Variable-Smoothing Local Regularization Methods for First-Kind Integral Equations

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Abstract. We consider the local regularization problem for integral equations of the first kind, generalizing previous work which applied only to problems of Volterra type. Our approach allows for local control of the regularization process, allowing for resolution of fine/sharp features of solutions without having to resort to nondifferentiable optimization techniques. In addition we present examples illustrating the numerical implementation of one version of the resulting local regularization algorithm and show that, under quite reasonable assumptions, the operation count of the local method compares well with that of standard Tikhonov regularization.

1. Introduction

We consider the problem of solving

$$Au = f, \tag{1.1}$$

where for $\Omega = [0,1]$, A is the bounded linear operator on $L^2(\Omega)$ given by

$$Au(t) = \int_\Omega k(t,s)u(s)\,ds, \quad \text{a.a. } t \in \Omega. \tag{1.2}$$

Here $k \in L^2(\Omega\times\Omega)$ satisfies

$$|k(t,s) - k(\tau,s)| \le L_k(s)|t-\tau|^{\mu_k}, \quad \text{a.a. } t,\tau,s \in \Omega, \tag{1.3}$$

for $\mu_k > 0$ and $L_k \in L^2(\Omega)$. We assume that the "true" data f is in the range of A, and we will let $\bar{u} \in L^2(\Omega)$ denote the (unique) minimum norm solution of (1.1) associated with f. Clearly $f \in L^\infty(\Omega)$, and we will assume $\bar{u} \in L^\infty(\Omega)$.

As an example of the types of problems of interest here, consider an application from image processing in which equation (1.1) models the blurring of an image u. In this case, $f = f(t)$ and $u = u(t)$ denote the grey-level values of blurred and original (one-dimensional) images, respectively, over all $t \in \Omega$, and A denotes the blurring operator with Gaussian convolution kernel

$$k(t,s) = \frac{\gamma}{\pi}\exp(-\gamma(t-s)^2), \quad t,s \in \Omega, \tag{1.4}$$

($\gamma > 0$ denotes the amount of blurring) [3]. This choice of k leads to a very simple model of blurring, but one that nevertheless provides for many of the features present in more complex imaging problems (see [7], for example, for generalizations). As noted in [3], surveillance photo enhancement is handled with a similar model, while electrical impedance tomography requires a nonlinear operator with properties similar to the above A.

We note that A as defined in (1.2) is a compact operator on $L^2(\Omega)$, and that the problem of solving (1.1) is an ill-posed problem (due to lack of continuous dependence on data f) if and only if the kernel k is nondegenerate (see, e.g., [5]). The focus of this paper will be on the solution of the equation (1.1) in this case and in the usual situation where the data f is only known approximately, i.e., where we only have available a perturbation f^δ of f. We will present a local regularization method for the solution of (1.1) which allows for variable regularization of the solution over different parts of the domain Ω. We note that the ideas of local regularization have extension to suitable $\Omega \subset \mathbb{R}^m$, but the details of this extension will be studied elsewhere.

Our work in this paper is in contrast to earlier treatments of local regularization methods in which the operator A was restricted to be of Volterra type (i.e., A is given by (1.2) with the kernel such that $k(t,s) = 0$ for $s > t$); see [2, 9, 10, 11, 13, 14, 15] for various types of local regularization methods for Volterra problems. The goal of this paper is to analyze local regularization methods which are applicable to general non-Volterra integral equations. This is done in Sections 2 and 3, with numerical examples of the (non-Volterra) blurring problem given in Section 4. Although the theory we develop in Sections 3–4 automatically applies to Volterra problems, in Section 5 we will show how the theory may be tailored to give more efficient results in the Volterra case. We note that the resulting numerical method for Volterra problems is *iterative* in nature, in contrast to the *sequential* local regularization methods developed in [2, 9, 10, 11, 13, 14, 15]. However, the iterative theory developed in this paper applies to *general* Volterra problems while the theory for sequential methods is at present limited to special Volterra operators and indeed may not be applicable to all Volterra problems [17].

1.1. Motivation

In order to motivate the local regularization method of interest in this paper, we consider the following (formal) construction. We will make all definitions precise in the next section.

First we define the functional "local regularization parameter" $\mathbf{r} = (r_{-1}, r_1)$, where $r_i : \Omega \to [0, \infty)$ is a sufficiently smooth function for $i = \pm 1$, and where for each $t \in \Omega$ we assume that the t-dependent local regularization interval $(t - r_{-1}(t)\,,\, t + r_1(t)]$ is nontrivial and contained in Ω. We note that the local regularization interval could be more simply defined as the intersection of $(t - r, t + r]$ with Ω, for a fixed scalar $r > 0$, however such a construction would not allow for *variable* local regularization throughout the domain Ω of the solution $\bar{u}$.

For a.a. $t \in \Omega$, we define the t-dependent "local part" of $\bar{u}$ to be the restriction of $\bar{u}$ to the local regularization interval $(t - r_{-1}(t)\,,\, t + r_1(t)]$. More precisely, given $t \in \Omega$, we define the map $\rho \mapsto \bar{\varphi}_{\mathbf{r}}(t)(\rho)$, with $\bar{\varphi}_{\mathbf{r}}(t) \in L^2(-r_{-1}(t), r_1(t))$, via

$$\bar{\varphi}_{\mathbf{r}}(t)(\rho) = \bar{u}(t + \rho), \quad \text{a.a. } \rho \in (-r_{-1}(t), r_1(t)].$$

If $\bar{u}$ is smooth, then once we are given $\bar{\varphi}_{\mathbf{r}}(t)$ for all $t \in \Omega$, we can recover $\bar{u}(t)$ exactly from $\bar{\varphi}_{\mathbf{r}}(t)$ via a map $T_{\mathbf{r}}$, where, for example,

$$\begin{aligned} T_{\mathbf{r}} \bar{\varphi}_{\mathbf{r}}(t) &\equiv \bar{\varphi}_{\mathbf{r}}(t)(0) \\ &= \bar{u}(t). \end{aligned}$$

But such a $T_{\mathbf{r}}$ is unbounded when applied to nonsmooth functions, so in the general case we will only recover $\bar{u}$ from $\bar{\varphi}_{\mathbf{r}}$ approximately using a bounded linear operator $T_{\mathbf{r}}$. For example, $T_{\mathbf{r}} \bar{\varphi}(t)$ could compute an integral average of $\bar{\varphi}(t)(\cdot)$ over a small subinterval of $(-r_{-1}(t), r_1(t)]$. A more precise construction of $T_{\mathbf{r}}$ will be given in the next section.

We now look for an equation in $\bar{\varphi}_{\mathbf{r}}$. We know that $\bar{u}$ satisfies $A\bar{u}(t+\rho) = f(t+\rho)$, for a.a. $\rho \in (-r_{-1}(t), r_1(t)]$, or

$$\int_0^1 k(t+\rho, s)\bar{u}(s)\,ds = f(t+\rho), \quad \text{a.a. } \rho \in (-r_{-1}(t), r_1(t)],\, t \in \Omega. \tag{1.5}$$

That is,

$$\int_0^{t-r_{-1}(t)} k(t+\rho, s)\bar{u}(s)\,ds + \int_{t-r_{-1}(t)}^{t+r_1(t)} k(t+\rho, s)\bar{u}(s)\,ds + \int_{t+r_1(t)}^1 k(t+\rho, s)\bar{u}(s)\,ds$$
$$= f(t+\rho), \;\text{ a.a. } \rho \in (-r_{-1}(t), r_1(t)],\; t \in \Omega, \tag{1.6}$$

where we may rewrite

$$\int_{t-r_{-1}(t)}^{t+r_1(t)} k(t+\rho, s)\bar{u}(s)\,ds = \int_{-r_{-1}(t)}^{r_1(t)} k(t+\rho, t+s)\bar{u}(t+s)\,ds. \tag{1.7}$$

We can then use (1.6)–(1.7) to motivate an "approximating equation" for $\bar{\varphi}_{\mathbf{r}}$,

$$\int_0^{t-r_{-1}(t)} k(t+\rho, s)T_{\mathbf{r}}\bar{\varphi}_{\mathbf{r}}(s)\,ds + \int_{-r_{-1}(t)}^{r_1(t)} k(t+\rho, t+s)\bar{\varphi}_{\mathbf{r}}(t)(s)\,ds \tag{1.8}$$
$$+ \int_{t+r_1(t)}^1 k(t+\rho, s)T_{\mathbf{r}}\bar{\varphi}_{\mathbf{r}}(s)\,ds = f(t+\rho), \;\text{ a.a. } \rho \in (-r_{-1}(t), r_1(t)],\; t \in \Omega.$$

The middle term on the left-hand side of equation (1.8) represents the action of A on the t-dependent "local part" of $\bar{u}$, so that equation (1.8) suggests a decomposition of the operator A into "global" and "local" parts, for each $t \in \Omega$. This splitting of A, which will be made more precise in the next section, is the basis of the local regularization method.

2. Basic Definitions

Throughout shall use use the notation $|\cdot|$ and $\langle\cdot,\cdot\rangle$ to indicate the usual norm and inner product, respectively, on $L^2(\Omega)$. We shall also use the notation $\mathcal{L}(H)$ to denote the space of bounded linear operators on a Hilbert space H.

2.1. The local regularization parameters $\mathbf{r}$ *and* α

Let $\Delta > 0$ be fixed, and let $\mathbf{r} \equiv (r_{-1}, r_1)$ denote a local regularization parameter in S,

$$S \equiv \{\mathbf{r} = (r_{-1}, r_1)\,|\, r_i \in C(\Omega) \cap C^1(\mathrm{int}(\Omega)),\, 0 < r_i(t) < \Delta,\, t \in \mathrm{int}(\Omega),\, i = \pm 1;$$
$$t - r_{-1}(t),\, t + r_1(t) \in \Omega,\, t \in \Omega\}.$$

Note that the possibility exists for $r_i(t) = 0$ at the endpoints $t = 0, 1$, of the interval $\Omega = [0, 1]$ for $i = \pm 1$, and indeed it is required that $r_{-1}(0) = 0$ and $r_1(1) = 0$. We will use the notation

$$\|\mathbf{r}\|_\infty \equiv \max_{i=\pm 1} \|r_i\|_\infty = \max_{i=\pm 1} \max_{t\in\Omega} |r_i(t)|.$$

For $t \in \mathrm{int}(\Omega)$ and $\mathbf{r}(t) \equiv (r_{-1}(t), r_1(t)) \in \mathbb{R}^2$, let $I_{-1}[\mathbf{r}(t)]$, $I_1[\mathbf{r}(t)]$, $I[\mathbf{r}(t)]$, and $I[\mathbf{r}(t)]^{\complement}$ designate intervals in $[-\Delta, \Delta]$, given respectively by

$$\begin{aligned} I_{-1}[\mathbf{r}(t)] &\equiv (-r_{-1}(t), 0], & I_1[\mathbf{r}(t)] &\equiv (0, r_1(t)], \\ I[\mathbf{r}(t)] &\equiv (-r_{-1}(t), r_1(t)], & I[\mathbf{r}(t)]^{\complement} &\equiv (-\Delta, \Delta] \setminus I[\mathbf{r}(t)]. \end{aligned}$$

A second local regularization parameter will be given by $\alpha \in \Lambda$, where

$$\Lambda \equiv \{\alpha \in L^\infty(\Omega) \,|\, \alpha(t) \geq \alpha_{\min} \equiv \inf_{s \in \Omega} \alpha(s) > 0, \text{ a.a. } t \in \Omega\}.$$

2.2. The spaces X, $\mathcal{X}$, and $\mathcal{X}_{\mathbf{r}}$

Let $X = L^2(-\Delta, \Delta)$ and let the usual norm and inner product on X be designated by $|\cdot|_X$ and $\langle \cdot, \cdot \rangle_X$, respectively. We shall define

$$\mathcal{X} \equiv L^2(\Omega; X)$$

to be the Hilbert space of measurable "functions" $\tilde{\varphi}$ on Ω with range in X, with norm

$$\|\tilde{\varphi}\|_{\mathcal{X}}^2 \equiv \int_\Omega |\tilde{\varphi}(t)|_X^2 \, dt = \int_\Omega \int_{-\Delta}^{\Delta} |\tilde{\varphi}(t)(\rho)|^2 \, d\rho \, dt,$$

for $\tilde{\varphi} \in \mathcal{X}$. Let $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ denote the associated inner product on $\mathcal{X}$.

