# Day 26 Calculating determinants for larger matrices

We now proceed to define $\det A$ for $n \times n$ matrices $A$. As before, we are looking for a function of $A$ that satisfies the product formula

$$\det(AB) = \det A \cdot \det B.$$

The key to understanding determinants is not to focus on a formula, but instead to think about the properties of determinants in terms of the rows of the matrix. The following theorem lists three properties that completely characterize the determinant function (without giving a formula for it!).

**Determinant Theorem.** *There is a one and only one function* $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ *which, when considered as a function of the row vectors*

$$\det A \;=\; \det \begin{pmatrix} -\mathbf{v}_1- \\ -\mathbf{v}_2- \\ \vdots \\ -\mathbf{v}_n- \end{pmatrix}$$

*satisfies*

1. *It is linear in each row. Thus*

$$\det \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ r\,\mathbf{v}_i \\ \vdots \\ \mathbf{v}_n \end{pmatrix} = r \det \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_n \end{pmatrix} \qquad\qquad \det \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i + \mathbf{w}_i \\ \vdots \\ \mathbf{v}_n \end{pmatrix} = \det \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_n \end{pmatrix} + \det \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{w}_i \\ \vdots \\ \mathbf{v}_n \end{pmatrix}$$

2. $\det A$ *changes sign when two rows of* $A$ *are interchanged.*
3. $\det I_n = 1$.

Before giving the proof, we give develop some consequences. Conditions 1, 2, and 3 imply:

4. $\det A = 0$ if one row of $A$ is all 0's, or if two rows are equal.
5. Adding a multiple of one row to another doesn't change $\det A$:

$$\det \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i + r\mathbf{v}_j \\ \vdots \\ \mathbf{v}_n \end{pmatrix} = \det \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_n \end{pmatrix} \qquad \text{for } i \neq j$$

6. The determinant of an upper triangular matrix is the product of the diagonal entries:

$$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a_{22} & \cdots & a_{2m} \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nm} \end{pmatrix} = a_{11}a_{22}\cdots a_{nn}$$

7. Determinants can be computed by expansion along any row: e.g. for the first row

$$\det A \;=\; \sum_{i=1}^{n} (-1)^i \; a_{1i} \; \det m_{1i}(A)$$

where $m_{1i}(A)$ is the $1i$ cofactor of $A$.

8. Repeatedly substituting the above formula into itself yields:

$$\det A \;=\; \sum_{R=i_1,i_2,\cdots i_n} \sum \cdots \sum (-1)^{\text{sign } R} \; a_{1i_i}a_{2i_2}\cdots a_{ni_n} \tag{0.1}$$

where the sum if over all $n!$ rearrangements of the numbers $\{1, 2, \cdots, n\}$.

*Proof. of the Determinant Theorem.* We have just shown that any function $f(A)$ that satisfies Properties 1-3 must be given by the formula (0.1). Thus there is at most one such function. On the other hand, it is easy to check that the function $\det A$ defined by formula (0.1) satisfies Properties 1-3, giving existence. $\square$

**Important Tip – how to calculate.** While $\det A$ is defined by (0.1), this ugly and complicated formula is seldom used! In practice, *one calculates determinants by row-reducing the matrix, using Properties 1-6 to see how the determinant changes at each step.*

**Example.**

$$\det \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 2 & 8 \end{pmatrix} = 2\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 2 & 8 \end{pmatrix} = -4\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 3 & 1 \end{pmatrix} = -4\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -11 \end{pmatrix} = -4(-11) = 44.$$

**Determinant Invertibility Test.** A square matrix $A$ is invertible (i.e. non-singular) if and only if $\det A \neq 0$.

*Proof.* We can convert $A$ to a nearly row echelon matrix $B$ by interchanging rows and adding a multiple of one row to another. By Properties 2, 5 and 6 we then have

$$\det A = \pm \det B = \pm \det \begin{pmatrix} b_{11} & * & \cdots & * \\ 0 & b_{22} & * & \\ 0 & \cdots & & \\ 0 & \cdots & & b_{nn} \end{pmatrix} = \pm b_{11} b_{22} \cdots b_{nn}.$$

Thus $\det A \neq 0$ if and only if every pivot $b_{kk}$ of $B$ is non-zero. But having non-zero pivots on the diagonal is exactly the criterion for $A$ to be invertible.
$\square$

**Determinant Product Theorem.** $\det AB = \det A \cdot \det B$ for any two $n \times n$ matrices $A, B$.

This very important fact is very difficult to verify directly from the formula (0.1). The trick to proving it is to use the properties, rather than the formula.

