcal{F}$-isomorphism classes of objects I lying over I^+. A given I is indecomposable exactly if its automorphism group contains no semi-simple elements except scalars.

For I of dimension $\mathbf{d}_1$, the possible I^+ are:

$$I_1^+, \quad I^{\mathrm{sym}}_{((110)(110)(110))}, \quad I^{\mathrm{sym}}_{((110)(110)(011))}, \quad I^{\mathrm{sym}}_{((110)(011)(110))}, \quad I^{\mathrm{sym}}_{((110)(011)(011))} .$$

For $\mathbf{d}_2$, they are:

$$I_2^+, \quad I^{\mathrm{sym}}_{((120)(111)(11100))}, \quad I^{\mathrm{sym}}_{((120)(111)(10110))},$$
$$I^{\mathrm{sym}}_{((120)(111)(01101))}, \quad I^{\mathrm{sym}}_{((120)(111)(00111))} .$$

The analysis of these ten cases completes the proof of Lemma 2.6 and Theorem 1.4. It turns out that only for $I^+ = I_1^+$ and $I^+ = I_2^+$ do we obtain any GL-indecomposable classes in the lifting: two classes in each case.

We work out two typical cases.

(i) $I^+ = I_1^+$. The lifted objects $I = (V; A, B, C)$ are of the form:

$$V = (e_1, e_2, e_3, e_4)$$
$$A = ((e_1 + e_2 + e_4) \subset (e_2, e_1 + e_4, e_1 + e_3)) \qquad B = ((e_1) \subset (e_1, e_2, e_3)),$$
$$C = C(s:t) = ((e_4) \subset (se_2 + te_3, e_4) \subset (e_2, e_3, e_4)) .$$

The automorphism $M(a, b)$ of I_1^+ takes $C(s, t)$ to $C(s + \frac{b}{a}t : t)$. Thus there are two orbits in the fiber: $I^1_{\mathbf{d}_1}$ represented by $(s : t) = (1 : 0)$, and $I^2_{\mathbf{d}_1}$ represented by $(s : t) = (0 : 1)$. Both are indecomposable, since $\mathrm{Aut}(I^1_{\mathbf{d}_1}) = \mathrm{Aut}(I_1^+)$, and $\mathrm{Aut}(I^2_{\mathbf{d}_1})$ consists of scalars.

(ii) $I^+ = I^{\mathrm{sym}}_{((110)(110)(011))}$. The lifted objects are: $V = (e_1, e_2, e_3, e_4)$,

$$A = ((e_1 + e_2) \subset (e_1, e_2, e_3 - e_4)), \qquad B = ((e_1) \subset (e_1, e_2, e_3)),$$
$$C = C(s:t) = ((e_4) \subset (se_2 + te_3, e_4) \subset (e_2, e_3, e_4)) .$$

Then

$$\mathrm{Aut}(I^+) = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & c & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \middle| \; a, b \neq 0 \right\} ,$$

which has two orbits on the set of I, both clearly decomposable. Thus, there are no indecomposables I lying over I^+.

The other eight cases are similar. □

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