We make the definition $j_{\mathbf{r}} \in \mathcal{L}(\mathcal{X})$ via

$$j_{\mathbf{r}}\varphi(t)(\rho) \equiv \begin{cases} \varphi(t)(\rho), & \text{a.a. } \rho \in I[\mathbf{r}(t)], \quad t \in \Omega \\ 0, & \text{a.a. } \rho \in I[\mathbf{r}(t)]^{\complement}, \quad t \in \Omega, \end{cases}$$

and define $\tilde{\mathcal{X}}_{\mathbf{r}} \equiv j_{\mathbf{r}}\mathcal{X}$. Finally, we define $\mathcal{X}_{\mathbf{r}} \subseteq \tilde{\mathcal{X}}_{\mathbf{r}}$ to be the completion of $\tilde{\mathcal{X}}_{\mathbf{r}} \cap C(\Omega; C[-\Delta, \Delta])$ with respect to the norm

$$\|\varphi\|_{\mathbf{r}}^2 \equiv \int_\Omega \sum_{i=-1,1} \left[\frac{1}{r_i(t)} \int_{I_i[\mathbf{r}(t)]} |\varphi(t)(\rho)|^2 \, d\rho \right] dt.$$

Given $\alpha \in \Lambda$, it will also be useful to define an equivalent weighted norm $\|\cdot\|_{\mathbf{r},\alpha}$ on $\mathcal{X}_{\mathbf{r}}$. We define

$$\|\varphi\|_{\mathbf{r},\alpha}^2 \equiv \int_\Omega \alpha(t) \sum_{i=-1,1} \left[\frac{1}{r_i(t)} \int_{I_i[\mathbf{r}(t)]} |\varphi(t)(\rho)|^2 \, d\rho \right] dt,$$

for $\varphi \in \mathcal{X}_{\mathbf{r}}$. Whenever the norm is clear from the context we will use the notation $\mathcal{X}_{\mathbf{r}}$ to designate both $(\mathcal{X}_{\mathbf{r}}, \|\cdot\|_{\mathbf{r}})$ and $(\mathcal{X}_{\mathbf{r}}, \|\cdot\|_{\mathbf{r},\alpha})$.

2.3. The isomorphism $E_{\mathbf{r}} : \mathcal{X}_{\mathbf{r}} \mapsto \mathcal{X}$

For simplicity and without loss of generality we shall henceforth take

$$\Delta \equiv 1.$$

Let $E_{\mathbf{r}}$ be defined, for $\varphi \in \mathcal{X}_{\mathbf{r}}$, via

$$E_{\mathbf{r}}\varphi(t)(\rho) \equiv \begin{cases} \varphi(t)(\rho\, r_{-1}(t)), & \text{a.a. } \rho \in (-1, 0], \quad t \in \Omega, \\ \varphi(t)(\rho\, r_1(t)), & \text{a.a. } \rho \in (0, 1], \quad t \in \Omega. \end{cases}$$

Then $E_{\mathbf{r}}$ performs a coordinate transformation and is a bounded linear operator from $\mathcal{X}_{\mathbf{r}}$ to $\mathcal{X}$ (with respect to either $\|\cdot\|_{\mathbf{r}}$ or $\|\cdot\|_{\mathbf{r},\alpha}$ on $\mathcal{X}_{\mathbf{r}}$); in fact,

$$\|E_{\mathbf{r}}\varphi\|_{\mathcal{X}} = \|\varphi\|_{\mathbf{r}}, \quad \varphi \in \mathcal{X}_{\mathbf{r}}.$$

2.4. Construction of $F_{\mathbf{r}}$, $\bar{F}_{\mathbf{r}}$, $F_{\mathbf{r}}^{\delta}$, $\bar{F}_{\mathbf{r}}^{\delta}$, $U_{\mathbf{r}}$, $\bar{U}_{\mathbf{r}} \in \mathcal{X}_{\mathbf{r}}$

The data function f may be used to define $F_{\mathbf{r}}, \bar{F}_{\mathbf{r}}$ via

$$F_{\mathbf{r}}(t)(\rho) \equiv \begin{cases} f(t+\rho), & \text{a.a. } \rho \in I[\mathbf{r}(t)], \quad t \in \Omega, \\ 0, & \text{a.a. } \rho \in I[\mathbf{r}(t)]^{\complement}, \quad t \in \Omega, \end{cases} \tag{2.1}$$

$$\bar{F}_{\mathbf{r}}(t)(\rho) \equiv \begin{cases} f(t), & \text{a.a. } \rho \in I[\mathbf{r}(t)], \quad t \in \Omega, \\ 0, & \text{a.a. } \rho \in I[\mathbf{r}(t)]^{\complement}, \quad t \in \Omega. \end{cases} \tag{2.2}$$

It is easy to show that $F_{\mathbf{r}}, \bar{F}_{\mathbf{r}} \in \mathcal{X}_{\mathbf{r}}$, with $\|F_{\mathbf{r}}\|_{\mathbf{r}}^2 \leq 2\|f\|_{\infty}^2$ and $\|\bar{F}_{\mathbf{r}}\|_{\mathbf{r}}^2 = 2|f|^2 \leq 2\|f\|_{\infty}^2$. Definitions similar to (2.1)–(2.2) may be made for $F_{\mathbf{r}}^{\delta}$ and $\bar{F}_{\mathbf{r}}^{\delta}$, respectively (where $f^{\delta} \in L^{\infty}(\Omega)$ replaces f in the above), and for $U_{\mathbf{r}}$ and $\bar{U}_{\mathbf{r}}$ (where $\bar{u}$ replaces f).

2.5. Operators $A_{\mathbf{r}}$, $B_{\mathbf{r}}$, ℓ, T, $T_{\mathbf{r}}$ *and* $C_{\mathbf{r}}$

The operators $A_{\mathbf{r}}$ and $B_{\mathbf{r}}$ are defined to facilitate a decomposition of the original operator A, following the discussion in Section 1. In particular, $A_{\mathbf{r}} : \mathcal{X}_{\mathbf{r}} \mapsto \mathcal{X}_{\mathbf{r}}$ is given, for $\varphi \in \mathcal{X}_{\mathbf{r}}$ and a.a. $t \in \Omega$, by

$$A_{\mathbf{r}}\varphi(t)(\rho) \equiv \begin{cases} \int_{-r_{-1}(t)}^{r_1(t)} k(t+\rho, t+s)\varphi(t)(s)\, ds, & \text{a.a. } \rho \in I[\mathbf{r}(t)], \\ 0, & \text{a.a. } \rho \in I[\mathbf{r}(t)]^{\complement}. \end{cases} \tag{2.3}$$

Then

$$\begin{aligned} \|A_{\mathbf{r}}\varphi\|_{\mathbf{r}}^2 &= \int_{\Omega} \sum_{i=-1,1} \left[\frac{1}{r_i(t)} \int_{I_i[\mathbf{r}(t)]} |A_{\mathbf{r}}\varphi(t)(\rho)|^2\, d\rho \right] dt \\ &\leq \int_{\Omega} \sum_{i=-1,1} \frac{1}{r_i(t)} \int_{I_i[\mathbf{r}(t)]} \left(\int_{-r_{-1}(t)}^{r_1(t)} k^2(t+\rho, t+s)\, ds \right) \left(\int_{-r_{-1}(t)}^{r_1(t)} (\varphi(t)(s))^2\, ds \right) d\rho\, dt \\ &\leq 4\|k\|_{\infty}^2 \|\mathbf{r}\|_{\infty}^2 \|\varphi_{\mathbf{r}}\|_{\mathbf{r}}^2, \end{aligned}$$

so that $A_{\mathbf{r}}$ is a bounded linear operator on $(\mathcal{X}_{\mathbf{r}}, \|\cdot\|_{\mathbf{r}})$, with

$$\|A_{\mathbf{r}}\| \leq C\|k\|_{\infty}\, \|\mathbf{r}\|_{\infty}, \tag{2.4}$$

for $\|\cdot\|$ the operator norm and for C independent of $\mathbf{r}$.

We define $B_{\mathbf{r}} : L^2(\Omega) \mapsto \mathcal{X}_{\mathbf{r}}$ by the following, for $\eta \in L^2(\Omega)$ and a.a. $t \in \Omega$,

$$B_{\mathbf{r}}\eta(t)(\rho) \equiv \begin{cases} \int_0^{t-r_{-1}(t)} k(t+\rho, s)\eta(s)\, ds + \int_{t+r_1(t)}^1 k(t+\rho, s)\eta(s)\, ds, & \text{a.a. } \rho \in I[\mathbf{r}(t)], \\ 0, & \text{a.a. } \rho \in I[\mathbf{r}(t)]^{\complement}. \end{cases} \tag{2.5}$$

Using arguments similar to those for $A_{\mathbf{r}}$, we have that $B_{\mathbf{r}}$ is a bounded linear operator from $L^2(\Omega)$ to $(\mathcal{X}_{\mathbf{r}}, \|\cdot\|_{\mathbf{r}})$ with operator norm

$$\|B_{\mathbf{r}}\| \leq C\|k\|_{\infty}, \tag{2.6}$$

for C again independent of $\mathbf{r}$.

Let $\ell \in X^\star$ be fixed and normalized so that $\ell(\mathbf{1}) = 1$, where $\mathbf{1} \in X$ is given by $\mathbf{1}(\rho) = 1$, for a.a. ρ. Let γ_ℓ denote the unique (nonzero) element of X satisfying

$$\ell(x) = \langle x, \gamma_\ell \rangle_X, \quad x \in X. \tag{2.7}$$

Define $T \in \mathcal{L}(\mathcal{X}, L^2(\Omega))$ by

$$T\tilde{\varphi}(t) \equiv \ell(\tilde{\varphi}(t)), \quad \text{a.a. } t \in \Omega, \tag{2.8}$$

for $\tilde{\varphi} \in \mathcal{X}$, and the bounded linear operator $T_\mathbf{r}$ from $(\mathcal{X}_\mathbf{r}, \|\cdot\|_\mathbf{r})$ to $L^2(\Omega)$ via

$$T_\mathbf{r} \equiv T E_\mathbf{r}.$$

We note that $\ell(\mathbf{1}) = 1$ gives

$$T_\mathbf{r} \bar{U}_\mathbf{r} = \bar{u} \tag{2.9}$$

where $\bar{U}_\mathbf{r}$ is defined above.

Example 2.1 *Consider, for example, $\ell \in X^\star$ given by the local averaging operator*

$$\ell(x) = \frac{1}{2c} \int_{-c}^{c} x(\rho)\, d\rho, \quad x \in X,$$

for some $0 < c \ll 1$. Then, in this case, $T_\mathbf{r}\varphi(t)$ computes the mean of the integral average of $\varphi(t)$ over a small subinterval of $I_{-1}[\mathbf{r}(t)]$ and $I_1[\mathbf{r}(t)]$, respectively, for a.a. $t \in \Omega$*; that is,*

$$T_\mathbf{r}\varphi(t) = \frac{1}{2}\left[\frac{1}{c\,r_{-1}(t)} \int_{-c\,r_{-1}(t)}^{0} \varphi(t)(\rho)\, d\rho + \frac{1}{c\,r_1(t)} \int_{0}^{c\,r_1(t)} \varphi(t)(\rho)\, d\rho\right],$$

for a.a. $t \in \Omega$ *and $\varphi \in \mathcal{X}_\mathbf{r}$.*

Example 2.2 *For the discrete algorithm described in Section 4, it is useful to consider another example, namely,*

$$\ell(x) = \frac{1}{c} \int_{-c}^{0} x(\rho)\, d\rho, \quad x \in X,$$

for $0 < c \ll 1$, from which we obtain

$$T_\mathbf{r}\varphi(t) = \frac{1}{c\,r_{-1}(t)} \int_{-c\,r_{-1}(t)}^{0} \varphi(t)(\rho)\, d\rho,$$

for a.a. $t \in \Omega$ *and $\varphi \in \mathcal{X}_\mathbf{r}$. We note (for the later discussion on discretization) that for x constant on $(-c, 0]$, we have $\ell(x) = x(0)$.*

Finally, we make the definition of the bounded linear operator $C_\mathbf{r}$ on $(\mathcal{X}_\mathbf{r}, \|\cdot\|_\mathbf{r})$ via

$$C_\mathbf{r} \equiv A_\mathbf{r} + B_\mathbf{r} T_\mathbf{r}.$$

2.6. The local regularization problem $\mathcal{P}^{\delta}_{\mathbf{r},\alpha}$

Using the spaces and operators constructed above, we define the local regularization problem of interest in this paper.

Definition 2.1 *Let $\mathbf{r} \in S$, $\alpha \in \Lambda$, and $\delta > 0$, and assume that $f^{\delta} \in L^{\infty}(\Omega)$ is given satisfying $\|f - f^{\delta}\|_{\infty} < \delta$. Then* Problem $\mathcal{P}^{\delta}_{\mathbf{r},\alpha}$ *is the problem of finding $\varphi^{\delta}_{\mathbf{r},\alpha} \in \mathcal{X}_{\mathbf{r}}$ such that*

$$\varphi^{\delta}_{\mathbf{r},\alpha} = \arg\min_{\varphi \in \mathcal{X}_{\mathbf{r}}} \left\{ \| C_{\mathbf{r}}\varphi - F^{\delta}_{\mathbf{r}} \|^2_{\mathbf{r}} + \|\varphi\|^2_{\mathbf{r},\alpha} \right\}$$

where $F^{\delta}_{\mathbf{r}}$ and $C_{\mathbf{r}}$ are defined in §2.4 and §2.5, respectively.

