Math 309– Notes and Homework for Days 4-6

Day 4 Read Section 1.2 and the notes below. The following is the main definition of the course.

Definition. A **vector space** is a set V (whose elements are called vectors) endowed with

- a rule for addition that associates to each pair $\mathbf{v}, \mathbf{w} \in V$ an element $\mathbf{v} + \mathbf{w} \in V$, and
- a rule for scalar multiplication that associates to each $\mathbf{v} \in V$ and $r \in \mathbb{R}$ an element $r\mathbf{v} \in V$,

such that, for all $\mathbf{v}, \mathbf{w}, \mathbf{x} \in V$ and $r, s \in \mathbb{R}$,

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$	$+C$	Addition is commutative
2. $(\mathbf{v} + \mathbf{w}) + \mathbf{x} = \mathbf{w} + (\mathbf{v} + \mathbf{x})$	$+A$	Addition is associative
3. $\exists$ a vector $\mathbf{0} \in V$ s.t. $\mathbf{v} + \mathbf{0} = \mathbf{v}$	$+0$	Additive Identity Property
4. $1 \cdot \mathbf{v} = \mathbf{v}$	$\times 1$	Multiplicative Identity Property
5. $r(s\mathbf{v}) = (rs)\mathbf{v}$	$\times A$	Multiplication is associative
6. For each $\mathbf{v} \in V$, $\exists$ an "opposite vector" $-\mathbf{v} \in V$ s.t. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	$+Inv$	Additive Inverse Property
7-8. $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$ and $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$	*Dist.*	Distributive Property

We will refer to these axioms using the abbreviations given in middle column and the words given in the last column.

Notes (a) These axioms implicitly assume that the properties of the real numbers, of sets, and of the symbol = (e.g. adding the same thing to both sides preserves equality) are known and will be used freely. Thus in proofs you will have occasion to use the abbreviations

prop of $\mathbb{R}$ *prop of sets*   *prop of* =

for "properties of the real numbers" , "properties of sets" and "properties of equality".

(b) In Axioms 3 and 4, the word "identity" means "vector that makes no change".

(c) When checking whether a given set is a vector space, always begin by checking the two bulleted requirements at the top. In fact, these are the most important.

Examples of Vector Spaces

1. **The real numbers** $\mathbb{R}$ with the usual addition and scalar multiplication.

2. **The vector spaces $\mathbb{R}^n$, Euclidean space with coordinate grid**, is the set of n-tuples or real numbers (written either vertically or horizontally). For $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, the sum is defined by adding corresponding components, and the scalar multiplication by $r \in \mathbb{R}$ is defined by multiplying every component by r:

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n) \qquad \text{and} \qquad r\mathbf{v} = (rv_1, rv_2, \ldots, rv_n)$$

Similarly, the vector space $\mathbb{C}^n$ consisting of all n-tuples of complex numbers. The textbook uses the notation F^n to mean either $\mathbb{R}^n$ or $\mathbb{C}^n$.

3. **The vector space $\mathbf{M}_{m,n}(F)$ of all $m \times n$ matrices** (m rows and n columns) with entries in $F = \mathbb{R}$ or $\mathbb{C}$. Matrices are added by adding corresponding entries. A matrix A is multiplied by a scalar $r \in \mathbb{R}$ by multiplying each entry in A by r.

Notation: Write A_{ij} for the number in the ith row and jth column of the matrix A. Then addition and scalar multiplication can be written as $(A+B)_{ij} = A_{ij} + B_{ij}$ and $(rA)_{ij} = rA_{ij}$.

4. **The vector space $P_n(F)$ of polynomials of degree $\leq n$** consists of all polynomials p with coefficients in in $F = \mathbb{R}$ or $\mathbb{C}$ of the form

$$p(x) \;=\; a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

in the variable x. Note that some, or even all, of the coefficients can be 0. Addition and scalar multiplication are done on each coefficient separately: if $q(x) \;=\; b_0 + b_1 x + \cdots + b_n x^n$ then

$$(p+q)(x) \;=\; (a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n \qquad (rp)(x) \;=\; ra_0 + (ra_1)x + \cdots + (ra_n)x^n,$$

This is the same as thinking of p and q as functions of x and adding and scalar multiplying as in the next example.