*Proof.* Fix a matrix $B$. If $\det B = 0$ then $B$ is singular by the Determinant Invertibility Test, so there is a vector $\mathbf{v}$ with $B\mathbf{v} = 0$. But then $AB(\mathbf{v}) = 0$, so $AB$ is singular and hence $\det AB = 0$. Thus the formula $\det AB = \det A \cdot \det B$ is trivially true in this case. We can therefore that $\det B \neq 0$ for the rest of the proof.

Define a function $f$ of $A$ by

$$f(A) = \det(AB)$$

This satisfies properties 1. and 2. of the Determinant Theorem, and $f(I_n) = \det(I_n B) = \det B$. Therefore the function

$$g(A) = \frac{1}{\det B} f(A)$$

satisfies 1. 2. and 3., so by the uniqueness in the Determinant Theorem must be the function $g(A) = \det A$. Thus

$$\frac{1}{\det B} \det(AB) = \det A$$

as needed. $\square$

# Homework 26

1. Use the product theorem to prove that the following hold for any $n \times n$ matrices $A$ and $B$
   (a) $\det AB = \det BA$.
   (b) If $A$ is non-singular then $\det A^{-1} = (\det A)^{-1}$.
   (c) If $B$ is a non-singular then $\det(B^{-1}AB) = \det A$.
2. Use row operations to find the determinant of the matrix $A$ in Problem 18 on page 222 of the textbook.
3. Do the same for the matrix in Problem 21.
4. (a) Use the formula for $\det A$ to prove that $\det(rA) = r^n \det A$ for any $n \times n$ matrix $A$ and $r \in \mathbb{R}$.
   (b) Given a second explanation using the fact that $\det A$ is linear in the row vectors of $A$ (i.e. Property 1 of the Determinant Theorem).
5. **Read Section 4.4 of the textbook.** Then answer all parts of Problem 1 on page 236 (these are True-False questions).

## Day 27 Eigenvalues and Eigenvectors

Linear transformations $T : V \to V$ that go from a vector space to itself are often called linear *operators*. Many linear operators can be understood geometrically by identifying directions in which $T$ acts as a dilation: for each vector $\mathbf{v}$ in these directions, $T(\mathbf{v})$ is a multiple of $\mathbf{v}$. Such vectors $\mathbf{v}$ are called *eigenvectors* and the corresponding multiples are called *eigenvalues*. Chapter 5 explains how to find eigenvalues and eigenvectors, and how to use them to understand linear operators.

Here is the main definition (memorize this!)

**Definition.** *Let $T : V \to V$ be a linear operator. A scalar $\lambda$ ("lambda") is called an* **eigenvalue** *of $T$ if there is a non-zero vector $\mathbf{v}$ in $V$ such that*

$$T(\mathbf{v}) \;=\; \lambda\mathbf{v}.$$

*The vector $\mathbf{v}$ is called an* **eigenvector** *of $T$ corresponding to the eigenvalue $\lambda$.*

Each $n \times n$ matrix $A$ specifies an operator $A : \mathbb{R}^n \to \mathbb{R}^n$, so we can express the above definition in terms of the matrix: A scalar $\lambda$ is an **eigenvalue** of $A$ if there is a non-zero $\mathbf{v} \in \mathbb{R}^n$ such that

$$A\mathbf{v} \;=\; \lambda\mathbf{v}.$$

The vector $\mathbf{v}$ is called an **eigenvector** of $A$ corresponding to $\lambda$.

**Finding Eigenvalues.** To solve the equation $A\mathbf{v} = \lambda\mathbf{v}$, we treat both $\mathbf{v}$ and $\lambda$ as unknowns — think of $\lambda$ as a variable much like one treats $x$ in high school algebra. The key fact is:

**Theorem 5.2** *Let $A$ be an $n \times n$ matrix. Then a scalar $\lambda$ is an eigenvalue of $A$ if and only if* $\det(A - \lambda I_n) = 0$.

*Proof.*

$$\begin{aligned} A\mathbf{v} = \lambda\mathbf{v} &\iff (A-\lambda I)\mathbf{v} = 0 \\ &\iff (A-\lambda I) \text{ is singular} \\ &\iff \det(A-\lambda I) = 0. \end{aligned}$$

□

**Definition.** The **characteristic polynomial** of a square matrix $A$ is

$$p_A(\lambda) = \det(A-\lambda I)$$

If $A$ is an $n \times n$ matrix, $p_A(\lambda)$ is a polynomial in $\lambda$ of the form

$$p_A(\lambda) \;=\; (-1)^n\lambda^n \;+\; \text{lower degree terms}.$$

Theorem 5.2 can then be rephrased as follows.