The following theorem follows from classical Tikhonov regularization theory (see, for example, [5, 6]) and the continuity of the operator $T_{\mathbf{r}}$.

Theorem 2.1 *Let $\mathbf{r} \in S$, $\alpha \in \Lambda$, $\delta > 0$, and let $f^{\delta} \in L^{\infty}(\Omega)$ be given satisfying $\|f - f^{\delta}\|_{\infty} < \delta$. Then there exists a unique solution $\varphi^{\delta}_{\mathbf{r},\alpha}$ of Problem $\mathcal{P}^{\delta}_{\mathbf{r},\alpha}$. Both $\varphi^{\delta}_{\mathbf{r},\alpha} \in \mathcal{X}_{\mathbf{r}}$ and $\eta^{\delta}_{\mathbf{r},\alpha} \equiv T_{\mathbf{r}}\varphi^{\delta}_{\mathbf{r},\alpha} \in L^2(\Omega)$ depend continuously on $F^{\delta}_{\mathbf{r}} \in \mathcal{X}_{\mathbf{r}}$ and thus on data $f^{\delta} \in L^{\infty}(\Omega)$.*

In the next lemma we use the optimality of $\varphi^{\delta}_{\mathbf{r},\alpha}$ to establish bounds on some relevant quantities.

Lemma 2.1 *Let $\mathbf{r}$, α, δ, and f^{δ} satisfy the conditions of Theorem 2.1, and let $\varphi^{\delta}_{\mathbf{r},\alpha}$ denote the solution of Problem $\mathcal{P}^{\delta}_{\mathbf{r},\alpha}$. Then*

$$\| C_{\mathbf{r}}\varphi^{\delta}_{\mathbf{r},\alpha} - F^{\delta}_{\mathbf{r}} \|^2_{\mathbf{r}} + \| \varphi^{\delta}_{\mathbf{r},\alpha} \|^2_{\mathbf{r},\alpha} \le C\left[(\|k\|^2_{\infty}\|\mathbf{r}\|^2_{\infty} + \|\alpha\|_{\infty})\|\bar{u}\|^2_{\infty} + \delta^2 \right],$$

for some $C > 0$ independent of $\mathbf{r}$, α, and δ.

Proof: The definition of $\varphi^{\delta}_{\mathbf{r},\alpha}$ gives

$$\begin{aligned}\| C_{\mathbf{r}}\varphi^{\delta}_{\mathbf{r},\alpha} - F^{\delta}_{\mathbf{r}} \|^2_{\mathbf{r}} + \| \varphi^{\delta}_{\mathbf{r},\alpha} \|^2_{\mathbf{r},\alpha} &\le \| C_{\mathbf{r}}\bar{U}_{\mathbf{r}} - F^{\delta}_{\mathbf{r}} \|^2_{\mathbf{r}} + \| \bar{U}_{\mathbf{r}} \|^2_{\mathbf{r},\alpha} \\ &\le 2(\| A_{\mathbf{r}}(\bar{U}_{\mathbf{r}} - U_{\mathbf{r}}) \|_{\mathbf{r}} + \| A_{\mathbf{r}}U_{\mathbf{r}} + B_{\mathbf{r}}T_{\mathbf{r}}\bar{U}_{\mathbf{r}} - F_{\mathbf{r}} \|_{\mathbf{r}})^2 + 2\| F_{\mathbf{r}} - F^{\delta}_{\mathbf{r}} \|^2_{\mathbf{r}} + \|\bar{U}_{\mathbf{r}}\|^2_{\mathbf{r},\alpha},\end{aligned}$$

where (2.9) gives

$$\begin{aligned}&\| A_{\mathbf{r}}U_{\mathbf{r}} + B_{\mathbf{r}}T_{\mathbf{r}}\bar{U}_{\mathbf{r}} - F_{\mathbf{r}} \|^2_{\mathbf{r}} \\ &\quad = \int_{\Omega} \sum_{i=-1,1} \frac{1}{r_i(t)} \int_{I_i[\mathbf{r}(t)]} \left[\int_{-r_{-1}(t)}^{r_1(t)} k(t+\rho, t+s)\bar{u}(t+s)\,ds \right. \\ &\qquad \left. + \int_0^{t-r_{-1}(t)} k(t+\rho, s)\bar{u}(s)\,ds + \int_{t+r_1(t)}^{1} k(t+\rho, s)\bar{u}(s)\,ds - f(t+\rho) \right]^2 d\rho\,dt \\ &\quad = \int_{\Omega} \sum_{i=-1,1} \left[\frac{1}{r_i(t)} \int_{I_i[\mathbf{r}(t)]} \left(\int_{\Omega} k(t+\rho, s)\bar{u}(s)\,ds - f(t+\rho) \right)^2 d\rho \right] dt \\ &\quad = 0.\end{aligned}$$

Thus,

$$\| C_{\mathbf{r}}\varphi^{\delta}_{\mathbf{r},\alpha} - F^{\delta}_{\mathbf{r}} \|^2_{\mathbf{r}} + \| \varphi^{\delta}_{\mathbf{r},\alpha} \|^2_{\mathbf{r},\alpha} \leq 2\| A_{\mathbf{r}} \|^2 (\| \bar{U}_{\mathbf{r}} \|_{\mathbf{r}} + \| U_{\mathbf{r}} \|_{\mathbf{r}})^2 + 4\| f - f^{\delta} \|^2_{\infty} + \|\alpha\|_{\infty} \| \bar{U}_{\mathbf{r}} \|^2_{\mathbf{r}}.$$

The result then follows from the bound on $\|A_{\mathbf{r}}\|$ in (2.4), and from the fact that $\|\bar{U}_{\mathbf{r}}\|^2_{\mathbf{r}}$, $\|U_{\mathbf{r}}\|^2_{\mathbf{r}} \leq 2\|\bar{u}\|^2_{\infty}$. □

3. Convergence

Our main convergence result is as follows:

Theorem 3.1 *Let $\{\delta_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^+$ with $\delta_n \to 0$ as $n \to \infty$. Let $\{\mathbf{r}_n\}_{n=1}^{\infty} \subseteq S$, $\mathbf{r}_n = ((r_n)_{-1}, (r_n)_1)$, and $\{\alpha_n\}_{n=1}^{\infty} \subseteq \Lambda$ be given such that $\|\mathbf{r}_n\|_{\infty}$, $\|\alpha_n\|_{\infty} \to 0$ as $n \to \infty$. Assume further that there is $M > 0$ such that*

(i) $\delta_n^2/\alpha_{n,\min} \to 0$.

(ii) $\|\mathbf{r}_n\|_{\infty}/\delta_n \leq M$, and

(iii) $\|\alpha_n\|_{\infty}/\alpha_{n,\min} \to 1$,

as $n \to \infty$. For each $n = 1, 2, \ldots$, let $f^{\delta_n} \in L^{\infty}(\Omega)$ be given with $\|f - f^{\delta_n}\|_{\infty} < \delta_n$, let $\varphi^{\delta_n}_{\mathbf{r}_n,\alpha_n} \in \mathcal{X}_{\mathbf{r}_n}$ denote the solution of problem $\mathcal{P}^{\delta_n}_{\mathbf{r}_n,\alpha_n}$ associated with f^{δ_n}, and let

$$\eta_n \equiv T_{\mathbf{r}_n}\varphi^{\delta_n}_{\mathbf{r}_n,\alpha_n}. \tag{3.1}$$

Then

$$\eta_n \to \bar{u} \quad \text{in } L^2(\Omega)$$

as $n \to \infty$, where $\bar{u}$ is the solution of the original problem (1.1).

Remark 3.1 *The assumptions (i) and (ii) in Theorem 3.1 are analogous to those in standard regularization theorems in that the regularization parameters $\mathbf{r}_n$ and α_n must converge to zero at a rate which is relative to the level δ_n of noise in the problem. Assumption (iii) says that α_n may vary with t so long as there is noise in the problem (i.e., for $\delta_n > 0$); however, in the limit as $\delta_n \to 0$, α_n must tend toward zero like a constant function.*

To simplify the notation in what follows, we will henceforth write $\varphi_n \equiv \varphi^{\delta_n}_{\mathbf{r}_n,\alpha_n}$, $\mathcal{P}_n \equiv \mathcal{P}^{\delta_n}_{\mathbf{r}_n,\alpha_n}$, $\mathcal{X}_n \equiv \mathcal{X}_{\mathbf{r}_n}$, $F_n \equiv F_{\mathbf{r}_n}$, $F^{\delta}_n \equiv F^{\delta_n}_{\mathbf{r}_n}$, $U_n \equiv U_{\mathbf{r}_n}$, $\bar{U}_n \equiv \bar{U}_{\mathbf{r}_n}$, $E_n \equiv E_{\mathbf{r}_n}$, $T_n \equiv T_{\mathbf{r}_n}$, $A_n \equiv A_{\mathbf{r}_n}$, and so on.

We will postpone the proof of Theorem 3.1 until the end of the section, after a number of preliminary lemmas have been established. The first result establishes weak subsequential convergence of η_n defined in (3.1) above.

Lemma 3.1 *Let $\{\delta_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^+$, $\{\mathbf{r}_n\}_{n=1}^{\infty} \subseteq S$, and let $\{\alpha_n\}_{n=1}^{\infty} \subseteq \Lambda$ and assume there exists $M > 0$ such that*

(i) $\delta_n^2/\alpha_{n,\min} \leq M$,

(ii) $\|\mathbf{r}_n\|_\infty/\delta_n \leq M$,

(iii) $\|\alpha_n\|_\infty/\alpha_{n,\min} \leq M$,

as $n \to \infty$. For each $n = 1, 2, \ldots$, let $f^{\delta_n} \in L^\infty(\Omega)$ be given with $\|f - f^{\delta_n}\|_\infty < \delta_n$, and let $\varphi_n \in \mathcal{X}_n$ denote the solution of problem $\mathcal{P}_n$ associated with f^{δ_n}, with $\eta_n \equiv T_n\varphi_n \in L^2(\Omega)$.

Let $\tilde{\varphi}_n \equiv E_n\varphi_n \in \mathcal{X}$. Then there is $\tilde{\varphi} \in \mathcal{X}$ and a subsequence of $\{\tilde{\varphi}_n\}$ which converges weakly in $\mathcal{X}$ to $\tilde{\varphi}$. That is, relabelling the subsequential indices,

$$\tilde{\varphi}_n \rightharpoonup \tilde{\varphi} \quad \text{in } \mathcal{X} \text{ as } n \to \infty.$$

Further, $\eta \in L^2(\Omega)$ defined by

$$\eta \equiv T\tilde{\varphi}$$

is such that (using the same relabelling of indices as above)

$$\eta_n \rightharpoonup \eta \quad \text{in } L^2(\Omega) \text{ as } n \to \infty.$$

Proof: We note that

$$\begin{aligned}\|E_n\varphi_n\|_{\mathcal{X}}^2 &= \|\varphi_n\|_{\mathbf{r}_n}^2 \\ &\leq \frac{1}{\alpha_{n,\min}}\left(\|C_n\varphi_n - F_n^\delta\|_{\mathbf{r}_n}^2 + \|\varphi_n\|_{\mathbf{r},\alpha_n}^2\right),\end{aligned}$$

we may use Lemma 2.1 and assumptions (i)–(iii) (along with the fact that the quantity $\|\mathbf{r}_n\|_\infty^2/\alpha_{n,\min} = (\|\mathbf{r}_n\|_\infty^2/\delta_n^2)\cdot(\delta_n^2/\alpha_{n,\min})$ is uniformly bounded from hypotheses (i) and (ii)) to obtain that $\|\tilde{\varphi}_n\|_{\mathcal{X}} = \|E_n\varphi_n\|_{\mathcal{X}}$ is uniformly bounded for all $n = 1, 2, \ldots$. The remaining statements of the lemma follow from the fact that $\mathcal{X}$ is a Hilbert space, and the observation that $\eta_n \equiv T_n\varphi_n = T\tilde{\varphi}_n$ for $n = 1, 2, \ldots$, where we recall that $T : \mathcal{X} \to L^2(\Omega)$ is a bounded linear operator. □

Our next lemma examines properties of the quantities η and $\tilde{\varphi}$ defined in Lemma 3.1.

