# Day 31 Applications of diagonalization

Many computations with matrices become easier if one can diagonalize the matrices. Geometrically, this means thinking of the matrix as a linear transformation and switching to a basis in which the linear transformation is a dilation in each direction.

Suppose that $A$ is an $n \times n$ matrix that can be diagonalized. This means that there is an $n \times n$ matrix $Q$ so that $Q^{-1}AQ$ is the diagonal matrix $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \ddots \end{pmatrix}$. Multiplying by Q on the left and $Q^{-1}$ on the right then gives

$$A = QDQ^{-1}. \tag{31.1}$$

**Powers.** From (31.1) we obtain $A^3 = QDQ^{-1} \cdot QDQ^{-1} \cdot QDQ^{-1} = QD^3Q^{-1}$ and similarly for the $k^{th}$ power of $A$

$$A^k = QD^kQ^{-1}. \tag{31.2}$$

**Polynomials.** Applying (31.2) to each term in a polynomial, we have

$$A^3 + 4A^2 - 7A + 2I_n = Q\left(D^3 + 4D^2 - 7D + 2I_n\right)Q^{-1}$$

and similarly for any polynomial in $A$. Note that *polynomials in $D$ are wasy to calculate.*

**Exponentials.** For real numbers $x$, $e^x$ can be defined in 3 ways:

(a) As repeated multiplication, e.g. $e^3 = e \cdot e \cdot e$.

(b) By the power series $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$.

(c) As the unique solution of the initial value problem $y'(x) = y(x)$, $y(0) = 1$.

Each has advantages: (a) is intuitive, (b) is good for calculating, and (c) is good for showing the basic property $e^{a+b} = e^a \cdot e^b$, which is not clear from (b)! If we replace $x$ by a square matrix $A$, then definition (a) makes little sense, but (b) and (c) both do, as follows.

**Definition.** *For a square matrix $A$, set*

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{n!}A^n + \cdots \tag{31.3}$$

This series converges absolutely for all matrices $A$.

**Theorem 31.1.** *Suppose that $A = QDQ^{-1}$ where $D$ is diagonal. Then $e^A = Qe^DQ^{-1}$. Thus if $A$ has a basis of eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_n$ then*

$$e^{tA} = Q\begin{pmatrix} e^{t\lambda_1} & & 0 \\ & e^{t\lambda_2} & \\ 0 & & \ddots \end{pmatrix}Q^{-1} \qquad \text{where } Q\text{= matrix with the e-vectors as columns (in order)}$$

## Systems of first order ODEs.

Consider two functions $y_1(t), y_2(t)$ that satisfy the coupled differential equations

$$y_1'(t) = 3y_1 + 4y_2$$
$$y_2'(t) = 3y_1 + 2y_2$$

This system can be written as $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ or simply as

$$\mathbf{Y}'(t) = A\mathbf{Y}(t) \qquad \text{where} \qquad A = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}$$

This generalizes in the obvious way: replace $A$ by an $n \times n$ matrix and take $\mathbf{Y}(t)$ to be a column vector of $n$ unknown functions $y_i(t)$. Such a system is called a *first order linear system of ODEs with constant coefficients*. Often, one wants to consider the *initial value problem* in which we are given the values of the functions $y_i$ as some time, say $t = 0$:

$$\frac{d\mathbf{Y}'}{dt} = A\mathbf{Y} \qquad \text{with} \qquad \mathbf{Y}(0) = \mathbf{Y_0}$$

**Theorem 31.2.** For any square matrix $A$,

(a) the set $S_A$ of all solutions of $\mathbf{Y}' = A\mathbf{Y}$ is a vector space.

(b) If $\mathbf{v}$ is an eigenvector with $A\mathbf{v} = \lambda\mathbf{v}$, then $\mathbf{Y}(t) = c\, e^{\lambda t}\mathbf{v}$ is a solution for any $c \in \mathbb{R}$.

(c) If $A$ is diagonalizable, then the solution of the initial value problem is

$$\mathbf{Y}(t) = e^{tA}\mathbf{Y}_0.$$

**Example.** Solve the initial value problem $\mathbf{Y} = A\mathbf{Y}$ where $A = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}$ and $\mathbf{Y}(0) = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$.