5. **The vector spaces $C[a,b]$ of continuous functions.** Let $\mathcal{C}[a,b]$ denote the set of all real-valued functions defined and continuous on the interval $a \leq x \leq b$. The sum of functions $f+g$ is defined by taking the sum of the real numbers $f(x)$ and $f(x)$ at each x: of gegree less than n. Define $p+q$ and rp by

$$(f+g)(x) \;=\; f(x) \;+\; g(x)$$

This is an element in $C[a,b]$ because of the calculus fact that the sum of continuous function is continuous. Scalar multiplication by $r \in \mathbb{R}$ is defined similarly:

$$(rf)(x) \;=\; r \cdot f(x).$$

Homework 4: Do the following problems (most are taken from Section 1.2 of the textbook):

1. The following are augmented matrices, already in reduced row echelon form. Write down the solution sets S in terms of free variables $r, s, \ldots$. *Use the procedure described in the Day 3 notes.*

$$(a)\quad \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix} \qquad\qquad (b)\quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 5 \end{bmatrix}$$

2. There is a "quick method" to solve linear systems, described below. Use this method to write down the Solution Set to the following two linear systems.

$$(a)\quad \begin{aligned} x + 2y + z &= 5 \\ 2x - 3y + 5z &= -17 \\ 4x + 8y - z &= 30 \end{aligned} \qquad\qquad (b)\quad \begin{aligned} x_1 + x_2 - 5x_3 + 3x_4 - 4x_5 &= 9 \\ 2x_1 - x_2 + 6x_3 - 3x_4 + x_5 &= 18 \\ -x_1 + 2x_2 + 4x_3 - 4x_4 + 6x_5 &= 19 \end{aligned}$$

   Quick Method: Go to WolframAlpha.com. Type in the equations, separated by commas. Press Return.

   For (b), give the variables simpler names before entering. Write the answer in both the "exact form" and the "approximate form" given by WolframAlpha.

3. Compare the definition of vector space given above to the one on page 7 of your textbook. Choose one or the other and *copy the entire definition onto your HW sheet.* Include the names and abbreviations given above. *This is intended to help you learn the definition.*

Do the following problems from pages 12-13 of the textbook.

4. Problem 1 parts a-i.
5. Problem 2
6. Problem 3
7. Problems 4a, 4c, 4e and 4h.
8. Problem 12
9. Problem 13
10. Problem 16

Day 5 Consequences of the axioms.

Read the notes below and do HW 5.

From the axioms, one can derive numerous simple consequences that are useful in calculations. Each is proved from the axioms *and from previously proved facts.* Once a fact is proved, it gets added to our basket of "known facts" and can be used from then on.

Here are several examples. Each is called a "Lemma", and a proof is given 2-column format. You will do similar proofs in HW.

Lemma 1. $\mathbf{0}+\mathbf{v}=\mathbf{v} \quad \forall \mathbf{v}\in V.$ (Note that 0 is a number, but $\mathbf{0}$ (also written $\vec{\mathbf{0}}$) is a vector.

Proof:
$$\begin{aligned} \mathbf{0}+\mathbf{v} &= \mathbf{v}+\mathbf{0} && \textit{+C} \\ &= \mathbf{v} && \textit{+ 1} \qquad \square \end{aligned}$$

Lemma 2. $\mathbf{v}+\mathbf{v}=2\mathbf{v} \qquad \forall \mathbf{v}\in V.$

Proof:
$$\begin{aligned} \mathbf{v}+\mathbf{v} &= 1\cdot\mathbf{v}+1\cdot\mathbf{v} && \times 1 \\ &= (1+1)\mathbf{v} && \textit{Dist} \\ &= 2\mathbf{v} && \textit{prop of } \mathbb{R} \qquad \square \end{aligned}$$

Lemma 3. (a) $0\cdot\mathbf{v}=\mathbf{0}$ for all $\mathbf{v}\in V$, and

(b) $s\cdot\mathbf{0}=\mathbf{0}$ for all $s\in\mathbb{R}$.