Lemma 3.2 *Assume $\{\delta_n\}$, $\{\mathbf{r}_n\}$, $\{\alpha_n\}$ and $\{f^{\delta_n}\}$ are given satisfying the conditions of Lemma 3.1, where we additionally assume that $\delta_n \to 0$, $\|\mathbf{r}_n\|_\infty \to 0$, and $\|\alpha_n\|_\infty \to 0$, as $n \to \infty$. Then for η and $\tilde{\varphi}$ given by Lemma 3.1, it follows that η is a solution of $Au = f$ and $\tilde{\varphi}$ is a solution of $\tilde{A}\psi = f$, for $\tilde{A} \in \mathcal{L}(\mathcal{X}, L^2(\Omega))$ defined by $\tilde{A} = AT$.*

Proof: For $n = 1, 2, \ldots$, define the operator $\bar{A}_n \in \mathcal{L}(L^2(\Omega), \mathcal{X}_n)$ via

$$\bar{A}_n x(t)(\rho) = \begin{cases} Ax(t), & \text{a.a. } \rho \in I[\mathbf{r}_n(t)], \quad t \in \Omega, \\ 0, & \text{a.a. } \rho \in I[\mathbf{r}_n(t)]^{\complement}, \quad t \in \Omega, \end{cases}$$

for A the original integral operator defined in (1.2) and $x \in L^2(\Omega)$. Then, for η given by Lemma 3.1, we have

$$\begin{aligned}|A\eta - f|^2 &= \frac{1}{2}\int_\Omega \sum_{i=-1,1} \frac{1}{(r_n)_i(t)} \int_{I_i[\mathbf{r}_n(t)]} |A\eta(t) - f(t)|^2 \, d\rho \, dt \\ &= \frac{1}{2}\int_\Omega \sum_{i=-1,1} \frac{1}{(r_n)_i(t)} \int_{I_i[\mathbf{r}_n(t)]} |\bar{A}_n\eta(t)(\rho) - \bar{F}_n(t)(\rho)|^2 \, d\rho \, dt \\ &= \frac{1}{2}\|\bar{A}_n\eta - \bar{F}_n\|^2_{\mathbf{r}_n},\end{aligned}$$

for $\bar{F}_n \equiv \bar{F}_{\mathbf{r}_n}$ defined as usual. But

$$\|\bar{A}_n\eta - \bar{F}_n\|_{\mathbf{r}_n} \leq T_1^n + T_2^n + T_3^n + T_4^n + T_5^n,$$

where, using φ_n and η_n as defined in Lemma 3.1,

$$\begin{aligned}T_1^n &= \|A_n\varphi_n + B_nT_n\varphi_n - F_n^\delta\|_{\mathbf{r}_n}, \\ T_2^n &= \|A_n\varphi_n\|_{\mathbf{r}_n}, \\ T_3^n &= \|\bar{A}_n\eta - B_nT_n\varphi_n\|_{\mathbf{r}_n}, \\ T_4^n &= \|F_n^\delta - F_n\|_{\mathbf{r}_n}, \\ T_5^n &= \|F_n - \bar{F}_n\|_{\mathbf{r}_n}.\end{aligned}$$

In the remainder of the proof we will show that $T_i^n \to 0$ as $n \to \infty$, $i = 1, \ldots, 5$, so that $|A\eta - f| = 0$, or η is a solution of $Au = f$. The fact that $\tilde{\varphi}$ solves $\tilde{A}\psi = f$ follows immediately since $\tilde{A}\tilde{\varphi} = AT\tilde{\varphi} = A\eta = f$.

We first note that

$$(T_1^n)^2 = \|C_n\varphi_n - F_n^\delta\|^2_{\mathbf{r}_n},$$

so that Lemma 2.1 may be used along with $\delta_n \to 0$, $\|\mathbf{r}_n\|_\infty \to 0$, $\|\alpha_n\|_\infty \to 0$, to show $T_1^n \to 0$ as $n \to \infty$. Also,

$$T_2^n \leq \|A_n\|\|\varphi_n\|_{\mathbf{r}_n}$$

where $\|\varphi_n\|_{\mathbf{r}_n} = \|E_n\varphi_n\|_{\mathcal{X}} = \|\tilde{\varphi}_n\|_{\mathcal{X}}$ is bounded (using the results in the proof of Lemma 3.1) and, using (2.4), $\|A_n\| \leq C\|k\|_\infty\|\mathbf{r}_n\|_\infty \to 0$ as $n \to \infty$. Further, for $\eta_n = T_n\varphi_n \in L^2(\Omega)$,

$$\begin{aligned}T_3^n &= \|\bar{A}_n\eta - B_n\eta_n\|_{\mathbf{r}_n} \\ &\leq \|\bar{A}_n(\eta - \eta_n)\|_{\mathbf{r}_n} + \|(\bar{A}_n - B_n)\eta_n\|_{\mathbf{r}_n}\end{aligned} \tag{3.2}$$

where it is easily seen that

$$\|\bar{A}_n(\eta - \eta_n)\|^2_{\mathbf{r}_n} = 2|A(\eta - \eta_n)|^2$$

and $|A(\eta - \eta_n)| \to 0$ as $n \to \infty$ from compactness of A on $L^2(\Omega)$ and the weak convergence of $\eta_n \rightharpoonup \eta$ in $L^2(\Omega)$, from Lemma 3.1. In bounding the second term in

(3.2) we note that for a.a. $\rho \in I[\mathbf{r}_n(t)]$, $t \in \Omega$,

$$\begin{aligned}\left[(\bar{A}_n - B_n)\eta_n(t)(\rho)\right]^2 &= \left[\int_0^1 k(t,s)\eta_n(s)\,ds - \int_0^{t-(r_n)_{-1}(t)} k(t+\rho,s)\eta_n(s)\,ds \right.\\ &\qquad \left. - \int_{t+(r_n)_1(t)}^1 k(t+\rho,s)\eta_n(s)\,ds\right]^2 \\ &= \left[\int_0^1 [k(t,s)-k(t+\rho,s)]\eta_n(s)\,ds + \int_{t-(r_n)_{-1}(t)}^{t+(r_n)_1(t)} k(t+\rho,s)\eta_n(s)\,ds\right]^2 \\ &\le 2\int_0^1 [k(t,s)-k(t+\rho,s)]^2\,ds \int_0^1 \eta_n^2(s)\,ds \\ &\qquad + 2\int_{t-(r_n)_{-1}(t)}^{t+(r_n)_1(t)} k^2(t+\rho,s)\,ds \int_{t-(r_n)_{-1}(t)}^{t+(r_n)_1(t)} \eta_n^2(s)\,ds \\ &\le 4|\eta_n|^2\left(\rho^{2\mu_k}|L_k|^2 + \|\mathbf{r}_n\|_\infty \|k\|_\infty^2\right),\end{aligned}$$

where we have used (1.3) in the first term above. Thus

$$\begin{aligned}\|(\bar{A}_n - B_n)\eta_n\|_{\mathbf{r}_n}^2 &\le 4|\eta_n|^2 \int_\Omega \sum_{i=-1,1} \frac{1}{(r_n)_i(t)} \int_{I_i[\mathbf{r}_n(t)]} \left(\rho^{2\mu_k}|L_k|^2 + \|\mathbf{r}_n\|_\infty \|k\|_\infty^2\right) d\rho\,dt \\ &\le 8|\eta_n|^2 \int_\Omega \sum_{i=-1,1}\left(\frac{|L_k|^2}{2\mu_k+1}(r_n)_i^{2\mu_k}(t) + \|\mathbf{r}_n\|_\infty^2\|k\|_\infty^2\right) dt.\end{aligned}$$

Since $|\eta_n| = |T\tilde{\varphi}_n| \le \|T\|\|\tilde{\varphi}_n\|_{\mathcal{X}}$, where $\|\tilde{\varphi}_n\|_{\mathcal{X}}$ is bounded as $n \to \infty$, it follows that $\|(\bar{A}_n - B_n)\eta_n\|_{\mathbf{r}_n}^2 \to 0$ and $T_3^n \to 0$ as $n \to \infty$.

Finally,

$$(T_4^n)^2 \le 2\|f^{\delta_n} - f\|_\infty^2 \le 2\delta_n^2,$$

and

$$\begin{aligned}(T_5^n)^2 &= \int_\Omega \sum_{i=-1,1} \frac{1}{(r_n)_i(t)} \int_{I_i[\mathbf{r}_n(t)]} \left[f(t+\rho) - f(t)\right]^2\,d\rho\,dt \\ &= \int_\Omega \sum_{i=-1,1} \frac{1}{(r_n)_i(t)} \int_{I_i[\mathbf{r}_n(t)]} \left[\int_0^1 [k(t+\rho,s)-k(t,s)]\,\bar{u}(s)\,ds\right]^2 d\rho\,dt \\ &\le |L_k|^2|\bar{u}|^2 \int_\Omega \sum_{i=-1,1} \frac{1}{(r_n)_i(t)} \int_{I_i[\mathbf{r}_n(t)]} \rho^{2\mu_k}\,d\rho\,dt \\ &\le \frac{|L_k|^2|\bar{u}|^2}{2\mu_k+1} 2\|\mathbf{r}_n\|_\infty^{2\mu_k},\end{aligned}$$

so that T_4^n, $T_5^n \to 0$ as $n \to \infty$. This completes the proof of the lemma. □

We conclude the section with the proof of Theorem 3.1.

Proof of Theorem 3.1: From Lemmas 3.1 and 3.2 we have that $\eta_n \rightharpoonup \eta$, for $\eta \in L^2(\Omega)$ a solution of $Au = f$, and $\tilde{\varphi}_n \equiv E_n\varphi_n \rightharpoonup \tilde{\varphi}$ for $\tilde{\varphi} \in \mathcal{X}$ a solution of $\tilde{A}\psi \equiv AT\psi = f$, $\psi \in \mathcal{X}$. In both cases the convergence is subsequential (the indices have been relabeled).

We will begin by proving the claim that $\tilde{\varphi} = \tilde{U}$ where $\tilde{U} \in \mathcal{X}$ is given by

$$\tilde{U}(t)(\rho) = \frac{\bar{u}(t)\gamma_\ell(\rho)}{|\gamma_\ell|_X^2}, \quad \text{a.a. } \rho \in [-1,1],\ t \in \Omega. \tag{3.3}$$

(The quantity $\gamma_\ell \in X$ is defined in (2.7).) The proof of this claim will take several steps.

First of all, $T\tilde{U}(t) = \ell\left(\bar{u}(t)\gamma_\ell(\cdot)/|\gamma_\ell|_X^2\right) = \bar{u}(t)$ for a.a. $t \in [0,1]$, so it follows that that $\tilde{U}$ solves $\tilde{A}\psi = AT\psi = f$. In fact, we will show that $\tilde{U} \in (\ker\tilde{A})^\perp \subseteq \mathcal{X}$, i.e., that $\tilde{U}$ is the minimum norm solution of $\tilde{A}\psi = f$ [16]. Indeed, let $\tilde{\psi} \in \ker\tilde{A}$. Then

$$\begin{aligned}
\langle \tilde{U}, \tilde{\psi}\rangle_{\mathcal{X}} &= \frac{1}{|\gamma_\ell|_X^2}\int_\Omega \bar{u}(t)\int_{-1}^1 \gamma_\ell(\rho)\tilde{\psi}(t)(\rho)\,d\rho\,dt \\
&= \frac{1}{|\gamma_\ell|_X^2}\int_\Omega \bar{u}(t)\ell(\tilde{\psi}(t))\,dt \\
&= \frac{1}{|\gamma_\ell|_X^2}\langle \bar{u}, T\tilde{\psi}\rangle_{L^2(\Omega)}.
\end{aligned}$$

But $\tilde{\psi} \in \ker\tilde{A}$ implies that $T\tilde{\psi} \in \ker A$. Since $\bar{u}$ is a minimum norm solution of $Au = f$ and thus $\bar{u} \in (\ker A)^\perp$, it follows that $\langle\tilde{U}, \tilde{\psi}\rangle_{\mathcal{X}} = 0$, or $\tilde{U} \in (\ker\tilde{A})^\perp$.