**Theorem 5.2** *The eigenvalues of $A$ are precisely the roots of the characteristic polynomial.*

This gives a way of finding the eigenvalues:

**Finding Eigenvectors.** Looking at the eigenvalue $\lambda_i$ one at a time, one solves the equation $A\mathbf{v} = \lambda\mathbf{v}$ for each. The set of solutions is a subspace:

**Definition.** For each eigenvalue $\lambda$ of a linear operator $T : V \to V$, the corresponding **eigenspace** is the set of all eigenvectors for with eigenvalue $\lambda$:

$$E_\lambda(T) \;=\; \left\{\mathbf{v} \in V \;\middle|\; T(\mathbf{v}) = \lambda\mathbf{v}\right\}.$$

This is a subspace for each $\lambda$ (see Problem 1 below).

**Method for finding eigenvalues and eigenspaces:** Given a square matrix $A$,

1. Find the characteristic polynomial by writing down the matrix $A-\lambda I$ (= $A$ with $\lambda$ subtracted from each diagonal entry), and calculating the determinant (and changing sign if necessary).

2. Factor the characteristic polynomial as much as possible; each linear factor $(\lambda - c)$ then gives an eigenvalue $c$.

3. For each eigenvalue $\lambda_i$, the eigenspace $E_A(\lambda_i)$ is the set of solutions to $(A-\lambda_i I)\mathbf{v} = \mathbf{0}$. Solve this system by Gaussian elimination or by inspection.

Caution: For some matrices $A$, the characteristic polynomial has no linear factors, so there are no eigenvalues.

# Homework 27

Read Section 5.1 of the textbook through the end of Example 7.

1. Let $T : V \to V$ be a linear operator. Prove that $E_\lambda(T)$ is a subspace (not just a subset) of $V$.

2. (a) Explain why $E_0(T)$ is the null space $N(T)$ of $T$ (recall that the null space is often called the kernel $\ker T$ of $T$).

   (b) Similarly explain why $E_\lambda(T)$ is the null space $N(T-\lambda I)$ of the linear transformation $T - \lambda I$.

3. The matrix $A = \begin{pmatrix} 1 & 2 \\ 5 & -2 \end{pmatrix}$ has $\lambda = 3$ as an eigenvalue. Find the corresponding eigenvector.

4. Let $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Find a basis for $E_2(A)$ and a basis for $E_{-1}(A)$.

5. Let $L : P_5 \to P_5$ by $L = \frac{d^2}{dx^2}$.

   (a) Find a basis for $E_0(L)$. *Hint: polynomials in $E_0(L)$ must satisfy what equation?*

   (b) Show that $L$ has no non-zero eigenvalues.

6. Find the eigenvalues of the following matrices. For each eigneblaue, find a basis for the corresponding eigenspace.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 6 & -24 & -4 \\ 2 & -10 & -2 \\ 1 & 4 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & -7 & -4 \\ -1 & 9 & 4 \\ 2 & -14 & -6 \end{pmatrix} \qquad D = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

7. Find all eigenvalues and eigenvectors of the following matrices.

$$E = \begin{pmatrix} -2 & 7 \\ 0 & 3 \end{pmatrix} \qquad F = \begin{pmatrix} 5 & -4 & 3 \\ 0 & -1 & 9 \\ 0 & 0 & 1 \end{pmatrix} \qquad G = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \qquad H = \begin{pmatrix} 3 & 7 & 4 & 1 \\ 0 & -1 & 2 & 8 \\ 0 & 0 & 2 & -7 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

   (e) What feature of these matrices make it easy to find their eigenvalues?

8. Write down a $2 \times 2$ matrix with eigenvalues 2 and 5. Make your matrix as simple as possible.

9. (a) Show that any symmetric $2 \times 2$ matrix, that is one of the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ has eigenvalues.

   (b) Under what conditions on $a$ and $b$ will the matrix have two distinct eigenvalues?

10. Prove that a square matrix is singular if and only 0 is one of its eigenvalue.

## Day 28 Similarity and Diagonalizability

Let $V$ be the vector space $R^n$ or $\mathbb{C}^n$ with standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Then each linear operator $T : V \to V$ can be written as a matrix $A = [T]_\alpha^\alpha$ and, going backwards, each $n \times n$ matrix $A$ determines a linear operator $T : V \to V$. Furthermore, if $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is another basis, then the matrix $B = [T]_\beta^\beta$ of $T$ in this $\beta$ basis is

$$B = Q^{-1}AQ \tag{0.2}$$

where $Q = [I]_\beta^\alpha$ is the change-of-basis matrix (whose $i$th column is the coordinates of $\mathbf{v}_i$ in the standard basis). This formula motivates the following definition.

**Definition** *Two $n \times n$ matrices $A$ and $B$ are* **similar** *if there is an invertible $n \times n$ matrix $Q$ such that $B = Q^{-1}AQ$.*

**Proposition 1.** *Two similar matrices have (a) the same determinant, (b) the same characteristic polynomial, and (c) the same eigenvalues.*

*Proof.* As shown in class, all three statements follow from the fact that $\det(AB) = \det A \cdot \det B$. $\square$

**Example 1.** The matrices $A = \begin{pmatrix} 2 & -4 \\ 3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 1 \\ 4 & 2 \end{pmatrix}$ are not similar because $\det A = 22$, while $\det B = 10$.