Proof: (a) was proved in class.

(b)
$$\begin{aligned} \mathbf{0} &= s\mathbf{0}+-(s\mathbf{0}) && \textit{addInv} \\ &= s(\mathbf{0}+\mathbf{0})+-(s\mathbf{0}) && \textit{+0} \\ &= (s\mathbf{0}+s\mathbf{0})+-(s\mathbf{0}) && \textit{Dist} \\ &= s\mathbf{0}+(s\mathbf{0}+-(s\mathbf{0})) && \textit{+C} \\ &= s\mathbf{0}+\mathbf{0} && \textit{+Inv} \\ &= s\mathbf{0} && \textit{+0} \qquad \square \end{aligned}$$

Lemma 4. (Cancellation Law) If $\mathbf{v}+\mathbf{x}=\mathbf{w}+\mathbf{x}$ then $\mathbf{v}=\mathbf{w}$. Read the proof

Lemma 5. (Uniqueness of Additive inverse) If $\mathbf{v} + \mathbf{x} = \mathbf{0}$ then $\mathbf{x} = -\mathbf{v}$.

Lemma 6. If $s\mathbf{v} = \mathbf{0}$ then $s = 0$ or $\mathbf{v} = \mathbf{0}$.

Proof: If $s \neq 0$ then $\frac{1}{s}$ exists. Then

$$\begin{aligned} \mathbf{v} &= 1 \cdot \mathbf{v} && \underline{\qquad\quad} \\ &= \vdots \\ &= \mathbf{0} && \square \end{aligned}$$

Background: Axiomatic Systems

Most areas of modern mathematics are organized axiomatically. This is the approach that Euclid took to developing geometry: he began with 5 axioms (Euclid's "postulates") and built the entire subject from there. In the 19th and early 20th centuries this scheme was applied to other areas of mathematics and refined into what is known as the *axiomatic system*. In particular, linear algebra is organized as an axiomatic system.

There are four parts of an axiomatic system.

1. Terms and definitions. Each axiomatic system begins with a few *undefined terms*. The undefined terms in linear algebra are *point*, *vector*, *set*, *element*, and a few others. After that, new terms are introduced with precise definitions.

2. Axioms. An *axiom* or *postulate* (the words are interchangeable) is a statement that is accepted without proof. The subject (linear algebra for us) begins with a short list of axioms. Everything else is logically derived from them.

Sometimes axioms refer to other mathematical subjects or objects. The axioms of linear algebra assume that you are familiar with the properties of the real numbers, and of sets (which themselves can be developed as axiomatic systems).

3. Theorems. The largest part, by far, of an axiomatic system consists of theorems and their proofs. A *theorem* is a "if–then" statement that has been proved to be a logical consequence of the axioms. Vocabulary: the words *theorem* and *proposition* are synonyms, a *lemma* is a theorem that is stated as a step toward some more important result, and a *corollary* is a theorem that can be quickly and esaily deduced from a previously-proved theorem.

Theorems are organized in a strict logical order, building on each other. The first theorem is proved using only the axioms. The second theorem is proved using only the first theorem and the axioms, etc..

4. Models. An axiomatic system is a purely logical construction. The human brain is not good at understanding complicated logical constructions. Thus it is advantageous to have a way of thinking about or visualizing the undefined terms that is compatible with human experience.

An *interpretation* of an axiomatic system is a particular way of giving meaning to the undefined terms in that system. An interpretation is called a *model* if the axioms are true statements in that interpretation. As a result, all of the theorems are also true for the model.

A good model makes it possible to visualize and guess theorems, and it provides guidance in developing proofs. For linear algebra, the best model is the pictures of vector addition by parallelograms and scalar multiplication by rescaling vectors described on Day 1.