We next show that $\|\tilde{\varphi}\|_{\mathcal{X}} \le \|\tilde{U}\|_{\mathcal{X}}$. Indeed, since $\tilde{\varphi}_n = E_n\varphi_n$, it follows that

$$\begin{aligned}
\|\tilde{\varphi}_n\|_{\mathcal{X}}^2 &= \|\varphi_n\|_{\mathbf{r}_n}^2 \\
&\le \frac{1}{\alpha_{n,\min}}\left[\|C_n\varphi_n - F_n^\delta\|_{\mathbf{r}_n}^2 + \|\varphi_n\|_{\mathbf{r}_n,\alpha_n}^2\right] \\
&\le \frac{1}{\alpha_{n,\min}}\left[\|C_nE_n^{-1}\tilde{U} - F_n^\delta\|_{\mathbf{r}_n}^2 + \|E_n^{-1}\tilde{U}\|_{\mathbf{r}_n,\alpha_n}^2\right]
\end{aligned}$$

where $E_n^{-1}\tilde{U} \in \mathcal{X}_n$ with $\|E_n^{-1}\tilde{U}\|_{\mathbf{r}_n,\alpha_n}^2 \le \|\alpha_n\|_\infty\|E_n^{-1}\tilde{U}\|_{\mathbf{r}_n}^2 = \|\alpha_n\|_\infty\|\tilde{U}\|_{\mathcal{X}}^2 = \|\alpha_n\|_\infty|\bar{u}|^2/|\gamma_\ell|_X^2$. Further, since $T_nE_n^{-1}\tilde{U} = T\tilde{U} = \bar{u}$,

$$\begin{aligned}
\|C_nE_n^{-1}\tilde{U} - F_n^\delta\|_{\mathbf{r}_n}^2 &= \|A_nE_n^{-1}\tilde{U} + B_n\bar{u} - F_n^\delta\|_{\mathbf{r}_n}^2 \\
&\le 4\|A_n(E_n^{-1}\tilde{U} - U_n)\|_{\mathbf{r}_n}^2 + 4\|A_nU_n + B_n\bar{u} - F_n\|_{\mathbf{r}_n}^2 + 2\|F_n - F_n^\delta\|_{\mathbf{r}_n}^2
\end{aligned}$$

where $\|A_nU_n + B_n\bar{u} - F_n\|_{\mathbf{r}_n}^2 = 0$ (see the proof of Lemma 2.1), $\|F_n - F_n^\delta\|_{\mathbf{r}_n}^2 \le 2\delta_n^2$, and

$$\begin{aligned}
\|A_n(E_n^{-1}\tilde{U} - U_n)\|_{\mathbf{r}_n}^2 &\le 2\|A_n\|^2\left(\|E_n^{-1}\tilde{U}\|_{\mathbf{r}_n}^2 + \|U_n\|_{\mathbf{r}_n}^2\right) \\
&\le 2C^2\|k\|_\infty^2\|\mathbf{r}_n\|_\infty^2\left(\frac{|\bar{u}|^2}{|\gamma_\ell|_X^2} + 2\|\bar{u}\|_\infty^2\right).
\end{aligned}$$

Therefore,

$$\|\tilde{\varphi}_n\|_{\mathcal{X}}^2 \le 8C^2\|k\|_\infty^2\frac{\|\mathbf{r}_n\|_\infty^2}{\alpha_{n,\min}}\left(\frac{|\bar{u}|^2}{|\gamma_\ell|_X^2} + 2\|\bar{u}\|_\infty^2\right) + 4\frac{\delta_n^2}{\alpha_{n,\min}} + \frac{\|\alpha_n\|_\infty}{\alpha_{n,\min}}\frac{|\bar{u}|^2}{|\gamma_\ell|_X^2}.$$

It follows that

$$\|\tilde{\varphi}\|^2_{\mathcal{X}} \le \liminf_{n\to\infty} \|\tilde{\varphi}_n\|^2_{\mathcal{X}} \tag{3.4}$$

$$\le \limsup_{n\to\infty} \left\{ 8C^2\|k\|^2_\infty \frac{\|\mathbf{r}_n\|^2_\infty}{\delta_n^2} \frac{\delta_n^2}{\alpha_{n,\min}} \left(\frac{|\bar{u}|^2}{|\gamma_\ell|^2_X} + 2\|\bar{u}\|^2_\infty \right) + 4\frac{\delta_n^2}{\alpha_{n,\min}} + \frac{\|\alpha_n\|_\infty}{\alpha_{n,\min}} \frac{|\bar{u}|^2}{|\gamma_\ell|^2_X} \right\} \tag{3.5}$$

$$= |\bar{u}|^2/|\gamma_\ell|^2_X \tag{3.6}$$

$$= \|\tilde{U}\|^2_{\mathcal{X}}, \tag{3.7}$$

under the assumptions of the theorem, so that $\|\tilde{\varphi}\|_{\mathcal{X}} \le \|\tilde{U}\|_{\mathcal{X}}$. By uniqueness of the minimum norm solution of $\tilde{A}\psi = f$, it follows that $\tilde{\varphi} = \tilde{U}$.

All the inequalities in (3.4)–(3.7) must then be equalities, so it follows that

$$\|\tilde{\varphi}_n\|_{\mathcal{X}} \to \|\tilde{\varphi}\|_{\mathcal{X}} \text{ as } n \to \infty.$$

This fact combined with the weak convergence $\tilde{\varphi}_n \rightharpoonup \tilde{\varphi}$ in the Hilbert space $\mathcal{X}$ guarantees the strong convergence in $\mathcal{X}$ of $\tilde{\varphi}_n \to \tilde{\varphi}$ as $n \to \infty$. Thus

$$\eta_n = T\tilde{\varphi}_n \to T\tilde{\varphi} = T\tilde{U} = \bar{u}, \text{ as } n \to \infty,$$

so that $\eta = \bar{u}$.

Finally, all convergence statements above are subsequential convergence. A standard argument (see, e.g., #11, p. 37 of [18]) may be used to extend these results to full sequential convergence. □

4. Numerical Implementation

4.1. The discrete local regularization problem

We describe here a practical implementation of the local regularization method studied in the previous sections. Let $\Delta t = 1/N$ for fixed $N = 1, 2, \ldots$, and let $t_j = j\Delta t$, $j = 0, 1, \ldots, N$, $\rho_\ell = \ell\Delta t$, $\ell = -N, -N+1, \ldots, -1, 0, 1, \ldots, N$. For $\ell = -N+1, -N+2, \ldots, 0, \ldots, N$, $t \in [0, 1]$, we will let $\chi_\ell(t) \equiv \chi_{(\rho_{\ell-1}, \rho_\ell]}(t)$, denote the usual characteristic function; i.e., $\chi_\ell(t) = 1$, $t \in (\rho_{\ell-1}, \rho_\ell]$, and $\chi_\ell(t) = 0$ otherwise.

As a discrete approximation of the regularization parameter α, we will let

$$\alpha(t) = \sum_{j=1}^{N} \alpha_j \, \chi_j(t), \quad t \in [0, 1], \tag{4.1}$$

where $\alpha_j > 0$, $j = 1, \ldots, N$, and corresponding to the regularization parameters r_i, $i = \pm 1$, we will define the discretizations

$$r_i(t) = \sum_{j=1}^{N} (R_{i,j}\, \Delta t)\ \chi_j(t), \quad t \in [0, 1],\ i = \pm 1, \tag{4.2}$$

where, for $i = \pm 1$ and $j = 1, \ldots, N$, the $R_{i,j}$ are non-negative integers. Note that by definition $r_i(0) = 0$ for $i = \pm 1$. We reformulate the conditions on r_i given in Section 2.1 to make them more appropriate in the setting of a discrete approximation. In this case we require

$$\begin{aligned} r_i(t_j) &\in (0,1), \quad j = \quad 1, \ldots, N-1, \quad i = \pm 1, \\ t_j - r_{-1}(t_j) &\in [0,1], \quad j = 0, 1, \ldots, N, \\ t_j + r_1(t_j) &\in [0,1], \quad j = 0, 1, \ldots, N. \end{aligned} \tag{4.3}$$

The smoothness conditions on r_i given in Section 2.1 are not enforced in our construction given above. Indeed we could have ensured $r_i \in C([0,1]) \cap C^1(0,1)$ by taking a C^1 piecewise polynomial as the discrete form of r_i; however this adds significantly to the complexity of the algorithm we describe below. We will instead aim for simplicity and adjust certain definitions to accommodate the less regular parameters $r_i(t)$ that we use here. (A more complete algorithm development and discrete convergence theory will be given elsewhere.) Fortunately, the discrete algorithms appear to perform quite well in numerical tests even without the stronger hypotheses on r_i needed in earlier sections describing the convergence theory in the continuous setting.

Returning to the conditions on r_i, $i = \pm 1$, given in (4.3) above, we note that these conditions lead to requirements on the $R_{i,j}$, $i = \pm 1$, $j = 1, \ldots, N$, namely

$$\begin{aligned} 1 \le R_{-1,j} &\le j, \quad j = 1, \ldots, N; \\ 1 \le R_{1,j} &\le N - j, \quad j = 1, \ldots, N-1; \qquad R_{1,N} = 0. \end{aligned} \tag{4.4}$$

Strictly speaking one only requires $0 \le R_{-1,N} \le N$, but because $R_{1,N} = 0$, the choice of $R_{-1,N} = 0$ means that there is no regularization whatsoever around $t_N = 1$, a situation that we choose to avoid here.

We note that, for $m = 1, 2, \ldots, N-1$,

$$\begin{aligned} I[\mathbf{r}(t_m)] &= (-R_{-1,m}\Delta t, R_{1,m}\Delta t] \\ &= \bigcup_{\ell \in \mathbf{i}[m]} (\rho_{\ell-1}, \rho_\ell] \end{aligned}$$

where $\mathbf{i}[m] = \{-R_{-1,m}+1, \ldots, 0, \ldots, R_{1,m}\}$. Similarly, for $m = 1, 2, \ldots, N-1$,

$$\begin{aligned} I_j[\mathbf{r}(t_m)] &= \bigcup_{\ell \in \mathbf{i}_j[m]} (\rho_{\ell-1}, \rho_\ell], \quad j = \pm 1, \\ I[\mathbf{r}(t_m)]^{\complement} &= \bigcup_{\ell \in \mathbf{i}[m]^{\complement}} (\rho_{\ell-1}, \rho_\ell], \end{aligned}$$

where

$$\begin{aligned} \mathbf{i}_{-1}[m] &= \{-R_{-1,m}+1, \ldots, 0\} \\ \mathbf{i}_1[m] &= \{1, \ldots, R_{1,m}\} \\ \mathbf{i}[m]^{\complement} &= \{-N+1, \ldots, -R_{-1,m},\ R_{1,m}+1, \ldots, N\}. \end{aligned}$$

Since $r_{-1}(t_N) \neq 0$, the intervals $I[\mathbf{r}(t_N)]$, $I_{-1}[\mathbf{r}(t_N)]$, and $I[\mathbf{r}(t_N)]^{\complement}$ are well-defined so we will make the additional definitions

$$\begin{aligned} \mathbf{i}[N] &\equiv \{-R_{-1,N}+1,\ldots,0\} \\ \mathbf{i}_{-1}[N] &\equiv \mathbf{i}[N] \\ \mathbf{i}[N]^{\complement} &\equiv \{-N+1,\ldots,-R_{-1,N},\, 1,\ldots,N\}, \end{aligned}$$

where we have used the fact that $r_1(t_N) = 0$.

Letting $\mathrm{len}(m)$ denote the number of elements in $\mathbf{i}[m]$, we have

$$\begin{aligned} &\mathrm{len}(m) = R_{-1,m} + R_{1,m}, \quad m = 1,\ldots,N-1, \\ &\mathrm{len}(N) = R_{-1,N}. \end{aligned}$$

We will make the standing hypothesis that $\mathrm{len}(m) < N$ for all $m = 1,\ldots,N$, so that the regularization is truly "local".