It is natural to ask whether a given matrix is similar to one that has an especially simple form, specifically whether it is similar to a diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

**Definition** *A square matrix $A$ is* **diagonalizable** *if it is similar to a diagonal matrix.*

**Lemma 2.** *Let $T : V \to V$ be a linear operator. Then any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of non-zero eigenvectors of $T$ with distinct eigenvalues are linearly independent.*

*Proof.* This is proved on page 261 of the textbook, and was done in a slightly different way in class. $\square$

**Theorem 3.** *An $n \times n$ matrix $A$ is has distinct eigenvalues, then $A$ is diagonalizable.*

*Proof.* Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear operator whose matrix in the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is $A$. The hypothesis that $A$ has distinct eigenvalues means that there are vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and constants $\{\lambda_1, \mathbf{v}_2, \ldots, \lambda_n\}$, all different, so that $T\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for each $i$. By Lemma 2, $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of $n$ linearly independent vectors in an $n$-dimensional space, so is a basis. Since $T$ multiplies each $\mathbf{v}_i$ by $\lambda_i$, the matrix of $T$ in the $\beta$ basis is

$$D = [T]_\beta^\beta = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

The change-of-basis formula (0.2) above then shows that $D = Q^{-1}AQ$, so $A$ is diagonalizable.
$\square$

**Procedure:** To diagonalize an $n \times n$ matrix $A$ with distinct eigenvalues:

1. Find the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$.
2. Factor as much as possible to find the eigenvalues $\lambda_i$ (check the discriminant $b^2 - 4ac$ to see if quadratics factor).

If the eigenvalues are distinct, then $A$ is diagonalizable. To diagonalize:

3. Find an eigenvector for each eigenvalue $\lambda_i$.
4. Write down the matrix $Q$ whose columns are your basis eigenvectors.
5. Compute $Q^{-1}$.
6. Then $D = Q^{-1}AQ$ is the diagonal matrix with the eigenvalues down the diagonal.

Notice that *you can find $D$ without first finding $Q$* because $D$ is a diagonal matrix with the eigenvalues $\lambda_i$ (found in Step 2) on the diagonal. Furthermore, Homework Problem 4 below shows that the order to the $\lambda_i$ doesn't matter: if $A$ is similar to the diagonal matrix $D$ with $\lambda$s down the diagonal in one order, it is also similar to the diagonal matrix $D'$ with the same $\lambda$s on the diagonal in any other order.

**Example 2.** Diagonalize $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$.

The characteristic polynomial is $p(\lambda) = \det \begin{pmatrix} 5-\lambda & 6 \\ -2 & -2-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$, so the eigenvalues are $\lambda = 1, 2$. These are different, so $A$ is diagonalizable.

Next solve $A\mathbf{v} = \lambda\mathbf{v}$ in the form $(A - \lambda I)\mathbf{v} = 0$ using augmented matrices in the two cases:

- For $\lambda = 1$, solve $(A - I)\mathbf{v} = 0$ $\begin{pmatrix} 4-\lambda & 6 & 0 \\ -2 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ so one solution is $\mathbf{v}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.
- For $\lambda = 2$, solve $(A - 2I)\mathbf{v} = 0$ $\begin{pmatrix} 3 & 6 & 0 \\ -2 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ so one solution is $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

$Q$ is the matrix whose columns are these eigenvectors, and we can write down $Q^{-1}$ by the usual trick:

$$Q = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \qquad\qquad Q^{-1} = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}$$

and then $D = Q^{-1}AQ = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is diagonal.

# Homework 28

1. Do Problem 1 on page 256 of the textbook.
2. Do Problem 2(a)(c) on the same page.
3. Do Problem 3 on page 257 of the textbook.
4. Show that $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is similar to $B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$. *Hint: write down the change-of-basis matrix $P$ that changes the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{e}_2, \mathbf{e}_1\}$.*
5. Do Problem 8a on page 258.
6. Then do Problem 8b.
7. Do Problem 9 on the same page.
8. For the matrix $A = \begin{pmatrix} 2 & -3 \\ 2 & -5 \end{pmatrix}$ follow the procedure and the Example above to find:
   (a) Find the characteristic polynomial and the eigenvalues.
   (b) Find eigenvectors for each eigenvalue.
   (c) Write down a matrix $Q$ so that $Q^{-1}AQ$ is diagonal.
   (d) Find $Q^{-1}$ and explicitly calculate $Q^{-1}AQ$ to show that it is diagonal.