*Solution.* Calculations show that the eigenvalues of $A$ are $\lambda_1 = 6$ and $\lambda_2 = -1$ with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus

$$A = QDQ^{-1} \qquad \text{where} \qquad D = \begin{pmatrix} 6 & 0 \\ 0 & -1 \end{pmatrix} \qquad Q = \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}$$

Therefore

$$\mathbf{Y}(t) = e^{tA}\mathbf{Y}_0 = Qe^{tD}Q^{-1}\mathbf{Y}_0 = \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} e^{6t} & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot \tfrac{1}{7} \begin{pmatrix} 1 & 1 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 4e^{6t} + 2e^{-t} \\ 3e^{6t} - 2e^{-t} \end{pmatrix}.$$

Hint for homework: when doing this last calculation, doing the multiplications right-to-left is easier.

# Homework 31

1. Find $e^{tA}$ for $A = \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix}$.

2. Find $e^{tB}$ for $B = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$.

*Solve Problems 3 and 4 by finding a basis of eigenvectors and using part (b) of Theorem 31.2.*

3. Solve the linear system

$$\begin{aligned} y_1'(t) &= y_1 + y_2 \\ y_2'(t) &= -2y_1 + 4y_2 \end{aligned}$$

4. Solve the linear system

$$\begin{aligned} y_1'(t) &= 2y_1 + 4y_2 \\ y_2'(t) &= -1y_1 - 3y_2 \end{aligned}$$

*Solve Problems 5 and 6 by diagonalizing the matrix and using part (c) of Theorem 31.2*

5. Solve the initial value problem

$$\begin{aligned} y_1'(t) &= -y_1 + 2y_2 & \qquad y_1(0) &= 3 \\ y_2'(t) &= 2y_1 - y_2 & \qquad y_2(0) &= 1. \end{aligned}$$

**Some Solutions:**

1. $\begin{pmatrix} 3 - 2e^{-t} & -6 + 6e^{-t} \\ 1 - e^{-t} & -2 + 3e^{-t} \end{pmatrix}$.

3. $\begin{pmatrix} ae^{2t} + be^{3t} \\ ae^{2t} + 2e^{3t} \end{pmatrix}$.

4. $\begin{pmatrix} -ae^{-2t} - 4be^{t} \\ ae^{-2t} + be^{t} \end{pmatrix}$.

5. $\begin{pmatrix} e^{-3t} + 2e^{t} \\ -e^{-3t} + 2e^{t} \end{pmatrix}$.

# Day 32 Complex Eigenvalues

Many polynomials with real coefficients have no real roots. Consequently, there are many matrices with real entries that have no real eigenvalues.

**Example 1.** The characteristic polynomial of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is $p_A(\lambda) = \lambda^2 + 1$, which has no real roots.

You may recognize this matrix $A$ as a rotation of the plane counter-clockwise by $90°$ (draw a sketch, noting that $A$ takes $\mathbf{e}_1$ to $\mathbf{e}_2$ and $\mathbf{e}_2$ to $-\mathbf{e}_1$). Thus $A$ is not a "disguised dilation" — there is no basis in which $A$ is diagonal.

There is a simple mathematical trick that gets around this problem: regard real matrices as special cases of complex matrices, and find *complex* eigenvalues and eigenvectors. This is called working "over $\mathbb{C}$".

**Example 1'.** The above characteristic polynomial $p_A(\lambda) = \lambda^2 + 1$ factors over $\mathbb{C}$ as $\lambda^2 + 1 = (\lambda - i)(\lambda + i)$, so $A$, *regarded as a complex matrix*, has eigenvalues $\lambda = \pm i$.

In fact, eigenvalue problems can always be solved over $\mathbb{C}$. This was one of the main reasons why the definition of vector space allowed, from the beginning, scalars to be in a field $F$ which could be $\mathbb{R}$ or $\mathbb{C}$.