Given the discretized parameters r_i and α, we define the *discrete regularization problem* to be that of finding a solution of the minimization problem

$$\min_{\varphi^N \in \mathcal{X}_{\mathbf{r}}^N} \left\{ \| \, C_{\mathbf{r}} \varphi^N - F_{\mathbf{r}}^{N,\delta} \, \|_{N,\mathbf{r}}^2 + \| \varphi^N \|_{N,\mathbf{r},\alpha}^2 \right\}. \tag{4.5}$$

where $\mathcal{X}_{\mathbf{r}}^N \subset \mathcal{X}_{\mathbf{r}}$ is given by

$$\mathcal{X}_{\mathbf{r}}^N = \left\{ \varphi \, : \, \varphi(t)(\rho) = \sum_{j=1}^{N} \sum_{\ell \in \mathbf{i}[j]} c_{j\ell} \, \chi_j(t) \chi_\ell(\rho), \quad \text{a.a. } \rho \in [-1,1],\, t \in [0,1],\; c_{j\ell} \in \mathbb{R} \right\}.$$

In addition, $F_{\mathbf{r}}^{N,\delta} \in \mathcal{X}_{\mathbf{r}}^N$ is given by

$$F_{\mathbf{r}}^{N,\delta}(t)(\rho) \equiv \sum_{j=1}^{N} \sum_{\ell \in \mathbf{i}[j]} f^\delta(t_j + \rho_\ell) \, \chi_j(t) \chi_\ell(\rho), \quad \text{a.a. } \rho \in [-1,1],\, t \in [0,1],$$

while $\| \cdot \|_{N,\mathbf{r}}$ and $\| \cdot \|_{N,\mathbf{r},\alpha}$ are discrete forms of the integral norms $\| \cdot \|_{\mathbf{r}}$ and $\| \cdot \|_{\mathbf{r},\alpha}$, respectively, which we define below. For smooth $g \in \mathcal{X}_{\mathbf{r}}$ we may approximate for $i = \pm 1$,

$$\begin{aligned} \int_{I_i[\mathbf{r}(t_m)]} |g(t_m)(\rho)|^2 \, d\rho &\doteq \int_{I_i[\mathbf{r}(t_m)]} \sum_{n \in \mathbf{i}_i[m]} |g(t_m)(\rho_n)|^2 \chi_n(\rho) \, d\rho \\ &= \sum_{n \in \mathbf{i}_i[m]} |g(t_m)(\rho_n)|^2 \Delta t. \end{aligned}$$

Making a similar approximation when integrating with respect to the t variable, we define

$$\begin{aligned} \|g\|_{N,\mathbf{r},\alpha}^2 &= \int_\Omega \sum_{i=\pm 1} \left[\sum_{\substack{m=1 \\ r_i(t_m) \neq 0}}^{N} \frac{\alpha(t_m)}{r_i(t_m)} \left(\int_{I_i[\mathbf{r}(t_m)]} \sum_{n=1}^{N} |g(t_m)(\rho_n)|^2 \chi_n(\rho) \, d\rho \right) \right] \chi_m(t) \, dt \\ &= \Delta t \sum_{m=1}^{N-1} \sum_{i=\pm 1} \frac{\alpha_m}{R_{i,m}} \sum_{n \in \mathbf{i}_i[m]} |g(t_m)(\rho_n)|^2 + \Delta t \frac{\alpha_N}{R_{-1,N}} \sum_{n \in \mathbf{i}_{-1}[N]} |g(t_N)(\rho_n)|^2, \end{aligned}$$

where the form of the last term in $\|\cdot\|^2_{N,\mathbf{r},\alpha}$ is necessitated by the fact that $R_{1,N} = 0$. The norm $\|\cdot\|_{N,\mathbf{r}}$ is defined similarly (using $\alpha_m = 1$ in the above, for all $m = 1, \ldots, N$).

We let $C_\mathbf{r} = A_\mathbf{r} + B_\mathbf{r} T_\mathbf{r}$ as usual, making use here of the definition of $T_\mathbf{r}$ as given in Example 2.2, with $c \equiv \Delta t$ in that example. Thus, for $\varphi \in \mathcal{X}^N_\mathbf{r}$ and $t \in (0, 1]$,

$$\begin{aligned} T_\mathbf{r}\varphi(t) &= \frac{1}{\Delta t\, r_{-1}(t)} \int_{-\Delta t \cdot r_{-1}(t)}^{0} \sum_{j=1}^{N} \sum_{\ell \in \mathbf{i}[j]} c_{j\ell}\, \chi_j(t)\chi_\ell(\rho)\, d\rho \\ &= \sum_{j=1}^{N} c_{j0}\chi_j(t), \end{aligned}$$

so that in the discrete case,

$$T_\mathbf{r}\varphi(t) = \varphi(t)(0).$$

Fix $m = 1, 2, \ldots, N$. For $\varphi \in \mathcal{X}^N_\mathbf{r}$ we have $A_\mathbf{r}\varphi(t_m)(\rho_n) = 0$ when $n \in \mathbf{i}[m]^\complement$, while for $n \in \mathbf{i}[m]$,

$$\begin{aligned} A_\mathbf{r}\varphi(t_m)(\rho_n) &= \int_{-r_{-1}(t_m)}^{r_1(t_m)} k(t_m + \rho_n, t_m + s)\varphi(t_m)(s)\, ds \\ &= \sum_{j=1}^{N} \sum_{\ell \in \mathbf{i}[j]} c_{j\ell}\, \chi_j(t_m) \int_{-R_{-1,m}\Delta t}^{R_{1,m}\Delta t} k(t_m + \rho_n, t_m + s)\, \chi_\ell(s)\, ds, \\ &= \sum_{\ell \in \mathbf{i}[m]} c_{m\ell}\, \Delta_{m+n,m+\ell} \end{aligned}$$

where

$$\Delta_{i,j} \equiv \int_{\rho_{j-1}}^{\rho_j} k(t_i, s)\, ds. \tag{4.6}$$

Thus for fixed m and $n \in \mathbf{i}[m]$,

$$A_\mathbf{r}\varphi(t_m)(\rho_n) = (\mathbf{A}_m \mathbf{c}_m)_n,$$

the entry in $\mathbf{A}_m\mathbf{c}_m \in \mathbb{R}^{\mathrm{len}(m)}$ associated with component $n \in \mathbf{i}[m]$, where $\mathbf{c}_m \in \mathbb{R}^{\mathrm{len}(m)}$ is the column vector $\mathbf{c}_m = (\hat{c}_{im})_{i \in \mathbf{i}[m]}$, $\hat{c}_{im} \equiv c_{mi}$, and $\mathbf{A}_m$ is the $\mathrm{len}(m) \times \mathrm{len}(m)$ matrix,

$$\mathbf{A}_m = \left(\Delta_{m+i\,,\,m+j}\right)_{i,j \in \mathbf{i}[m]}.$$

Similarly for $\varphi \in \mathcal{X}^N_\mathbf{r}$ and fixed $m = 1, \ldots, N$, it follows that $B_\mathbf{r}T_\mathbf{r}\varphi(t_m)(\rho_n) = 0$ for $n \in \mathbf{i}[m]^\complement$, while for $n \in \mathbf{i}[m]$,

$$\begin{aligned} &B_\mathbf{r}T_\mathbf{r}\varphi(t_m)(\rho_n) \\ &\quad = \int_0^{t_m - r_{-1}(t_m)} k(t_m + \rho_n, s) \sum_{j=1}^{N} c_{j0}\, \chi_j(s)\, ds + \int_{t_m + r_1(t_m)}^{1} k(t_m + \rho_n, s) \sum_{j=1}^{N} c_{j0}\, \chi_j(s)\, ds, \\ &\quad = \sum_{j=1}^{m - R_{-1,m}} c_{j0}\, \Delta_{m+n,j} \;+ \sum_{j=m+R_{1,m}+1}^{N} c_{j0}\, \Delta_{m+n,j}, \end{aligned}$$

where we use the convention $\sum_{j=a}^{b} = 0$ for $a > b$. Thus, for fixed m and $n \in \mathbf{i}[m]$,

$$B_{\mathbf{r}} T_{\mathbf{r}} \varphi(t_m)(\rho_n) = (\mathbf{B}_m \mathbf{c})_n,$$

the entry of $\mathbf{B}_m \mathbf{c} \in \mathbb{R}^{\mathrm{len}(m)}$ associated with the component $n \in \mathbf{i}[m]$, where

$$\mathbf{c} = (c_{10}, c_{20}, \ldots, c_{N0})^T \in \mathbb{R}^N,$$

and $\mathbf{B}_m \in \mathbb{R}^{\mathrm{len}(m) \times N}$ is a partitioned matrix with the following construction: If $R_{-1,m}$ and $R_{1,m}$ satisfy $m - R_{-1,m} \geq 1$ and $m + R_{1,m} + 1 \leq N$, then $\mathbf{B}_m = (\mathbf{B}_{-1,m} \,|\, \mathbf{0} \,|\, \mathbf{B}_m)$, where

$$\mathbf{B}_{-1,m} = \begin{pmatrix} \Delta_{m-R_{-1,m}+1\,,\,1} & \cdots & \Delta_{m-R_{-1,m}+1\,,\,m-R_{-1,m}} \\ \Delta_{m-R_{-1,m}+2\,,\,1} & \cdots & \Delta_{m-R_{-1,m}+2\,,\,m-R_{-1,m}} \\ \vdots & \vdots & \vdots \\ \Delta_{m+R_{1,m}\,,\,1} & \cdots & \Delta_{m+R_{1,m}\,,\,m-R_{-1,m}} \end{pmatrix} \in \mathbb{R}^{\mathrm{len}(m) \times (m - R_{-1,m})}$$

$$\mathbf{B}_{1,m} = \begin{pmatrix} \Delta_{m-R_{-1,m}+1\,,\,m+R_{1,m}+1} & \cdots & \Delta_{m-R_{-1,m}+1\,,\,N} \\ \Delta_{m-R_{-1,m}+2\,,\,m+R_{1,m}+1} & \cdots & \Delta_{m-R_{-1,m}+2\,,\,N} \\ \vdots & \vdots & \vdots \\ \Delta_{m+R_{1,m}\,,\,m+R_{1,m}+1} & \cdots & \Delta_{m+R_{1,m}\,,\,N} \end{pmatrix} \in \mathbb{R}^{\mathrm{len}(m) \times (N - m - R_{1,m})},$$

and $\mathbf{0} \in \mathbb{R}^{\mathrm{len}(m) \times \mathrm{len}(m)}$. If $m - R_{-1,m} = 0$ and $m + R_{1,m} + 1 \leq N$, then $\mathbf{B}_m = (\mathbf{0} \,|\, \mathbf{B}_{1,m})$ for $\mathbf{B}_{1,m}$ given above and $\mathbf{0} \in \mathbb{R}^{\mathrm{len}(m) \times \mathrm{len}(m)}$. Finally if $m - R_{-1,m} \geq 1$ and $m + R_{1,m} + 1 = N + 1$, then $\mathbf{B}_m = (\mathbf{B}_{-1,m} \,|\, \mathbf{0})$ for $\mathbf{B}_{-1,m}$ given above and $\mathbf{0} \in \mathbb{R}^{\mathrm{len}(m) \times \mathrm{len}(m)}$. (We note that all possibilities have been exhausted since otherwise would give $[0,1] \subseteq I[\mathbf{r}(t_m)]$ which is impossible under the assumption that $\mathrm{len}(m) < N$).

Also we define for $m = 1, \ldots, N$, $\mathbf{f}_m \in \mathbb{R}^{\mathrm{len}(m)}$ via

$$\mathbf{f}_m = \left(f(t_m + \rho_n)\right)_{n \in \mathbf{i}[m]} \,. \tag{4.7}$$

Thus, for $\varphi \in \mathcal{X}_{\mathbf{r}}^N$,

$$\begin{aligned} &\|A_{\mathbf{r}}\varphi + B_{\mathbf{r}} T_{\mathbf{r}} \varphi - F_{\mathbf{r}}^{N,\delta}\|_{N,\mathbf{r}}^2 + \|\varphi\|_{N,\mathbf{r},\alpha}^2 \\ &\quad = \Delta t \sum_{m=1}^{N-1} \left[\frac{1}{R_{-1,m}} \sum_{n=-R_{-1,m}+1}^{0} \left|(\mathbf{A}_m \mathbf{c}_m + \mathbf{B}_m \mathbf{c} - \mathbf{f}_m)_n\right|^2 + \frac{1}{R_{1,m}} \sum_{n=1}^{R_{1,m}} \left|(\mathbf{A}_m \mathbf{c}_m + \mathbf{B}_m \mathbf{c} - \mathbf{f}_m)_n\right|^2 \right] \\ &\qquad + \Delta t \left[\frac{1}{R_{-1,N}} \sum_{n=-R_{-1,N}+1}^{0} \left|(\mathbf{A}_N \mathbf{c}_N + \mathbf{B}_N \mathbf{c} - \mathbf{f}_N)_n\right|^2 \right] \\ &\qquad + \Delta t \sum_{m=1}^{N-1} \left[\frac{\alpha_m}{R_{-1,m}} \sum_{n=-R_{-1,m}+1}^{0} |c_{mn}|^2 + \frac{\alpha_m}{R_{1,m}} \sum_{n=1}^{R_{1,m}} |c_{mn}|^2 \right] \\ &\qquad + \Delta t \left[\frac{\alpha_N}{R_{-1,N}} \sum_{n=-R_{-1,N}+1}^{0} |c_{Nn}|^2 \right] \end{aligned}$$

$$= \Delta t \sum_{m=1}^{N} H_m(\mathbf{c}_m; \mathbf{c}),$$

where for each $m = 1, \ldots, N$,

$$H_m(\mathbf{c}_m; \mathbf{c}) = \|\mathbf{A}_m \mathbf{c}_m + \mathbf{B}_m \mathbf{c} - \mathbf{f}_m\|_{N,m,\mathbf{r}}^2 + \alpha_m \|\mathbf{c}_m\|_{N,m,\mathbf{r}}^2,$$

for $\|\cdot\|_{N,m,\mathbf{r}}$ a weighted Euclidean norm on $\mathbb{R}^{\mathrm{len}(m)}$.

