

Day 7 Linear Combinations and Span

Read Section 1.4 of the textbook and the notes below. This section is build around two important definitions (Definitions 1 and 2 below).

Definition 1. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be vectors in a vector space V . Any sum of the form

$$\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_n\mathbf{v}_n$$

for some scalars $r_1, r_2, \dots, r_n \in \mathbb{R}$ is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

The following problem is a prototype for many of the HW problems of this section. Note that the problem *immediately translates into a problem of solving a linear system*.

Problem. In $V = \mathbb{R}^5$, is the vector $(8, 5, -2, 3, 1)$ a linear combination of the vectors $\mathbf{v}_1 = (1, 0, 0, 0, 0)$, $\mathbf{v}_2 = (1, 1, 0, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1, 0, 0)$, and $\mathbf{v}_4 = (1, 1, 1, 1, 0)$?

Solution. This is true if there are scalars a, b, c and d so that

$$a(1, 0, 0, 0, 0) + b(1, 1, 0, 0, 0) + c(1, 1, 1, 0, 0) + d(1, 1, 1, 1, 0) = (8, 5, -2, 3, 1).$$

Matching the first, second, etc. components gives us 5 equations to solve simultaneously:

$$\begin{aligned} a + b + c + d &= 8 \\ 0 + b + c + d &= 5 \quad \text{etc.} \end{aligned}$$

which can be written as the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 8 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

One would usual proceed to solve the system by Gaussian elimination, but in this case there is no need: looking at the bottom row one immediately sees that the system is inconsistent, i.e. has no solution. Thus $(8, 5, -2, 3, 1)$ is NOT a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

Definition 2. Let S be a non-empty subset of a vector space V , for example S may be a finite set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. The **span** of S is the set of all linear combinations of vectors in S :

$$\text{Span}(S) = \left\{ r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_n\mathbf{v}_n \mid r_i \in \mathbb{R} \text{ and } \mathbf{v}_i \in S \forall i \right\}.$$

We also define the span of the empty set to be $\text{Span}(\emptyset) = \{0\}$.

Examples.

1. The span of one non-zero vector $S = \{\mathbf{v}\}$ is the line $\text{Span}(\mathbf{v}) = \{r\mathbf{v} \mid r \in \mathbb{R}\}$ thru $\mathbf{0}$ and \mathbf{v} .
2. The span of two non-zero vectors $S = \{\mathbf{v}, \mathbf{w}\}$ is the plane $\text{Span}(\mathbf{v}, \mathbf{w}) = \{r\mathbf{v} + s\mathbf{w} \mid r, s \in \mathbb{R}\}$ thru $\mathbf{0}$, \mathbf{v} and \mathbf{w} .

Span Theorem. In a vector space V ,

- (a) The span of any set S is a subspace of V , and
- (b) Any subspace that contains S contains all of $\text{Span}(S)$.

Proof: Done in class and on page 30 of the textbook.

Definition 3. We say that a subset $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of a vector space V *generates* or *spans* V if every vector of V can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ (or equivalently, if $\text{Span}(S) = V$).

To show that a collection $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ generates V , pick an arbitrary vector $\mathbf{w} \in V$ and prove that we can find scalars $r_1, r_2, \dots, r_n \in \mathbb{R}$ so that $\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_n\mathbf{v}_n$. This often involves showing that a linear system can be solved.

Homework 7.

1. Problem 1, page 32 of the textbook. *To answer 1b), read line 5 on page 30.*
2. (a) Find scalars a, b and c so that

$$a(2, 3, -1) + b(1, 0, 4) + c(-3, 1, 2) = (7, 2, 5).$$

Begin by writing this as a linear system.