**Review of complex numbers.** Read the review of complex numbers in Appendix D of the textbook. Here is a summary.

The *complex plane* is $\mathbb{C} = \mathbb{R}^2$ with basis $\{1, i\}$. Elements of the complex plane are called complex numbers. Thus they can be written as $\alpha = a + bi$ where $a, b \in \mathbb{R}$, and are added as vectors in $\mathbb{R}^2$:

$$(a + bi) + (c + di) \;=\; (a + c) + (b + d)i.$$

Complex numbers can also be multiplied using the distributive rule and the formula $i^2 = -1$:

$$\alpha\beta \;=\; (a + bi)\cdot(c + di) \;=\; ac + adi + bci + bd\,i^2 \;=\; (ac - bd) + (ad + bc)i.$$

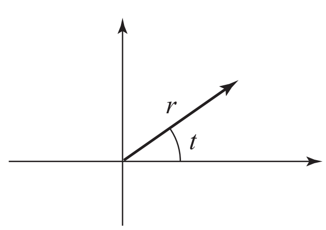
These operators are commutative, associative, and distributive. One also defines

- The *complex conjugate* of $\alpha = a + bi$ is $\overline{\alpha} = a - bi$.
- The *absolute value* or *modulus* or *norm* of $\alpha$ is $|\alpha| = \sqrt{a^2 + b^2}$. Note that $|\alpha|^2 = \alpha\overline{\alpha}$.

Geometrically, the map $\alpha \mapsto \overline{\alpha}$ is reflection through the real axis, and $|\alpha|$ is the distance from $\alpha \in \mathbb{C}$ to the origin. One then sees that each non-zero $\alpha = a + bi \in \mathbb{C}$ has a multiplicative inverse, namely

$$\frac{1}{\alpha} \;=\; \frac{\overline{\alpha}}{\alpha\overline{\alpha}} \;=\; \frac{\overline{\alpha}}{|\alpha|^2} \;=\; \frac{a - bi}{a^2 + b^2}$$

These properties mean that $\mathbb{C}$ is a *field.*



Complex numbers can also be written in polar form. As usual, points $(a, b)$ in the plane have polar coordinates $(r, \theta)$ as shown in the figure. Then $a = r\cos\theta$ and $b = r\sin\theta$; with complex number notation this becomes

$$\alpha \;=\; a + bi \;=\; r(\cos\theta + i\sin\theta) \;=\; re^{i\theta}$$

where $r = |\alpha|$ is the modulus and $\theta$ is called the *argument of* $\alpha$, and where the last equality comes from this famous fact:

**Euler's formula:** $e^{i\theta} \;=\; \cos\theta + i\sin\theta$.

*Proof.* $f(\theta) = e^{i\theta}$ is the unique solution of the initial value problem $f'(\theta) = if(\theta)$ and $f(0) = 1$. But $g(\theta) = \cos\theta + i\sin\theta$ satisfies $g'(\theta) = ig(\theta)$ and $g(0) = 1$, so $g(\theta) = e^{i\theta}$. $\square$

All of algebra extends to the complex numbers. For example, a polynomial of degree $n$ in a complex variable $z$ has the form

$$p(z) \;=\; a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

where the coefficients $a_i$ are complex numbers, i.e. $a_i \in \mathbb{C}\ \ \forall i$. A complex number $r \in \mathbb{C}$ is a *root* of $p(z)$ if $p(r) = 0$. The advantage of working over $\mathbb{C}$ is a result of the following remarkable fact.

**The Fundamental Theorem of Algebra.** *If $p(z)$ is a polynomial of degree $n \geq 1$ in a complex variable $z$, then there are $n$ complex numbers $r_1, \ldots, r_n$ (not necessarily distinct) such that*

$$p(z) \;=\; a_n(z - r_1)(z - r_2)\cdots(z - r_n).$$

Thus over $\mathbb{C}$, every degree $n$ polynomial has exactly $n$ complex roots. Applying this to the characteristic polynomial of matrix, we have:

**Corollary 32.1.** *Every $n \times n$ matrix $A$ over $\mathbb{C}$ has exactly $n$ eigenvalues – the roots of $p_A(\lambda) = \det(A - \lambda I)$.*

**Diagonalizing over the complex numbers.** Once we agree to work over $C$, the process of diagonalizing a matrix is exactly as before.