We will consider a relaxation type of minimization method for solving the discrete regularization problem. The idea is to set initial values for the vectors $\mathbf{c}_1, \ldots, \mathbf{c}_N$ (e.g., $\mathbf{c}_m = \mathbf{0} \in \mathbb{R}^{R_{-1,m}+R_{1,m}}$ for $m = 1, \ldots, N$), and to update the value of each $\mathbf{c}_m$ one by one. To set some notation useful in describing the resulting algorithm, we will let $m = 1, \ldots, N$ be fixed and assume for the moment that the vectors $\mathbf{c}_j$ are known for all $j \neq m$. It will then be useful to describe the dependence of the summation $\sum_{j=1}^{N} H_j(\mathbf{c}_j, \mathbf{c})$ on the remaining unknown vector $\mathbf{c}_m$. To this end we make the definitions

$$J_m(\mathbf{c}_m) \equiv H_m(\mathbf{c}_m; \mathbf{c}), \quad \mathbf{c}_m \in \mathbb{R}^{\mathrm{len}(m)},$$

noting that form of $\mathbf{B}_m$ gives that $H_m(\mathbf{c}_m; \mathbf{c})$ depends on the vector $\mathbf{c}_m$ only through the first component of H_m (i.e., $\mathbf{B}_m\mathbf{c}$ is independent of c_{m0} since the mth column of $\mathbf{B}_m$ is zero). The remaining H_j functions ($j \neq m$) depend on $\mathbf{c}_m$ only through the component c_{m0} in $\mathbf{c}$, and this is the case for only those j such that $\mathbf{B}_j\mathbf{c}$ depends on c_{m0}. We thus make the definition

$$\hat{J}_m(c_{m0}) = \sum_{j \in I_m} H_j(\mathbf{c}_j, (c_{10}, \ldots, c_{m0}, \ldots, c_{N0})^T).$$

where $I_m = \{j \,|\, m\text{th column of } \mathbf{B}_j \text{ nonzero}\}$. Thus, if $\mathbf{c}_j$ is known for $j \neq m$, then

$$\begin{aligned} \frac{1}{\Delta t} \left\{ \|A_{\mathbf{r}}\varphi + B_{\mathbf{r}}T_{\mathbf{r}}\varphi - F_{\mathbf{r}}^{N,\delta}\|_{N,\mathbf{r}}^2 + \|\varphi\|_{N,\mathbf{r},\alpha}^2 \right\} &= \sum_{m=1}^{N} H_m(\mathbf{c}_m; \mathbf{c}) \\ &= J_m(\mathbf{c}_m) + \hat{J}_m(c_{m0}) + K_m \end{aligned}$$

where K_m is a constant independent of $\mathbf{c}_m$.

We now turn to an an iterative relaxation-type minimization algorithm for the solution of the discrete regularization problem.

Local Tikhonov Regularization Algorithm:

(1) Initialize vectors $\mathbf{c}_1$, $\mathbf{c}_2$, ..., $\mathbf{c}_N$.

(2) Do for $m = 1, \ldots, N$:

  (a) Holding the previously determined values of $\mathbf{c}_j$, $j \neq m$, fixed, find $\bar{\beta} \in \mathbb{R}^{\mathrm{len}(m)}$ solving

$$\min \left\{ J_m(\beta) + \hat{J}_m(\beta_0), \;\; \beta = (\beta_i)_{i \in \mathbf{i}[m]} \in \mathbb{R}^{\mathrm{len}(m)} \right\}. \tag{4.8}$$

  (b) Set $\mathbf{c}_m = \bar{\beta}$.

(3) Go to step (2).

Under reasonable conditions, convergence of the relaxation-type minimization algorithm is guaranteed [1], and the quantities $c_{10}, c_{20}, \ldots, c_{N0}$, in the 0^{th} component of the converged vectors $\mathbf{c}_1, \ldots, \mathbf{c}_N$ correspond to approximations for $\bar{u}(t_1), \bar{u}(t_2), \ldots, \bar{u}(t_N)$. A more precise study of convergence (for the discrete regularization method) is beyond the scope of this paper, and will be presented elsewhere.

We also consider a variation of the iteration algorithm given above, which appears to work well in numerical tests (see examples given below). In this case we replace the minimization of $J_m(\beta) + \hat{J}_m(\beta_0)$ in (4.8) (Step 2(a) above) with the minimization of $J_m(\beta) = \|\mathbf{A}_m\beta + \mathbf{B}_m\mathbf{c} - \mathbf{f}_m\|_{N,m,\mathbf{r}} + \alpha_m\|\beta\|^2_{N,m,\mathbf{r}}$ only. We note that in $J_m(\beta)$, the term $\mathbf{B}_m\mathbf{c}$ depends only on previously computed vectors $\mathbf{c}_j$, $j \neq m$, and does not depend on $\mathbf{c}_m$. Thus we do not need *a priori* initialization of all of the vectors $\mathbf{c}_1, \ldots, \mathbf{c}_N$, only the 0th components in each (namely, $c_{10}, c_{20}, \ldots c_{N0}$).

Modified **Local Tikhonov Regularization Algorithm**:

(1) Initialize the scalars c_{10}, c_{20}, ... c_{N0}.

(2) Do for $m = 1, \ldots, N$:

  (a) Holding the previously determined values of c_{j0}, $j \neq m$, fixed and using these vectors to define

$$\mathbf{d}_m \equiv (c_{10}, \ldots, c_{m-R_{-1,m},0},\, 0, \ldots, 0,\, c_{m+R_{1,m}+1,0}, \ldots, c_{N0})^T,$$

  find $\bar{\beta} \in \mathbb{R}^{\mathrm{len}(m)}$ solving

$$\min \left\{ \|\mathbf{A}_m\beta + \mathbf{B}_m\mathbf{d}_m - \mathbf{f}_m\|^2_{N,m,\mathbf{r}} + \alpha_m\|\beta\|^2_{N,m,\mathbf{r}}, \;\; \beta \in \mathbb{R}^{\mathrm{len}(m)} \right\}. \tag{4.9}$$

  (b) Set $\mathbf{c}_m = \bar{\beta}$.

(3) Go to step (2).

Convergence of this modified algorithm will be the subject of a future study (although numerical examples below indicate that it works well in tests). As before, the quantities $c_{10}, c_{20}, \ldots, c_{N0}$, in the 0^{th} component of the vectors $\mathbf{c}_1, \ldots, \mathbf{c}_N$ correspond to approximations for $\bar{u}(t_1), \bar{u}(t_2), \ldots, \bar{u}(t_N)$. See Figure 1.

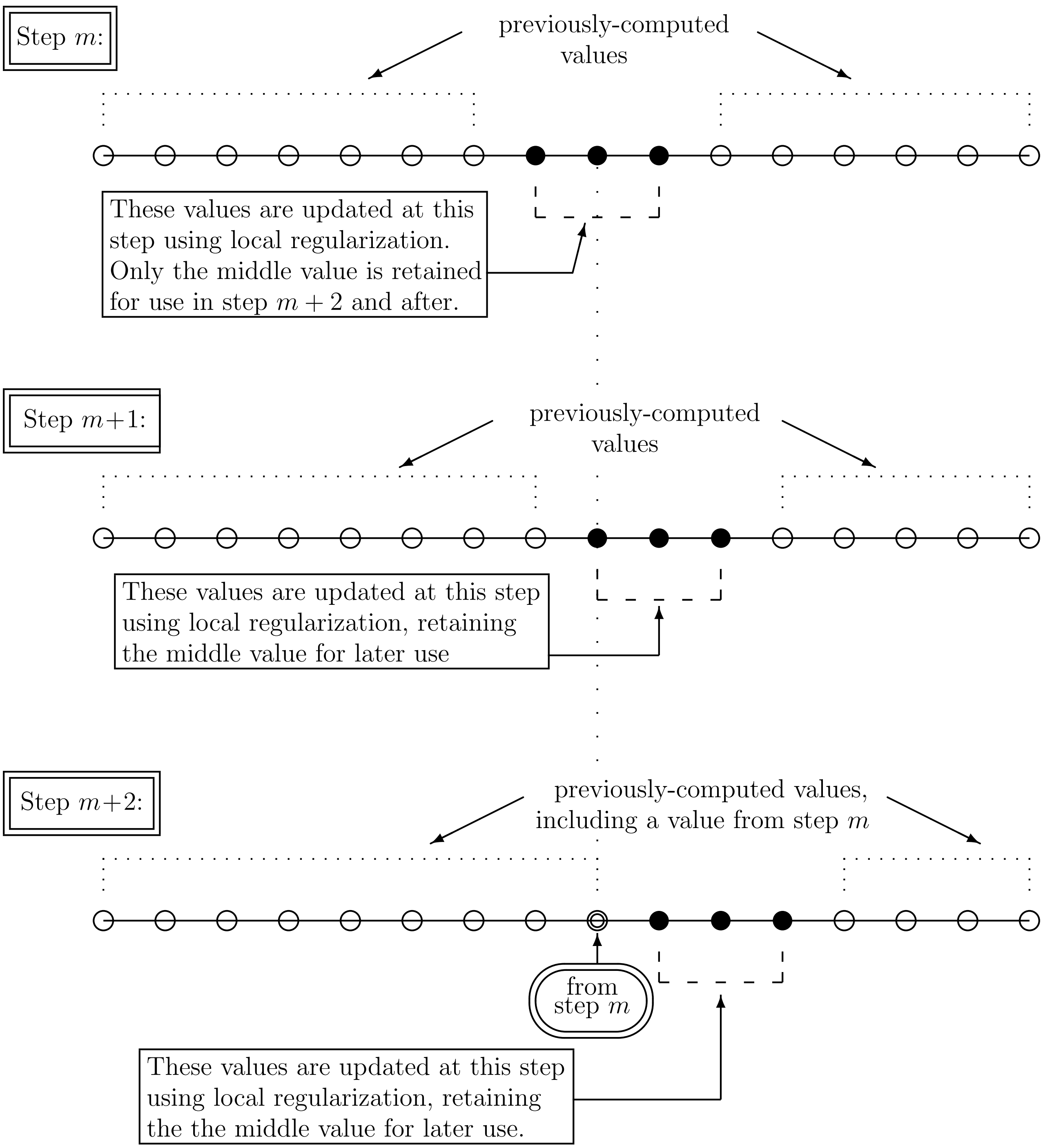


Figure 1. Example of the *Modified* Local Tikhonov Regularization Algorithm

4.2. Count of Floating Point Operations

Because we use the *modified* local Tikhonov regularization algorithm in our numerical examples below, we will briefly describe the operation count for this particular algorithm and compare the result to standard Tikhonov regularization. In what follows we assume that the quantities $\Delta_{i,j}$ in (4.6) and $f(t_m + \rho_n)$ in (4.7) are already computed. All operation counts are to highest order.

Fix a value of $m = 1, 2, \ldots N$ in the modified regularization algorithm. Then it is not difficult to show that the computation of $\mathbf{B}_m \mathbf{d}_m - \mathbf{f}_m \in \mathbb{R}^{\mathrm{len}(n)}$ requires $\mathcal{O}\left(N \mathrm{len}(m) - \mathrm{len}(m)^2\right)$ operations. Solving the Tikhonov problem (4.9) using the efficient method of [4] requires $\mathcal{O}(\mathrm{len}(m)^3)$ computations (see [5]). Thus for each full iteration of the entire local algorithm, the cost (to highest order) is

$$\mathcal{O}\left(\sum_{m=1}^{N} \left(\mathrm{len}(m)^3 + N\,\mathrm{len}(m)\right)\right)$$

operations. Letting $\bar{R} = \max_{i,j} R_{i,j}$ and letting K_{iter} denote the total number of iterations, we have that the overall cost of the modified Tikhonov regularization algorithm is

$$\mathcal{O}\left(N K_{\mathrm{iter}} \bar{R}^3 + N^2 K_{\mathrm{iter}} \bar{R}\right)$$

operations.

Thus if $\bar{R} = \mathcal{O}\left(N^{1/2}\right)$ and $K_{\mathrm{iter}} = \mathcal{O}\left(N^{1/2}\right)$ (reasonable assumptions for most of the numerical examples below), it follows that the modified local Tikhonov regularization algorithm requires $\mathcal{O}\left(N^3\right)$ operations, the same cost (to highest order) as standard Tikhonov regularization applied to the original problem (1.1) [5]. (We note that to compare the leading constants in these $\mathcal{O}\left(N^3\right)$ counts, it is necessary to have more information about the constants in the $\mathcal{O}\left(N^{1/2}\right)$ assumptions on $\bar{R}$ and K_{iter}.)

