Math 309 – Exam 2 Review

Exam 2 will be given in class, Friday Feb 21. The exam covers all the material done so far in the course, which corresponds to Sections 1.1–1.6, 2.1 and 2.2, and 3.4 in the textbook.

What to study:

1. Definitions. You should be able to give *precise* definitions of the following important terms:

vector space	subspace	linear combination
linearly independent	span$(\mathbf{v}_1, \dots \mathbf{v}_n)$	basis
linear transformation	one-to-one	onto
Null space $N(T)$	Range $R(T)$	coordinate vector $[\mathbf{v}]_\alpha$ of a vector in a basis α.
rank T	nullity T	matrix $[T]_\alpha^\beta$ of T with respect to bases α, β.

2. Theorems. You should be able to apply the lemmas and theorems stated in the class notes for Days 1-16.

3. Calculations. You should know how to

- Show that a given set of vectors is (i) linearly independent, (ii) spans and (iii) is a basis.
- Find a basis of $N(T)$ for a given linear transformation $T : V \to W$.
- Find a basis of $R(T)$ for a given linear transformation $T : V \to W$.
- Find the rank and nullity of a given linear transformation.
- Find the matrix of a given linear transformation with respect to given bases.

4. Proofs. There will be one or two proofs on the exam.

Review Problems for Exam 2

1. Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be the function defined by $T(f(x)) = xf'(x)$, where $f'(x)$ is the derivative of $f(x)$.

(a) Show that T is a linear transformation.

(b) Find the matrix of T with respect to the ordered basis $\beta = \{1, x, x^2, x^3\}$ of $\mathbb{P}_3$.

2. Let $T : \mathbb{M}(2;4) \to \mathbb{R}^8$ be a linear transformation that is one-to-one. Show that T is onto.

3. Let S be the subspace of $\mathbb{R}^4$ spanned by the the vectors

$$\mathbf{u}_1 = (2, -1, 3, 1), \qquad \mathbf{u}_2 = (7, -6, 5, 2), \qquad \mathbf{u}_3 = (-3, 4, 1, 0).$$

Find a basis for S and determine its dimension.

4. Let $T : V \to W$ be a linear transformation. Prove that the range $R(T)$ is a subspace of W.

5. Determine if the following 3 vectors form a basis for $\mathbb{R}^3$:

$$\mathbf{v}_1 = (1, 1, -2) \qquad \mathbf{v}_2 = (3, 2, -4), \qquad \mathbf{v}_3 = (0, 1, 0).$$

5. Use the Rank-Nullity Theorem to show that if a linear transformation $T : V \to W$ is onto, then $\dim V \geq \dim W$.

6. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$T(x_1, x_2, x_3) = (x_1 - 2x_2, -3x_1 + 6x_2, 2x_1 - 4x_2).$$

Find a basis for $N(T)$ and $R(T)$.

7. Determine if the following 3 vectors form a basis for $\mathbb{R}^3$:

$$\mathbf{v}_1 = (1, 1, -2) \qquad \mathbf{v}_2 = (3, 2, -4), \qquad \mathbf{v}_3 = (0, 1, 0).$$

8. Let W be a subspace of a vector space V. Prove that there is a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V whose first k vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis of W.

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x, y, z) = (x, x + y, x + y + z).$$

Show that T maps $\mathbb{R}^3$ *onto* $\mathbb{R}^3$. *Use the fact that the rank is the number of pivot columns.*

10. Prove that a linear transformation $T : V \to W$ is one-to-one if and only if $N(T) = \{0\}$.