**Example 2.** Diagonalize the matrix $A = \begin{pmatrix} 0 & 1 \\ -5 & 4 \end{pmatrix}$.

*Solution.* The characteristic polynomial is $p(\lambda) = -\lambda(4 - \lambda) + 5 = \lambda^2 - 4\lambda + 5$. Using the quadratic formula, the roots are

$$\lambda \;=\; \frac{4 \pm \sqrt{16 - 20}}{2} \;=\; 2 \pm \tfrac{1}{2}\sqrt{-4} \;=\; 2 \pm i.$$

To find an eigenvector for $\lambda_1 = 2 + i$ we must solve $(A - \lambda_1 I)\mathbf{v} = 0$, or

$$\begin{pmatrix} -2 - i & 1 \\ -5 & 2 - i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note that the bottom row is $2 - i$ times the top row because $(2 - i)(-2 - i) = -5$. This leaves only the top row, which gives the relation $-(2 + i)a + b = 0$. For one solution, take $a = 1$ to get the eigenvector

$$\mathbf{v}_1 \;=\; \begin{pmatrix} 1 \\ 2 + i \end{pmatrix}$$

Similarly for $\lambda_2 = 2 - i$ we solve

$$\begin{pmatrix} -2 + i & 1 \\ -5 & 2 + i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

to get

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

(in fact, this is just replacing $i$ by $-i$ everywhere.) The matrix is now diagonalized by $Q = \begin{pmatrix} 1 & 1 \\ 2+i & 2-i \end{pmatrix}$ which has $\det Q = (2-i) - (2-i) = -2i$. Then

$$Q^{-1}AQ = \frac{1}{-2i}\begin{pmatrix} 2-i & -1 \\ -2-i & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2+i & 2-i \end{pmatrix} = \begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}$$

# Homework 32

Read Appendix D of the textbook (pages 556-561) and answer do the following problems.

1. (a) What is $\dfrac{1}{i}$ ?
   (b) Write the number $4-4i$ in polar form.
2. A complex number $z$ is called a $n^{th}$ *root of unity* if $z^n = 1$.
   (a) How many $n^{th}$ roots of unity are there? (Apply the Fundamental Theorem of Algebra to $p(z) = z^n - 1$).
   (b) Write the $n^{th}$ roots of unity $\{z_k\}$ (all of them) in polar form, i.e. $z_k = re^{i\theta}$ for what $r$ and $\theta$?
3. (a) If $z = re^{i\theta}$ and $w = se^{i\phi}$, what is the polar form of the product $zw$?
   (b) For a fixed complex number $z = re^{i\theta}$, show that there are exactly two complex numbers $w$ with $w^2 = z$ and find the polar expressions of both of these numbers.
4. The transformation $T(z) = (2-3i)z$ from $\mathbb{C}$ to $\mathbb{C}$ defines a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by writing $z = a + bi$. Find the matrix of $T$ as a $2 \times 2$ real matrix.
5. Write the polynomial $p(z) = z^3 - 8z^2 + 25z - 26$ as a product of linear factors over $\mathbb{C}$.
   *Hint: $z = 2$ is a root. Divide by $(z-2)$ and use the quadratic formula.*
6. Find all eigenvalues of the following matrices.

$$A = \begin{pmatrix} 11 & -15 \\ 6 & -7 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

7. For each of the following matrices find
   (a) All complex eigenvalues.
   (b) Corresponding (complex) eigenvectors.
   (c) A matrix $Q$ so that $Q^{-1}CQ$ is diagonal.

$$C = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{pmatrix}$$

# Day 33 More complex eigenvalues – with tricks

Recall that an $n \times n$ matrix is diagonalizable if there is an invertible matrix $Q$ usch that

$$Q^{-1}AQ \;=\; \begin{pmatrix} \lambda_1 & 0 \\ 0 & \ddots \end{pmatrix} \tag{33.1}$$

is diagonal. This is equivalent to the existence of a basis $\mathbf{v}_1, \cdots \mathbf{v}_n$ of eigenvectors. There are three basic facts about diagonalization:

(1) If the characteristic polynomial $p_A(\lambda)$ has distinct real roots $\lambda_i$ then $A$ is diagonalizable.

