Day 23 Kernels, inverses and isomorphisms

Today we will cover Section 2.4 of the textbook, which asks and answers the question "When does a linear transformation have an inverse?"'

First, recall three basic definitions that apply to maps $T : V \to W$ between two sets.

- T is **onto** if $R(T) = W$, that is, for each $\mathbf{w} \in W$ there is a $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$.
- T is **one-to-one** if $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ implies that $\mathbf{v}_1 = \mathbf{v}_2$.
- The **inverse** of $T : V \to W$ is a transformation $S : W \to V$ such that

$$S \circ T = I_V \qquad \text{and} \qquad T \circ S = I_W \tag{23.1}$$

  where $I_V : V \to V$ is the identity transformation of V, and I_W is the identity transformation on W.

Often, no inverse exists. The above conditions – onto and one-to-one — are exactly what is needed to guarantee that an inverse exists:

Lemma. A map $T : V \to W$ between sets is invertible $\Longleftrightarrow$ it is onto and one-to-one.

Proof. If T is one-to-one and onto, then for each $\mathbf{w} \in W$ there is a unique $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$. Define S by $S(\mathbf{w}) = \mathbf{v}$. Then $T(S(\mathbf{w})) = T(\mathbf{v}) = \mathbf{w}$ and $S(T(\mathbf{v})) = S(\mathbf{w}) = \mathbf{v}$, so (23.1) holds.

Conversely, suppose that T is invertible. Then

- T is onto because for each $\mathbf{w} \in W$ the element $\mathbf{v} = S(\mathbf{w}) \in V$ satisfies $T(\mathbf{v}) = T(S(\mathbf{w})) = \mathbf{w}$.
- T is one-to-one because if $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ then $\mathbf{v}_1 = S(T(\mathbf{v}_1)) = S(T(\mathbf{v}_2)) = \mathbf{v}_2$.

  □

Theorem. If a linear transformation $T : V \to W$ is invertible, then T^{-1} is also linear.

Proof. For any $\mathbf{w}_1, \mathbf{w}_2 \in W$, set $\mathbf{v}_1 = T^{-1}(\mathbf{w}_1)$ and $\mathbf{v}_2 = T^{-1}(\mathbf{w}_2)$. Then $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$, so for any $a.b \in \mathbb{R}$ we have

$$\begin{aligned} T^{-1}(a\mathbf{w}_1 + b\mathbf{w}_2) \;=\; T^{-1}(aT(\mathbf{v}_1) + bT(\mathbf{v}_2)) \;&=\; T^{-1}(T(a\mathbf{v}_1 + b\mathbf{v}_2)) \\ &=\; a\mathbf{v}_1 + b\mathbf{v}_2 \\ &=\; aT^{-1}(\mathbf{w}_1) + bT^{-1}(\mathbf{w}_2). \end{aligned}$$

Thus T^{-1} is a linear transformation. □

Definition (a) A linear map $T : V \to W$ is called an *isomorphism* if it is invertible.

(b) Two vector spaces V and W are *isomorphic* if there exists an isomorphism $T : V \to W$. (If there is one, then there will be many such T).

Isomorphism Theorem. Let V and W be finite-dimensional vector spaces. Then

(a) V and W are isomorphic if and only if $\dim V = \dim W$.

(b) For a linear transformation $T : V \to W$ between vector spaces *of the same dimension*

$$T \text{ is an isomorphism } \Longleftrightarrow\; N(T) = 0 \;\Longleftrightarrow\; R(T) = W$$

Proof. (a) If V and W are isomorphic, then there is an invertible transformation $T : V \to W$. Invertible transformations are one-to-one and onto, so $\dim \ker T = 0$ and $\dim R(T) = \dim W$. Then the Rank-Nullity Theorem says: $\dim V = \dim N(T) + \dim R(T) = \dim W$.

Conversely, if $\dim V = \dim W$ then we can choose bases $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ of V and $\{\mathbf{w}_1, \ldots \mathbf{w}_n\}$ of W with the same number of elements. Then

$$T(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) \;=\; a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n.$$

defines a linear transformation that has an inverse, namely $T^{-1}(a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n) \;=\; a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$.

(b) If $\dim V = \dim W$, the Rank-Nullity Theorem says that $\dim N(T) + \dim R(T) = \dim W$, which implies that $N(T) = 0 \Leftrightarrow \dim R(T) = \dim W \Leftrightarrow R(T) = W$. We then have

$$T \text{ is an isomorphism} \quad\Longleftrightarrow\quad \begin{array}{c} N(T) = 0 \\ \text{and} \\ R(T) = 0 \end{array} \quad\Longleftrightarrow\quad T \text{ is one-to-one and onto}$$

where the first equivalence is from the Lemma above, and the second equivalence is true because, T is one-to-one if and only if $N(T) = 0$ and is onto if and only if $R(T) = W$. $\square$

Relating to matrices. After fixing bases $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V and $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ of W, each linear transformation $T : V \to W$ has an associated matrix $A = [T]_\alpha^\beta$. The properties of T with respect to inverses carry over to the matrix:

Matrix Invertibility Theorem. With the above notation,

(a) A is invertible if and only if T is invertible, and A^{-1} is $[T^{-1}]_\beta^\alpha$.