Homework 5: Give proofs, similar to those above, that each of the following facts holds for all $v \in V$ and $r \in \mathbb{R}$. For each, you are allowed to use the axioms and all previously proved Lemmas (Lemmas 1-6 for Problem 1, Lemmas 1-7 for Problem 2, etc.).

1. **Lemma 7.** $(-1)\mathbf{v} = -\mathbf{v}$

   Hint: start with the words "Let $\mathbf{x} = (-1)\mathbf{v}$*". Then show that* $\mathbf{v} + \mathbf{x} = \mathbf{0}$ *and quote Lemma 5.*

2. **Lemma 8.** If $\mathbf{v} = -\mathbf{v}$ then $\mathbf{v} = \mathbf{0}$.

3. **Lemma 9.** $-(-\mathbf{v}) = \mathbf{v}$.

4. **Lemma 10.** $r(-\mathbf{v}) = -(r\mathbf{v})$.

5. **Lemma 11.** $(-r)\mathbf{v} = -(r\mathbf{v})$.

6. **Lemma 12.** If $\mathbf{v} \neq \mathbf{0}$ and $r\mathbf{v} = s\mathbf{v}$, then $r = s$.

   Hint: start with $(r - s)\mathbf{v} = (r + (-s))\mathbf{v}$ *and use facts 11 and 6 above.*

Subtraction of vectors is defined in terms of addition in the obvious manner: $\mathbf{v} - \mathbf{w}$ is defined to be $\mathbf{v} + (-\mathbf{w})$. Use this definition to prove the following identities hold for each $\mathbf{v}, \mathbf{w}$ in V and $r, s \in \mathbb{R}$. *The reason for some steps will be "def. of subtr.".*

7. **Lemma 13.** $\mathbf{0} - \mathbf{v} = -\mathbf{v}$.

8. **Lemma 14.** $\mathbf{v} - (-\mathbf{w}) = \mathbf{v} + \mathbf{w}$.

9. **Lemma 15.** $(\mathbf{v} - \mathbf{w}) + \mathbf{w} = \mathbf{v}$.

Day 6 Subspaces. Read Section 1.4 of the textbook and do HW6.

Definition. A subset S of a vector space V is called a *vector subspace* of V if S is itself a vector space (using the addition and scalar multiplication of V).

Simple Examples: Any line through the origin is a subspace of $\mathbb{R}^2$. Each line through the origin, and each plane through the origin, is a subspace of $\mathbb{R}^3$.

Subspace Theorem. A subset S of a vector space V is a vector subspace if and only if it has the following three properties:

(i) S is not empty.

(ii) S is *closed under addition:* If $\mathbf{v}$ and $\mathbf{w}$ are both in S, then so is $\mathbf{v} + \mathbf{w}$.

(iii) S is *closed under scalar multiplication:* If $\mathbf{v} \in S$ then $r\mathbf{v}$ is in S for every scalar $r \in \mathbb{R}$.

Proof: Done in class and in the textbook page 17. $\square$

Remarks. 1. In a vector space V, one can readily check that $\{\mathbf{0}\}$ and V are vector subspaces. All other subspaces are called *proper subspaces.* We refer to $\{\mathbf{0}\}$ as the *zero subspace.*

2. Every subspace must contain the zero vector (by Axiom 3). Thus a simple way to verify that a given subset is a subspace is to

(a) Show that $\mathbf{0} \in S$, and

(b) Prove that $\mathbf{v}, \mathbf{w} \in S \implies \mathbf{v} + \mathbf{w} \in S$.

(c) Prove that $\mathbf{v} \in S \implies r\mathbf{v} \in S \quad \forall r \in \mathbb{R}$.

Alternatively, one can combine the two types of closure by checking closure under linear combinations, replacing steps (b) and (c) by

(bc) Prove that if $\mathbf{v}, \mathbf{w} \in S$ then $r\mathbf{v} + s\mathbf{w} \in S$ for all $r, s \in \mathbb{R}$.

Example. The set $S = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$ is a subspace of $\mathbb{R}^2$.

Proof: (a) $\mathbf{0} = (0, 0) \in S$ ✓ (i.e. the zero vector of $\mathbb{R}^2$ satisfies $x = y$).