(2) If $p_A(\lambda)$ has distinct complex roots $\lambda_i$ then $A$ is diagonalizable over $\mathbb{C}$, i.e. (33.1) holds with $Q$ and $D$ complex matrices.

(3) If $p_A(\lambda)$ has repeated roots then $A$ may not be diagonalizable.

Today we will do more on (2) and give some tricks involving trace and determinants. Next time we will consider (3).

**Lemma 33.1.** *For a matrix $A$ with real entries, the roots of the characteristic polynomial $p_A(\lambda)$ are of two types:*

- *real roots $\lambda_i$*
- *conjugate pairs $\lambda_j, \overline{\lambda_j}$.*

*and the corresponding eigenvectors are in corresponding conjugate pairs.*

*Proof.* First, since the coefficients of

$$p_A(x) \;=\; \det(A - xI) \;=\; a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are real we have $\overline{p_A(x)} = p_A(\overline{x})$. Hence is $\lambda$ is a root then $p_A(\overline{\lambda}) = \overline{p_A(\lambda)} = 0$, so $\overline{\lambda}$ is also a root.

Next, suppose that $A\mathbf{v} = \lambda\mathbf{v}$, so $\mathbf{v}$ is an eigenvector with eigenvalue $\lambda$. Then since $A$ is real,

$$A\overline{\mathbf{v}} \;=\; \overline{A\mathbf{v}} \;=\; \overline{\lambda\mathbf{v}} \;=\; \overline{\lambda}\,\overline{\mathbf{v}}$$

and thus $\overline{\mathbf{v}}$ is an eigenvector with eigenvalue $\overline{\lambda}$. Finally, note that $\mathbf{v} = \overline{\mathbf{v}}$ if and only if $\mathbf{v}$ is real. □

**Example 1.** The rotation matrix

$$R_\theta \;=\; \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{33.2}$$

has eigenvalues $e^{\pm i\theta}$ (a conjugate pair), and eigenvectors $\mathbf{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. One can then find (cf. (HW Problem 4) a complex matrix $Q$ so that

$$QR_\theta Q^{-1} \;=\; \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

By Lemma 33.1, any matrix $A$ with distinct eigenvalues can be diagonalized over $\mathbb{C}$ to a diagonal matrix $D$ (i.e. there is a complex matrix $Q$ such that $QAQ^{-1} = D$) with

$$D \;=\; \mathrm{diag}(r_1, \cdots r_k, \lambda_1, \overline{\lambda_1}, \ldots, \lambda_\ell, \overline{\lambda_\ell})$$

for real eigenvalues $r_i$ and non-real complex eigenvalues $\lambda_j$. Going backwards through Example 1 above yields the following fact.

**Theorem 33.2.** *If a real matrix $A$ has distinct roots, then there is a real matrix $Q$ so that $D = QAQ^{-1}$ has the form*

$$D = QAQ^{-1} = \begin{pmatrix} r_1 & & & & 0 \\ & r_k & & & \\ & & \square & & \\ & & & \square & \\ 0 & & & & \ddots \end{pmatrix}$$

*where each box is a $2 \times 2$ rotation matrix $R_{\theta_k}$ of the form (33.2) for some $\theta_k$.*

## Traces, determinants and eigenvalues.

**Definition.** *The* **trace** *of a square matrix $A$ is the sum of the diagonal entries of $A$:*

$$\operatorname{tr} A = \sum_{i=1}^{n} A_{ii}.$$

**Theorem 33.3.** *For any $n \times n$ matrices $A$ and $B$,*

*(a)* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ *and* $\operatorname{tr}(B^{-1}AB) = \operatorname{tr}(A)$.

*(b)* $\det(AB) = \det(BA)$ *and* $\det(B^{-1}AB) = \det(A)$.

