VARIABLE-SMOOTHING REGULARIZATION METHODS FOR INVERSE PROBLEMS

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Many inverse problems of practical interest are ill-posed in the sense that solutions do not depend continuously on data. To effectively solve such problems, regularization methods are typically used. One problem associated with classical regularization methods is that the solution may be oversmoothed in the process. We present an alternative "local regularization" approach in which a decomposition of the problem into "local" and "global" parts permits varying amounts of local smoothing to be applied over the domain of the solution. This allows for more regularization in regions where the solution is likely to be more smooth, and less regularization in regions where sharp features are likely to be present. We illustrate this point with several numerical examples.

1 A Volterra Inverse Problem

We consider here the inverse problem of finding $u \in L_2(0,1)$ solving

$$Au(t) = f(t), \quad \text{a.a. } t \in (0,1), \tag{1}$$

where A is a Volterra operator given by

$$Au(t) = \int_0^t k(t-\tau)u(\tau)\,d\tau, \quad t \in (0,1), \tag{2}$$

and where the kernel k is assumed to be uniformly Hölder continuous on the interval $[0,1]$, $k(t) > 0$ for $t \in (0,1]$. We also assume that f is Hölder continuous on $[0,1]$ and is such that u uniquely solves problem (1).

It is well-known that (1) is an ill-posed problem due to lack of continuous dependence of the solution u on data $f \in L_2(0,1)$. Thus, in the usual case where only a measured or computed approximation f^δ to f is available, with $\|f^\delta - f\| < \delta$, some kind of regularization or stabilization method is required in order to obtain a reasonable approximation u^δ to u. In the above, $\|\cdot\|$ denotes the usual $L_2(0,1)$ norm.

Classical Tikhonov regularization is based on finding u_α^δ solving the minimization problem

$$\min_{u \in dom\, L} \|Au - f^\delta\|^2 + \alpha\,\|Lu\|^2 \tag{3}$$

where L is a densely defined closed operator on $L_2(0,1)$ and $\alpha > 0$ is known as the Tikhonov regularization paper. Standard regularization theory guarantees that a choice of $\alpha = \alpha(\delta)$ may be made such that $\alpha(\delta) \to 0$ and $u_{\alpha(\delta)}^\delta \to u$ as $\delta \to 0$.

One problem associated with classical regularization methods such as the Tikhonov method above is that the regularized solution u_α^δ is usually oversmoothed in the process. This is due to the fact that smoothing occurs globally (via a shift in the spectrum of $A^\star A$) and is controlled by a single scalar parameter α.

As an alternative, the author has developed a family of "local regularization methods" for Volterra problems which lead to fast numerical methods and additionally allow for locally-controlled smoothing of the solution. The methods are based on a decomposition of A into t-dependent "global" and "local" parts, where, for $0 < r \ll 1$, the local operator acts on functions with support on a small subinterval $(t, t+r)$ of $(0,1)$. The idea is to impose regularization *locally* on each subinterval and to use a (local) regularization parameter α to control the amount of smoothing on each subinterval. Proofs of convergence of these methods and of various types of finite-dimensional discretizations have been provided elsewhere.[1,2,3,4]

In more recent work,[5,6] the author has extended this theory to handle the case of variable $r = r(t)$ and $\alpha = \alpha(t)$. Thus these functional regularization parameters allow for local, *variable* control of the regularization process, i.e., one may impose more smoothing on some parts of the domain of u and less smoothing elsewhere in the domain.

In this paper we will investigate the degree to which these variable parameters can be used to control smoothing via the examination of a several numerical examples. For fixed $r = r(t)$ we will also discuss selection of the optimal regularization parameter $\alpha = \alpha(t)$. This discussion will take place in a discrete framework.