(b) A is invertible if and only if it is *square* and its row echelon form has a pivot in every column.

Proof. Statement (a) is an immediate consequence of the matrix composition property:

$$[T^{-1}]_\beta^\alpha \cdot [T]_\alpha^\beta \;=\; [T^{-1} \circ T]_\alpha^\alpha \;=\; [I_V]_\alpha^\alpha \;=\; I_n$$

and similarly in the opposite order. Statement (b) then follows from what we already know about matrices. $\square$

Homework 23

1. Let $Q : \mathbb{R}^4 \to M(2,2)$ be the transformation $Q\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
   (a) Show that Q is linear.
   (b) Show that Q is invertible by defining a linear map $R : M(2,2) \to \mathbb{R}^4$ and showing that $Q \circ R = I$ and $R \circ Q = I$.

2. Prove that the vector spaces P_3 and $M(2,2)$ are isomorphic by defining a transformation $T : P_3 \to M(2,2)$ and showing that it is linear and invertible.

3. Which pairs of the following vector spaces are isomorphic?

$$\mathbb{R}^7 \qquad \mathbb{R}^{12} \qquad M(3,3) \qquad M(3,4) \qquad M(4,3) \qquad P_6 \qquad P_8 \qquad P_{11}.$$

4. Which of the following matrices is invertible? *Use the Matrix Invertibility Theorem stated above.*

$$(a)\quad \begin{pmatrix} 1 & 5 \\ -2 & -6 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix} \qquad (c)\quad \begin{pmatrix} 1 & 1 & 0 & -1 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & -1 \\ -1 & 0 & 1 & 1 \end{pmatrix}$$

5. Do Problems 2a-d on page 106 of the textbook.

6. Do Problems 3a-c on the next page.

7. Continuing, do Problem 6. *This can be done in one line!*

8. Do Problem 7 using these hints:

   (a) *Start by applying A^2 to any non-zero vector.*

   (b) *Find 2×2 matrices A and B with Range$(B) =$ span$(\mathbf{e}_1)$ and with $\mathbf{e}_1 \in N(A)$.*

9. Do Problem 17 on page 108 using these hints:

   (a) *For any $\mathbf{w}_1 = T(\mathbf{v}_1)$ and $\mathbf{w}_2 = T(\mathbf{v}_2)$ in $T(V_0)$, show that each linear combination $a\mathbf{w}_1 + b\mathbf{w}_2$ is in $T(V_0)$.*

   (b) *Pick a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of V_0 and show that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is a basis of V_0.*

Day 24 Changing Bases

Today we will cover Section 2.5 of the textbook. Read the beginning of this section to understand the motivation. The basic question to be solved is the following.

Suppose that $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\alpha' = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$ are two different bases of a vector space V. Given a vector $\mathbf{v} \in V$, we can write $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$ and form the column vector

$$[\mathbf{v}]_\alpha \;=\; \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

When the same vector $\mathbf{v}$ is expanded in the basis α' it has different coefficients

$$[\mathbf{v}]_{\alpha'} \;=\; \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Question: *How are the x's and y's related?*

To answer, recall that whenever we have a linear transformation $T : V \to W$ and bases α of V and β of W, there is an associated matrix $[T]_\alpha^\beta$. In particular, if T is the identity map $I : V \to V$ and we have two different bases on V, then there is an associated matrix

$$Q \;=\; [I_V]_{\alpha'}^{\alpha}$$

This matrix is easy to write down because

$$Q \;=\; [I_V]^{\alpha}_{\alpha'} \;=\; \text{the matrix whose } i^{th} \text{ column is } [\mathbf{v}'_i]_\alpha.$$

That is, expand each element of the α' basis in terms of the first basis, and assemble the coefficients as column vectors of a matrix Q.

Basis Change Lemma. The matrix $Q \;=\; [I_V]^{\alpha}_{\alpha'}$

(a) is invertible with $Q^{-1} = [I_V]^{\alpha'}_{\alpha}$

(b) For any $\mathbf{v} \in V$, we have $[\mathbf{v}]_\alpha = [T]^{\alpha}_{\alpha'}\,[\mathbf{v}]_{\alpha'}$, i.e. $[\mathbf{v}]_\alpha = Q[\mathbf{v}]_{\alpha'}$.