If $\mathbf{v} = (x_1, y_1)$ and $\mathbf{w} = (x_2, y_2)$ are in S then $x_1 = y_1$ and $x_2 = y_2$. Then

(b) $\mathbf{v} + \mathbf{w} = (x_1 + x_2, y_1 + y_2) \in S$ because $x_1 + x_2 = y_1 + y_2$.

(c) For any $r \in \mathbb{R}$, $r\mathbf{v} = (rx_1, ry_1) \in S$ because $rx_1 = ry_1$.

Thus S is a subspace of $\mathbb{R}^2$. □

Subspace Theorem. A subspace S of a vector space V is itself a vector space (using the addition and scalar multiplication of V).

Intersection Theorem. The intersection of two or more subspaces is a subspace.

Proof: Let $V_1, \ldots, V_k$ be subspaces of a vector space V. Consider $W = V_1 \cap V_2 \cap \cdots \cap V_k$.

(a) Because V_1 is a subspace, we know $\mathbf{0} \in V_1$. Similarly, $\mathbf{0} \in V_i$ for each $i = 2, \ldots k$. Therefore $\mathbf{0} \in V_1 \cap \cdots \cap V_n = W$.

(b) Exercise 5 below.

Therefore W is a subspace of V. □

This section also includes two useful definitions about matrices (these appear in the examples).

- The *transpose* of an $n \times n$ matrix A is the $m \times n$ matrix A^t obtained by interchanging the rows and columns of A. Thus $(A^t)_{ij} = A_{ji}$
- The *trace* of a square $n \times n$ matrix A, denoted trA, is the sum of the entries on the diagonal, that is $trA = \sum_{i=1}^{n} A_{ii}$.

Homework 6:

1. Finish HW Set 5 if you haven't already done so. Then give a similar proof of the following fact:

    Lemma 16. If $r\mathbf{v} + s\mathbf{w} = 0$ with $r \neq 0$, then $\mathbf{v}$ is a multiple of $\mathbf{w}$.

2. Determine whether the following sets are subspaces of $\mathbb{R}^2$. If not, explain why, if yes, give a proof. *Draw a graph and give a reason.*

    (a) $A = \{(x, y) \,|\, x + y = 0\}$

    (b) $B = \{(x, y) \,|\, xy = 0\}$

    (c) $C = \{(x, y) \,|\, y = 3x\}$

    (d) $D = \{(x, y) \,|\, y^2 = x^2\}$.

3. Determine whether the following sets are subspaces of $\mathbf{P}_4(\mathbb{R})$. If not, explain why, if yes, give a proof. *Be careful!*

    (a) The set E of polynomials in $\mathbf{P}_4(\mathbb{R})$ of even degree.

    (b) The set F of all polynomials of exactly degree 3.

    (c) The set G of all polynomials $p(x)$ in $\mathbf{P}_4(\mathbb{R})$ such that $p(0) = 0$.

4. Determine whether the following sets are subspaces of $C[-1, 1]$. If not, explain why, if yes, give a proof.

    (a) The set I of odd functions in $C[-1, 1]$, i.e. those with $f(-x) = -f(x)$ for all $x \in [-1, 1]$.

    (b) The set J of functions f in $C[-1, 1]$ such that $f(-1) = f(1)$.

    (c) The set K of all functions f in $C[-1, 1]$ with average value zero, i.e. $\int_{-1}^{1} f(x)\, dx = 0$.

5. Finish the proof of the Intersection Theorem (above).

6. The union of subspaces needn't be a subspace. Show this by giving an example of two subspaces of $\mathbb{R}^2$ whose union is not a subspace.

Do the following problems from pages 19-23 of the textbook.

7. Problem 1, all parts.

8. Problem 2a, 2d and 2h.

9. Problem 5   *Hint: use the notation* $(A^t)_{ij} = A_{ji}$.

10. Problem 6   *Hint: use the notation* $trA = \sum_{i=1}^{n} A_{ii}$.

11. Problem 20.