2 Discrete Formulation of the Inverse Problem

We assume that the matrix equation in $\mathbf{u}^\delta \in \mathbf{R}^n$,

$$\mathbf{A}\mathbf{u}^\delta = \mathbf{f}^\delta, \tag{4}$$

represents a discrete form of problem (1), where f in (1) has been replaced by its perturbed counterpart f^δ, and $\mathbf{f}^\delta \in \mathbf{R}^n$ is a discretized form of f^δ. We assume that the discretization method generates $\mathbf{A} \in \mathbf{R}^{n\times n}$ which is lower-triangular and Toeplitz (because the original operator A is Volterra and of convolution type), with $\mathbf{A}$ given by

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ a_2 & a_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & \dots & a_1 \end{pmatrix}, \qquad (5)$$

where $a_i > 0$ for $i = 1, \dots, n$.

2.1 Discrete Tikhonov Regularization

Standard (discrete) Tikhonov regularization applied to (4) finds the solution $\mathbf{u}_\alpha^\delta \in \mathbf{R}^n$ of the following minimization problem, which is a discrete analog of (3),

$$\min_{\mathbf{u} \in R^n} \left\{ \|\mathbf{A}\mathbf{u} - \mathbf{f}^\delta\|^2 + \alpha \, \|\mathbf{L}\mathbf{u}\|^2 \right\}, \qquad (6)$$

where $\alpha > 0$ is a given Tikhonov regularization parameter, $\|\cdot\|$ denotes the usual $\mathbf{R}^n$ norm, and $\mathbf{L}$ is a discretized version of L. In the case of $\mathbf{L}$ the $n \times n$ identity, problem (6) represents *0th order* (discrete) Tikhonov regularization; if $\mathbf{L}$ is a discrete analog of the differentiation operator, then (6) becomes *1st order* (discrete) Tikhonov regularization.

There are several methods for determining the optimal regularization parameter α to use in solving the discrete Tikhonov problem (6). One classical way is via a discrete Morozov discrepancy principle [7], where the optimal parameter α is found through the criterion

$$\|\mathbf{A}\mathbf{u}_\alpha^\delta - \mathbf{f}^\delta\|^2 = \beta \|\mathbf{d}\|^2, \qquad (7)$$

where $\mathbf{d} = (f_1^\delta - f_1, \dots, f_n^\delta - f_n)^\top \in \mathbf{R}^n$ is the discrete noise vector, $\mathbf{u}_\alpha^\delta \in \mathbf{R}^n$ is the solution of (6) for given α, and $\beta \equiv 1$. (In practice, β is often taken to be a fixed constant in the interval $[1, 2]$.) The optimal α is then found using (7) and a line search or simple optimization procedure. We will use this principle below to determine an optimal scalar α for standard 0th order and 1st order Tikhonov regularization in several examples.

2.2 Discrete Local Regularization

As an alternative to Tikhonov regularization, we now describe a (discrete) *local regularization method* that is a variation of that developed by Lamm and Eldén [4]. The method is a *sequential* variation of discrete Tikhonov regularization which allows for fast regularized solution of Volterra problems of the form (1); here we generalize the approach in Lamm and Eldén [4] by considering the possibility of variable regularization parameters defined by $\mathbf{r} = (r_1, r_2, \dots, r_n)$ and $\mathbf{a} = (\alpha_1, \alpha_2, \dots \alpha_n)$. In this case, $\alpha_i > 0$ and integers r_i satisfy $1 \le r_i \le n$ with the added restriction that $i + r_i - 1 \le n$ (needed to ensure that local subintervals do not extend beyond the interval $[0, 1]$).

Given *a priori* values of the vector regularization parameters $\mathbf{r}$ and $\mathbf{a}$, the basic local regularization algorithm is as follows. Assume that $u_1, u_2, \dots, u_{i-1}$ have already been found. Then at the i^{th} step, we determine u_i by first finding the vector $\mathbf{b}_{r_i,\alpha_i}^\delta \in \mathbf{R}^{r_i}$, which solves the reduced-dimension Tikhonov problem

$$\min_{\mathbf{b} \in R^{r_i}} \left\{ \|\mathbf{A}_{r_i}\mathbf{b} - \mathbf{h}^{(i)}\|^2 + \alpha_i \, \|\mathbf{b}\|^2 \right\}, \qquad (8)$$