For $\bar{R}$ and/or K_{iter} smaller than $\mathcal{O}\left(N^{1/2}\right)$, the modified local algorithm is an improvement over standard Tikhonov regularization. Furthermore, if A has special structure then the cost of the modified algorithm can potentially be reduced dramatically. For instance, if the kernel k is of convolution type ($k(t, s) = \kappa(t - s)$), then the matrices $\mathbf{A}_m$ are identical for those m with the same values of $R_{i,m}$, $i = \pm 1$. (For example, in our numerical simulations below the kernel is of convolution type and the $R_{i,m}$ are the same for nearly all values of m, $i = \pm 1$.) In such a situation a good bit of pre-processing of $\mathbf{A}_m$ can be performed in advance, resulting in considerable savings in numerical costs.

4.3. Numerical Examples

In the following examples we consider an operator A of the form (1.2) where the kernel k comes from the one-dimensional image deblurring example in Section 1, with the blurring parameter γ set at $\gamma = 5$. In each example, a "true" solution $\bar{u}$ was selected *a priori* and the data f^δ used in the regularization process is a (uniformly distributed) random perturbation of $f = \mathcal{A}\bar{u}$ (computed using Mathematica), where f^δ differs from f with approximately 1% relative error. In all examples, $N = 20$.

Example 4.1 Here we use $\bar{u}(t) = 3t(1-t)$ in order to compare standard Tikhonov regularization against local Tikhonov regularization on a problem with smooth solution $\bar{u}$. In Figure 2 we show the results of standard Tikhonov regularization using various choices of the associated Tikhonov parameter α. In Figure 3, we show the "converged" solution using the Modified Local Tikhonov Regularization Algorithm given above, for various choices of local regularization parameters $r_1(\cdot)$, $r_{-1}(\cdot)$, and $\alpha(\cdot)$. In this each figure we use these parameters of the form given by (4.1) and (4.2), where, for given values of $\texttt{alpha} > 0$ and $\texttt{r} \geq 1$ (integer-valued),

$$\alpha_j = \texttt{alpha}, \quad j = 1, \ldots, N, \tag{4.10}$$

$$R_{-1,j} = \min\{\texttt{r}, j\}, \quad j = 1, \ldots, N, \tag{4.11}$$

$$R_{1,j} = \min\{\texttt{r}, N - j\}, \quad j = 1, \ldots, N. \tag{4.12}$$

That is, the regularization function $\alpha(\cdot)$ is constant valued, taking the value `alpha`, while $r_{-1}(\cdot)$ and $r_1(\cdot)$ are constant-valued (taking the value `r`) except near the boundary of Ω.

In Figure 3 we show the results for (1) `r`=1 and `alpha`=.001, (2) `r`=2 and `alpha`=.00005, and (3) `r`=4 and `alpha`=.0002. In each case we show the 20th iterate in the local regularization algorithm, although "convergence" appears to occur much earlier (by about the 5th iterate in each example).

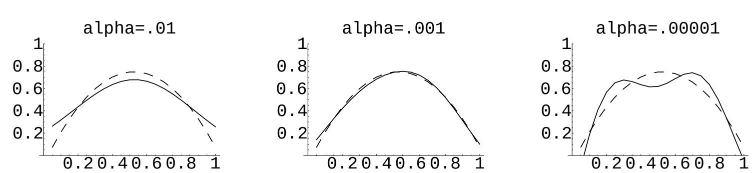


Figure 2. Results for Example 4.1 using Tikhonov regularization for various α values

Example 4.2 Our next example is of a $\bar{u}$ with both "steep" and "flat" features, something that is not handled well by standard Tikhonov regularization with its scalar

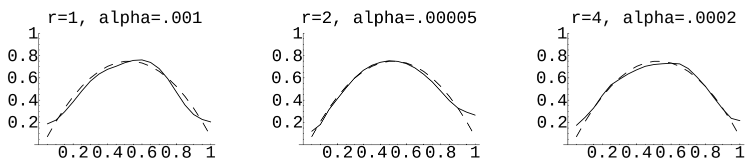


Figure 3. Results for Example 4.1 using the Modified Local Regularization Algorithm

regularization parameter. In this case,

$$\bar{u}(t) = \begin{cases} 0, & 0 \leq t \leq .3 \\ 10(t - .3) & .3 < t \leq .4 \\ 1 & .4 < t \leq .5 \\ 10(.6 - t) & .5 < t \leq .6 \\ 0 & .6 < t \leq 1 \end{cases}$$

See Figure 4 for the results of standard Tikhonov regularization using various choices of the regularization parameter. In Figure 5.2 we show that, using a constant-valued regularization function $\alpha(t) =$`alpha`, the performance of local regularization is roughly comparable to that of standard Tikhonov regularization with $\alpha = .000003$. We illustrate using $r_{-1}(\cdot)$, $r_1(\cdot)$, and $\alpha(\cdot)$ as defined in (4.10)–(4.12), with (1) `r`=1 and `alpha`= 5×10^{-7} (with "convergence" by iterate #8), and (2) `r`=2 and `alpha`= 10^{-5} (with "convergence" by iterate #3).

In Figure 5.2, we repeat the same choices of `r` as above, but now use a variable $\alpha(t)$.

5. The Volterra problem

The Volterra problem is a special case of the problem given in (1.1) where $k(t, s) = 0$ for $0 \leq t < s \leq 1$. In this case, the operator A becomes

$$Au(t) = \int_0^t k(t, s)u(s)\, ds, \quad \text{a.a. } t \in \Omega, \tag{5.1}$$

but all the theory developed in Sections 2–3 still applies. Thus from Theorem 3.1 we are guaranteed convergence of an *iterative*-type numerical method of local regularization for the Volterra problem under very general conditions on the kernel k. This is in contrast to results found in [2, 9, 10, 11, 14, 15, 17] for the *sequential* local regularization of Volterra problems. Indeed, despite the fact that sequential methods are easier to implement and result in very fast numerical schemes, the convergence theory for such methods is at present limited to *mildly smoothing* operators A (i.e., those A given by (5.1) with further restrictions on the kernel k along the line $t = s$; see, e.g., [11, 17]).

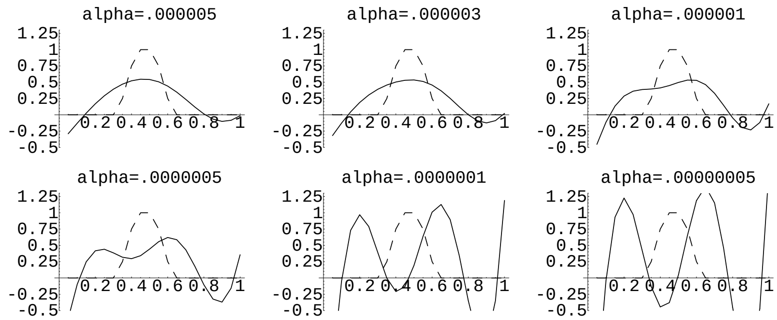
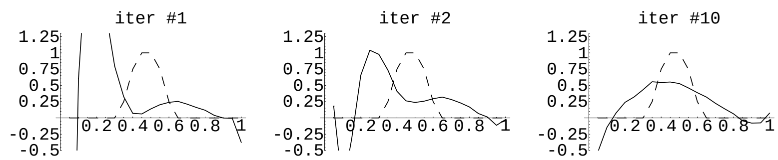
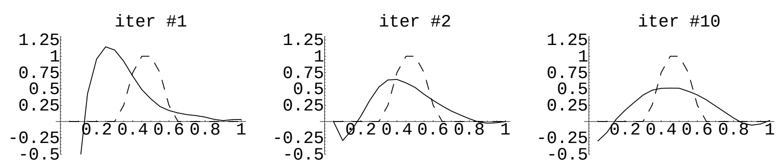


Figure 4. Results from *standard* Tikhonov regularization for various α values



`r`=1: *Iterates #1, #2, and #10 for this case, using* `alpha`$= 5 \times 10^{-7}$.

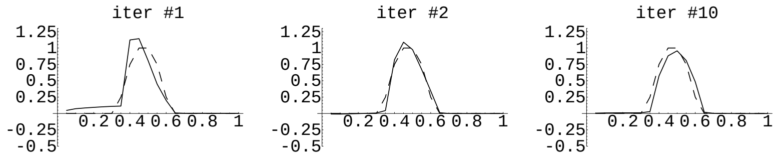
"Convergence" appears to have occurred by iterate #8, with only slight changes in iterates after #5.



`r`=2: *Iterates #1, #2, and #10 for this case, with using* `alpha`$= 1 \times 10^{-5}$.

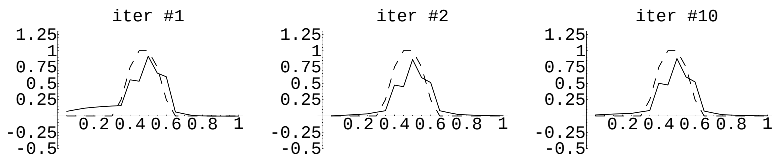
"Convergence" appears to have occurred by iterate #3.

Figure 5. Example 4.2: Local regularization for various choices of constant-valued regularization functions r_{-1}, r_1 (constant, except near the boundary of Ω), and α. Compare with Figure 5.2, where a variable $\alpha = \alpha(t)$ was used for this same example.



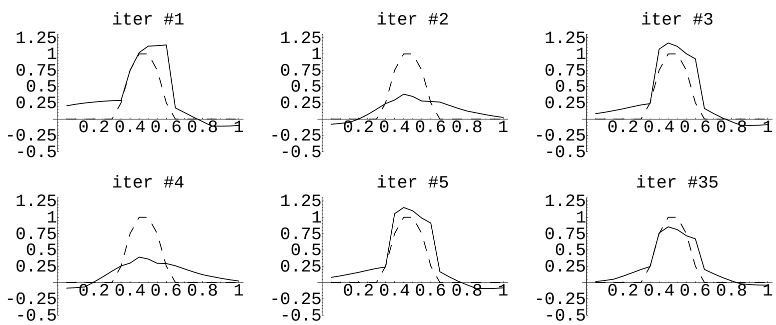
r=1: *Iterates #1, 2, and 10 for this case, using variable α.*

"Convergence" appears to have occurred by iterate #3.



r=2: *Iterates #1, 2, and 10 for this case, using variable α.*

"Convergence" appears to have occurred by iterate #2.



r=4: *Iterates #1–5 and #35 for this case, using variable α.*

"Convergence" apparently occurs at iterate #35, with only slight changes in iterates #21–35.

Figure 6. Local Tikhonov regularization for various choices of the (constant-valued) local regularization parameters r_{-1}, r_1, and t-dependent $\alpha = \alpha(t)$.

The theory developed in Sections 2–3 can be further simplified in the case of the Volterra operator. In this particular case it makes more sense to replace the local regularization intervals $(-r_{-1}(t), r_1(t)]$ by the half-intervals $(0, r_1(t)]$ due to the fact that the problem $Au = f$ is now a causal problem. That is, the value of the true solution $\bar{u}$ at time t has no impact on the data $f(t+\rho)$ for $\rho \in (-r_{-1}(t), 0)$, thus there is no reason to use data on this earlier interval to estimate $\bar{u}$ at time t. This simplification in the Volterra case results in the following revised definitions of spaces, norms, and operators needed in Sections 2–3:

$$
\begin{aligned}
&X = L^2(0,1); \quad \mathcal{X} = L^2(\Omega; X);\\
&\mathbf{r}(\cdot) = r_1(\cdot); \quad I[\mathbf{r}(t)] = (0, r_1(t)]; \quad I[\mathbf{r}(t)]^{\complement} = (r_1(t), 1];\\
&\|\varphi\|^2_{\mathbf{r},\alpha} = \int_\Omega \alpha(t) \int_{I[\mathbf{r}(t)]} |\varphi(t)(\rho)|^2 \, d\rho \, dt, \quad \varphi \in \mathcal{X}_{\mathbf{r}}\\
&E_{\mathbf{r}}\varphi(t)(\rho) = \varphi(t)(\rho\, r_1(t)), \quad \text{a.a. } \rho \in (0,1], t \in \Omega, \ \ \varphi \in \mathcal{X}_{\mathbf{r}}.
\end{aligned}
$$

The definitions of $A_{\mathbf{r}}$, $B_{\mathbf{r}}$, ℓ, $T_{\mathbf{r}}$, $C_{\mathbf{r}}$, $F_{\mathbf{r}}$, $\bar{F}_{\mathbf{r}}$, etc., are all unchanged from before, except that now they are constructed using the new definitions of $I[\mathbf{r}(t)]$ and $I[\mathbf{r}(t)]^{\complement}$ given here. It is straightforward to show that the theory carries over as before with these new definitions. For more details, see [12].

6. Conclusion

We have proposed a method for the local regularization of linear integral equations of the first kind. The approach allows for local control of the regularization process, meaning that there is potential for the resolution of fine/sharp features of solutions without having to resort to nondifferentiable optimization techniques. Regularization parameters for this method are *functional* in nature, and the selection of these parameters is a critical issue. See [13] for an initial study into this question, which is presently the subject of an on-going study.

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