*Thus similar matrices have the same trace and the same determinant.*

*Proof.* The $(ik)^{th}$ entry of the product matrix $AB$ is the dot product of the $i^{th}$ row of $A$ and the $k^{th}$ column of $B$, so the formula is $(AB)_{ik} = \sum_j A_{ij}B_{jk}$. Hence

$$\operatorname{tr}(AB) = \sum_i (AB)_{ii} = \sum_i \sum_j A_{ij}B_{ji} = \sum_i \sum_j B_{ji}A_{ij} = \sum_i (BA)_{jj} = \operatorname{tr}(BA).$$

This gives (a), and (b) follows (HW Problem 1). □

**Proposition 33.4.** *For any square matrix $A$,*

*(a) The sum of the eigenvalues of $A$ is $\operatorname{tr} A = \sum \lambda_i$.*

*(b) The product of the eigenvalues of $A$ is $\det A$.*

*Proof.* Suppose that $A$ is diagonalizable, so there is a matrix $Q$ such that $A = Q^{-1}DQ$ where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \ddots \end{pmatrix}$

Then

$$\operatorname{tr}(A) = \operatorname{tr}(Q^{-1}DQ) = \operatorname{tr}(D) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

and

$$\det(A) = \det(Q^{-1}DQ) = \det(D) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n.$$

The Proposition is still true for matrices that cannot be diagonalized; we will give a proof later. □

**Three useful tricks:** for finding the eigenvalues and determinants:

(1.) $\operatorname{tr}(A) =$ sum of the eigenvalues.

(2.) $\det A =$ product of the eigenvalues.

(3.) For matrices in block form $\quad \det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det A \cdot \det C = \det\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}.$

**Example 2.** To find the eigenvalues $\lambda_1, \lambda_2$ of

$$C = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix},$$

note that $\lambda_1 + \lambda_2 = \operatorname{tr} C = 3$ and $\lambda_1\lambda_2 = \det C = 2$; some trial-and-error then leads to $\lambda_1 = 1$ and $\lambda_2 = 2$.

For larger matrices, the eigenvalues are not determined by the trace and the determinant, but Proposition 33.4 can still be used to check your calculations and when you already know several eigenvalues.

**Example 3.**

$$\det\begin{pmatrix} 2 & 3 & 5 & 4 \\ 1 & 8 & 7 & -2 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 3 & 2 \end{pmatrix} = \det\begin{pmatrix} 2 & 3 \\ 1 & 8 \end{pmatrix} \cdot \det\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} = 13 \cdot 11 = 143.$$

# Homework 33

1. Use the fact that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ to prove that $\operatorname{tr}(Q^{-1}AQ) = \operatorname{tr}(A)$.
2. Use the "trick" described above (or some other method) to find the eigenvalues for the following matrices. you do not have to find the eigenvectors.

$$(a)\ \begin{pmatrix} 5 & 11 \\ 1 & -5 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} \qquad (c)\ \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}$$

3. Consider the matrix $A = \begin{pmatrix} 2 & 1 & -3 \\ 4 & -1 & 5 \\ -1 & 1 & 0 \end{pmatrix}$.
   (a) Check that $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ is an eigenvector. What is its eigenvalue?
   (b) Use Tricks 1 and 2 to find the other two eigenvectors of $A$.
4. Write out the details of Example 1 of this section.
5. Physics books define three standard $3 \times 3$ matrices that are usually called $J_x, J_y$ and $J_z$. The last is:

$$J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   (a) What are the eigenvalues of $J_x$?
   (b) Show that $\mathbf{v}_1 = \mathbf{e}_1 - i\mathbf{e}_2, \mathbf{v}_2 = \mathbf{e}_1 + i\mathbf{e}_2$ and $\mathbf{v}_3 = \mathbf{e}_3$ is a basis of eigenvectors for $J_z$.
   (c) Write down the diagonalization of $J_z$ over $\mathbb{C}$ (order the eigenvalues and eigenvectors as in (b) ).
   (d) What is $e^{tJ_z}$ as a complex matrix?