where $\mathbf{A}_{r_i}$ is the leading $r_i \times r_i$ block of $\mathbf{A}$ and $\|\cdot\|$ denotes the $\mathbf{R}^{r_i}$ norm. Here $\mathbf{h}^{(i)} \in \mathbf{R}^{r_i}$ with $\mathbf{h}^{(1)} = (f_1^\delta, \dots, f_{r_1}^\delta)^\top$ and $\mathbf{h}^{(i)} = (h_1^{(i)}, h_2^{(i)}, \dots, h_{r_i}^{(i)})^\top$ for $i \ge 2$, where

$$h_p^{(i)} = f_{i+p-1}^\delta - \sum_{j=1}^{i-1} a_{i+p-j} u_j, \qquad p = 1, \dots, r_i. \qquad (9)$$

After determining $\mathbf{b}_{r_i,\alpha_i}^\delta$ in this way, we take u_i to be the *first* component of the vector $\mathbf{b}_{r_i,\alpha_i}^\delta$, discarding all remaining components of $\mathbf{b}_{r_i,\alpha_i}^\delta$.

Thus the approximation is a type of "moving local Tikhonov problem" where a small (future) Tikhonov problem is solved at each step, with the size of the local problem (and future interval) determined by r_i and the local Tikhonov parameter given by $\alpha_i > 0$. The parameters r_i and α_i are thus local regularization parameters, and changing these parameters for each i allows for varying amounts of regularization over the discretized interval $[0, 1]$. Convergence of the finite-dimensional approximation has been established for the case of constant $r_i \equiv r$, $\alpha_i \equiv \alpha$, for $i = 1, \dots, n$ [4]. A discrete convergence theory in the case of variable $\mathbf{r}$ (with the restriction $i + r_i + 1 \le n$) and variable $\mathbf{a}$ will be considered elsewhere.

Below we present several numerical examples and compare the results of *standard* Tikhonov regularization with that of the *local* regularization method for fixed $\mathbf{r}$ (i.e., $\mathbf{r}$ is selected *a priori* as a fixed n-vector). The regularization parameter controlling smoothing is then the vector-valued $\mathbf{a} = (\alpha_1, \alpha_2, \dots, \alpha_n)$. In our examples below, we find the optimal α_i at the ith sequential step using a local Morozov principle. That is, assuming we have already determined solution components $u_1, u_2, \dots, u_{i-1}$ and positive regularization parameters $\alpha_1, \alpha_2, \dots \alpha_{i-1}$, we find the optimal local regularization parameter α_i solving the reduced-dimension Morozov principle (a reduced-dimension version of (7))

$$\|\mathbf{A}_{r_i}\mathbf{b}_{r_i,\alpha_i}^\delta - \mathbf{h}^{(i)}\|^2 = \beta \|\mathbf{d}_{r_i}\|^2, \qquad (10)$$

where $\mathbf{b}_{r_i,\alpha_i}^\delta \in \mathbf{R}^{r_i}$ is the solution to (8) given r_i, α_i; $\mathbf{d}_{r_i} = (f_i^\delta - f_i, \dots, f_{i+r_i-1}^\delta - f_{i+r_i-1})^\top \in \mathbf{R}^{r_i}$ is the ith noise vector; and the fixed constant β is as in (7).

3 Numerical Examples

In each example below, we specify the k appearing in (2) and a true solution u for (1). The true data f is defined via (1) from u, and the perturbed data is determined by the addition of random noise uniformly distributed in $[-\gamma, \gamma]$, where γ is selected so that the relative error in f^δ is a desired level (specified in each example below).

Throughout we use a uniform discretization of $[0, 1]$ into $n = 40$ subintervals, and form the approximation (4) via collocation of (1) over the space of piecewise constant functions, with collocation occurring at n gridpoints.[4] In addition, we shall always take $\mathbf{r} \in \mathbf{R}^n$ to be fixed and of the form $\mathbf{r} = (r, r, \ldots, r, r-1, r-2, \ldots, 1)$, where the value of integer $r \geq 2$ is specified in the examples below.