(b) Explain why the conversion to an augmented matrix is easier when vectors are written vertically, i.e. when the system is written as

$$a \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + c \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}$$

3. Problems 3a and 3b on pg. 33.
4. Problems 4a and 4b.
5. Consider the four vectors $\mathbf{v}_1 = (-1, 2, 3)$, $\mathbf{v}_2 = (3, 4, 2)$, $\mathbf{x} = (2, 6, 6)$ and $\mathbf{y} = (-9, -2, 5)$ in \mathbb{R}^3 .
 - (a) Is $\mathbf{x} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$?
 - (b) Is $\mathbf{y} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$?

Prove your answers.

6. Problems 5f and 5g on page 33.
7. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a spanning set for a vector space V .
 - (a) If we add another vector \mathbf{v}_{n+1} to the set, will we still have a spanning set? Explain.On the other hand, if we delete one of the vectors, we may or may not still have a spanning set. Taking $V = \mathbb{R}^2$ give examples of sets $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ (you get to choose the N) for which:
 - (b) Deleting one vector still leaves a spanning set for \mathbb{R}^2 .
 - (c) Deleting one vector leaves a set that no longer spans \mathbb{R}^2 .
8. Do Problem 9 on page 34 (see Definition 3 above.)

9. Do Problem 10. Begin by writing a general 2×2 symmetric matrix as $A = \begin{pmatrix} r & s \\ s & t \end{pmatrix}$.

Day 8 Linear Dependence and Linear Independence

Read Section 1.5 of the textbook and the notes below. Linear Independence is a tricky notion!

Definition 1. The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in a vector space V are **linearly dependent** if there exist scalars a_1, a_2, \dots, a_n **not all zero** such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n = \mathbf{0}.$$

A linearly dependent collection of vectors is, as explained in class, redundant. Thus being linearly dependent is a “bad” condition. We are more interested in the “good” case: sets of vectors that are *not* linearly dependent are said to be linearly independent. Here is a clear, more useful definition:

Definition 2. The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in a vector space V are **linearly independent** if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n = \mathbf{0}$$

implies that all scalars a_1, a_2, \dots, a_n are equal to 0.

★ To prove that a given collection of vectors is linearly independent, proceed in two steps:

1. Write down the equation $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n = \mathbf{0}$ for the vectors \mathbf{v}_i given in the problem.
2. Show, “by hand” or by Gaussian elimination, that the only solution is the one with all $a_i = 0$.

The augmented matrices that occur in Step 2 will always have all 0s in their last column.

Here are two useful facts give information about what happens when we add or subtract vectors from linearly independent sets.

Lemma 3. Every subset of a linearly independent set S is linearly independent.

Thus if $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent and we delete one of the vectors, say \mathbf{v}_n , the remaining collections of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1}\}$ is automatically linearly independent.

Lemma 4. Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set of vectors in a vector space V . Then for any $\mathbf{v} \in V$, the “enlarged” set $S \cup \{\mathbf{v}\} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} \in \text{Span}(S)$.

Proof: This is Theorem 1.7 on page 39 of the textbook, and was also proved in class. \square

Homework 8

1. Problem 1 on page 40 of the textbook.
2. Problems 2c, 2e, and 2g, pg. 41. Use the method ★ above.
3. Let V be a vector space. Prove that any set of vectors in V that contains the zero vector, say $\{\mathbf{0}, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, is linearly dependent.
4. Problem 4. In this notation $\mathbf{e}_1 = (1, 0, \dots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0)$ etc.
5. Problem 5. *Hint:* Repeatedly set $x = 0$ and differentiate.
6. Problem 7.

7. (a) Prove that if \mathbf{w} is a multiple of \mathbf{v} , $\mathbf{w} = rv$, then $\{\mathbf{v}, \mathbf{w}\}$ is a linearly dependent set of vectors.
(b) Prove that if two vectors v, w are linearly dependent, then one is a multiple of the other.
8. Problem 13a.

Then **study for Wednesday's exam.**

Day 9 Exam 1 will be given in class, Wednesday Jan 29. The exam covers all the material done on Days 1-8, which corresponds to Sections 1.1 – 1.5 and 3.4 in the textbook.