For each example we consider both 0th order and 1st order standard (discrete) Tikhonov regularization, using the Tikhonov regularization parameter α determined by the Morozov discrepancy principle (7). These (more classical) results are then compared to the results for *local* regularization (both 0th order and 1st order local methods) for various values of r. The vector regularization parameter $\mathbf{a} = (\alpha_1, \ldots, \alpha_n)$ needed for the local method is determined by a sequence of local Morozov discrepancy principles (10), as described earlier. We note that in (10) we do not take the entries in the local noise vector $\mathbf{d}_{r_i}$ to be the actual values, since in practice one typically does not know pointwise noise values. Instead we compute $d \equiv \text{mean}\{|f_i^\delta - f_i|, i = 1, \ldots, n\}$ and use $\mathbf{d}_{r_i} \equiv (d, d, \ldots, d)^\top \in \mathbf{R}^{r_i}$. Other definitions of d may be possible, but we do not investigate this here.

Finally, although the original problem (1) is defined on the interval $[0, 1]$, we only present results for the interval $[0, .8]$. The reason for this is that, due to the nature of the Volterra problem (1), it is impossible to perfectly recover the solution near the end of the interval (regardless of the method used).

3.1 Example 3.1

Here we take the kernel k to be $k(t) \equiv 1$, $t \in [0, 1]$, and we pick a continuous u as shown in dashed lines in Figures 1–4. We use random error to construct $\mathbf{f}^\delta$ so that the relative error in data is approximately 4%, and use $\beta = 2$ in the Morozov discrepancy principles (7) and (10). In Figures 1–4 we show our numerical findings for standard Tikhonov 0th order regularization, local 0th order regularization (using a fixed value of $r = 4$), standard Tikhonov 1st order regularization, and local 1st order regularization ($r = 4$), respectively. In all cases, approximations are graphed using solid curves. As Table 1 shows, the relative error in approximations is slightly better for standard (0th and 1st order) Tikhonov than it is for the local regularization results, however, this is not surprising given that the true u we are seeking is already smooth.

Table 1: Example 3.1 - Relative Errors in Approximate Solutions

Method	0th Order Relative Error	1st Order Relative Error
Std. Tikhonov	0.1511	0.1114
Local $r = 2$	0.1812	0.1644
Local $r = 3$	0.1672	0.1410
Local $r = 4$	0.1618	0.1325
Local $r = 5$	0.1531	0.1400
Local $r = 6$	0.1514	0.1427
Local $r = 7$	0.1576	0.1301
Local $r = 8$	0.1611	0.1274
Local $r = 9$	0.1627	0.1300

It is interesting to compare the regularization parameters α and $\mathbf{a}$ found for this example. For standard 0th order Tikhonov regularization, the optimal α is given by $\alpha = \mathcal{O}(10^{-4})$ while the vector-valued regularization parameter $\mathbf{a} = (\alpha_1, \ldots, \alpha_{40})$ for local 0th order regularization ($r = 4$) satisfies

$$\alpha_i = \begin{cases} \mathcal{O}(10^{-3}), \ i = 1, \ldots, 10 \\ \mathcal{O}(10^{-4}), \ i = 11, \ldots, 40. \end{cases} \tag{11}$$

Thus α_i decreases at a point corresponding to $t = .25$ in the interval $(0, 1)$; more regularization occurs before this point, and less after. This is quite reasonable when one considers the graph of the original solution.

The results are even more dramatic for 1st order regularization. For standard Tikhonov regularization, the optimal α is given by $\alpha = \mathcal{O}(10^{-7})$ while the vector-valued regularization parameter $\mathbf{a}$ for local 1st order regularization ($r = 4$) satisfies

$$\alpha_i = \begin{cases} \mathcal{O}(10^{+2}), \ i = 1, \ldots, 10 \\ \mathcal{O}(10^{-5}), \ i = 11, \ldots, 40 \end{cases} \tag{12}$$

with the decrease in α_i corresponding to a (sharp) decrease in regularization at approximately $t = .25$.

3.2 Example 3.2

For this example we again take $k(t) \equiv 1$, $t \in [0, 1]$, however we now pick a sharply discontinuous u which is shown in dashed lines in Figures 5–8. (We note that in order to better display the graphs of u and its approximations, we have artificially connected all jumps using line segments.) As in the last example, the relative error in perturbed data is approximately 4% and we use $\beta = 2$ in the Morozov principles (7) and (10).

In Figures 5–6 we illustrate the results of standard 0th order Tikhonov regularization and local 0th order

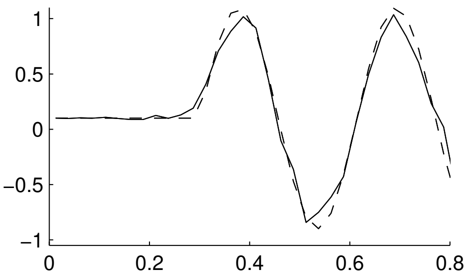

Figure 1: Example 3.1 - *Standard* 0th Order Tikhonov Regularization

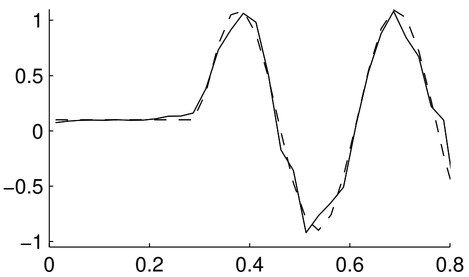

Figure 2: Example 3.1 - *Local* 0th Order Regularization ($r = 4$)

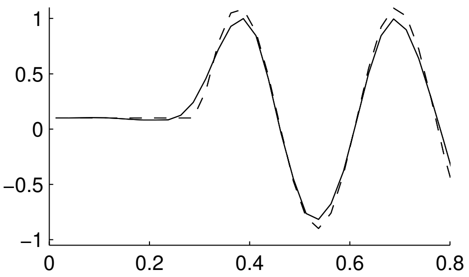

Figure 3: Example 3.1 - *Standard* 1st Order Tikhonov Regularization

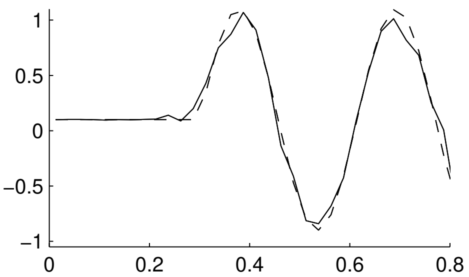

Figure 4: Example 3.1 - *Local* 1st Order Regularization ($r = 4$)

regularization for the case of $r = 2$. In Figures 7–8 we repeat these examples, instead using 1st order regularization throughout. The improvement of local regularization over standard Tikhonov regularization (which oversmooths solutions here) is illustrated in Table 2 where we give the relative error in solutions for standard 0th order and 1st order Tikhonov regularization and for local regularization (0th order and 1st order, for various values of r). From these tables we see that the best results are obtained using local regularization with $r = 2$.

Table 2: Example 3.2 - Relative Errors in Approximate Solutions

Method	0th Order Relative Error	1st Order Relative Error
Std. Tikhonov	0.4399	0.4891
Local $r = 1$	0.2905	0.2905
Local $r = 2$	0.2094	0.2049
Local $r = 3$	0.2131	0.2241
Local $r = 4$	0.2189	0.2243
Local $r = 5$	0.2473	0.2613
Local $r = 6$	0.2715	0.2891
Local $r = 7$	0.2737	0.2916
Local $r = 8$	0.2889	0.3069
Local $r = 9$	0.2946	0.3229

3.3 Example 3.3

Here we repeat Example 2 except that we now define $k(t) = t$, $t \in [0, 1]$. This example is considerably more ill-posed than Example 2, and thus we only perturb the data so that the relative error is approximately 0.5%. Because of the increased ill-posedness, it is not surprising that the results are not as good as they were for Example 3.2, regardless of the method applied.

In Figures 9–11, we show 0th order regularization (standard Tikhonov, local with $r = 4$, and local with $r = 5$) using $\beta = 2$. In Figure 12, we repeat the local 0th order regularization using $r = 4$, this time with $\beta = 1.44$. (The smaller β serves to damp the noise at the end of the interval, but also to damp the size of the jumps found by the approximation.) In Figures 13–16 we show the same results, but with 1st order regularization throughout.

In all cases the local regularization method finds the discontinuous jumps better than the standard Tikhonov regularization methods and does better overall on the interval $[0, .8]$. However, local regularization tends to perform worse toward the end of the interval as the propagated error in the sequential solution method begins to accumulate.

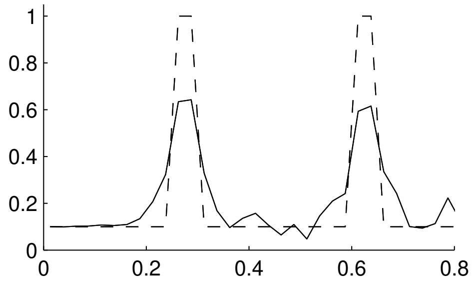

Figure 5: Example 3.2 - *Standard* 0th Order Tikhonov Regularization

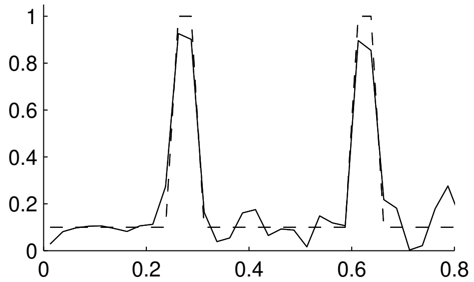

Figure 6: Example 3.2 - *Local* 0th Order Regularization ($r = 2$)

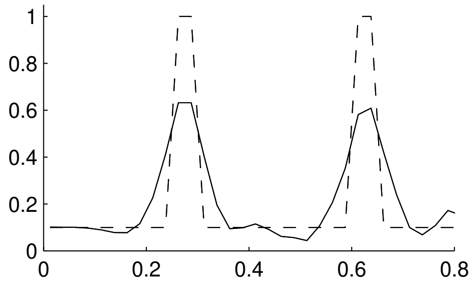

Figure 7: Example 3.2 - *Standard* 1st Order Tikhonov Regularization

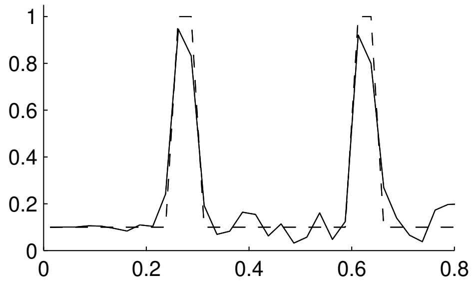

Figure 8: Example 3.2 - *Local* 1st Order Regularization ($r = 2$)

Acknowledgments

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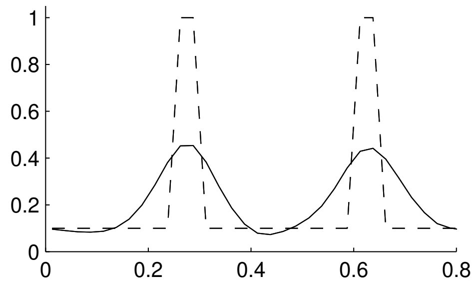

Figure 9: Example 3.3 - *Standard* 0th Order Tikhonov Regularization

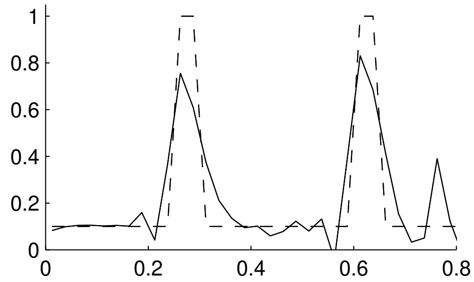

Figure 10: Example 3.3 - *Local* 0th Order Regularization ($r = 4$, $\beta = 2$)

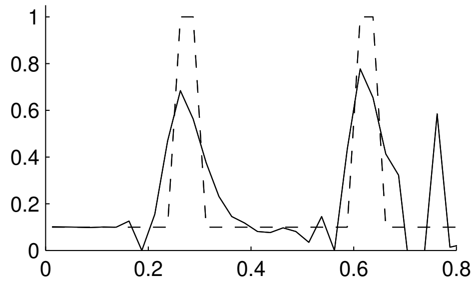

Figure 11: Example 3.3 - *Local* 0th Order Regularization ($r = 5$, $\beta = 2$)

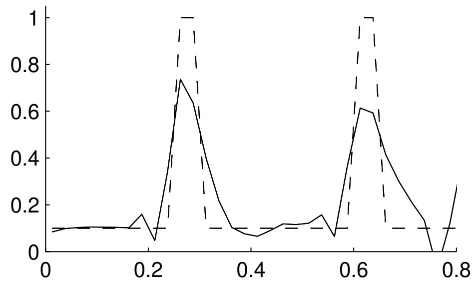

Figure 12: Example 3.3 - *Local* 0th Order Regularization ($r = 4$, $\beta = 1.44$)

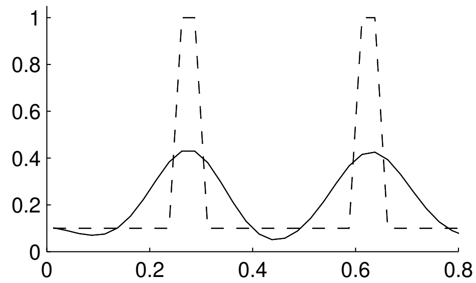

Figure 13: Example 3.3 - *Standard* 1st Order Tikhonov Regularization

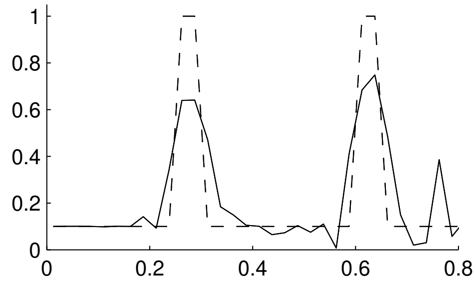

Figure 14: Example 3.3 - *Local* 1st Order Regularization ($r = 4$, $\beta = 2$)

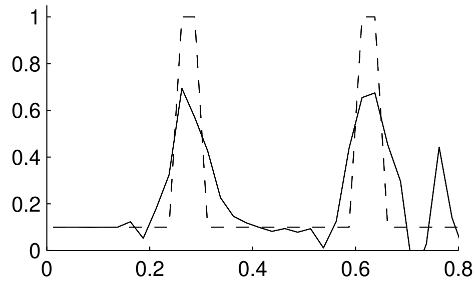

Figure 15: Example 3.3 - *Local* 1st Order Local Regularization ($r = 5$, $\beta = 2$)

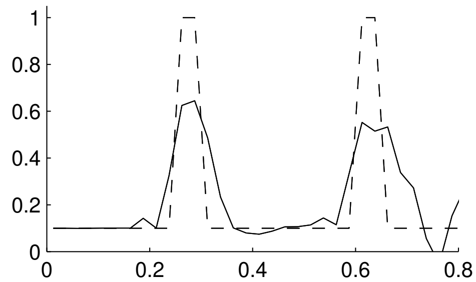

Figure 16: Example 3.3 - *Local* 1st Order Regularization ($r = 4$, $\beta = 1.44$)