This is proved on page 111 of the textbook. Note that (c) says that it solves the above problem: $\mathbf{x} = Q\mathbf{y}$. Accordingly, Q is called the **change of coordinate matrix** for **changing α'-coordinates to α-coordinates**.

Here are the main facts in this section:

Given two bases α and α' of a vector space V, write down the matrix

$$Q \;=\; [I_V]^{\alpha}_{\alpha'} \;=\; \text{the matrix whose } i^{th} \text{ column is } [\mathbf{v}'_i]_\alpha.$$

Then

- *Multiplication by Q changes α'-coordinates to α-coordinates:*

$$[\mathbf{v}]_\alpha = Q[\mathbf{v}]_{\alpha'}$$

- *Multiplication by Q^{-1} changes α-coordinates to α'-coordinates:*

$$[\mathbf{v}]_{\alpha'} = Q^{-1}[\mathbf{v}]_\alpha \qquad \boxed{\text{What's usually wanted}}$$

- *For each linear operator $T : V \to V$*

$$[T]^{\alpha'}_{\alpha'} \;=\; [I_V]^{\alpha'}_{\alpha} \cdot [T]^{\alpha}_{\alpha} \cdot [I_V]^{\alpha}_{\alpha'} \qquad \boxed{\text{Note how the subscripts are arranged.}}$$

Thus if T has matrix $A = [T]^{\alpha}_{\alpha}$ in the basis α, then its matrix in the basis α' is

$$B = Q^{-1}AQ.$$

Exercise. Verify the formula in the last bullet point by applying the righthand side to a the coordinate vector $[\mathbf{v}]_{\alpha'}$ of $\mathbf{v} \in V$, using the formulas from the first two bullet points, and showing that you obtain $[T\mathbf{v}]_{\alpha'}$.

According to the last formula in the box, the matrices A and $B = Q^{-1}AQ$ represent the *same linear operator* $T : V \to V$, but expressed using different bases. When this is the case, the matrices are called similar.

Defintion. Two $n \times n$ matrices A and B are said to be *similar* if there is an invertible $n \times n$ matrix Q such that $B = Q^{-1}AQ$.

Example. (a) Let $\mathbf{v}$ be the vector whose coordinates are $\binom{5}{-2}$ in the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$. What are its coordinates in the basis $\{\mathbf{f}_1, \mathbf{f}_2\}$ where $\mathbf{f}_1 = \binom{3}{1}$ and $\mathbf{f}_2 = \binom{-3}{4}$?

(b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\begin{pmatrix} 2 & 3 \\ 1 & 7 \end{pmatrix}$ in the standard basis, what is it in the basis $\{\mathbf{f}_1, \mathbf{f}_2\}$?

Solution: $Q = [I]^{\mathbf{f}}_{\mathbf{e}}$ is the matrix whose columns are the $\mathbf{f}$'s, so $Q = \begin{pmatrix} 3 & -3 \\ 1 & 4 \end{pmatrix}$. Using our trick for writing down the inverse of a 2×2 matrix, $Q^{-1} = \frac{1}{15}\begin{pmatrix} 4 & 3 \\ -1 & 3 \end{pmatrix}$. Then

$$\text{(a)} \qquad [\mathbf{v}]_{\mathbf{f}} = Q^{-1}[\mathbf{v}]_{\mathbf{e}} = \frac{1}{15}\begin{pmatrix} 4 & 3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \frac{1}{15}\begin{pmatrix} 14 \\ -11 \end{pmatrix}$$

$$\text{(b)} \qquad [T]^{\mathbf{f}}_{\mathbf{f}} = Q^{-1}AQ = \frac{1}{15}\begin{pmatrix} 4 & 3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 3 & -3 \\ 1 & 4 \end{pmatrix}$$

In most problems involving change-of-basis, Q is easy to write down but, unfortunately, *one usually has to compute* Q^{-1}.

Homework 24

1. In this exercise, you are given a basis $\mathbf{f} = \{\mathbf{f}_1, \mathbf{f}_2\}$ for $\mathbb{R}^2$ (or for $\mathbb{R}^3$) and the $\mathbf{e}$-coordinates of a vector $\mathbf{v}$. Find Q, Q^{-1} and $[\mathbf{v}]_f$. *Follow the example on this handout.*

   (a) $\mathbf{f}_1 = \binom{3}{2}$, $\mathbf{f}_2 = \binom{4}{3}$, and $[\mathbf{v}]_{\mathbf{e}} = \binom{2}{5}$.

   (b) $\mathbf{f}_1 = \binom{1}{3}$, $\mathbf{f}_2 = \binom{3}{1}$, and $[\mathbf{v}]_{\mathbf{e}} = \binom{3}{7}$.

   (c) $\mathbf{f}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{f}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{f}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, and $[\mathbf{v}]_{\mathbf{e}} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$.

2. Problems 2a and 2c on page 116 of the textbook (these ask you just to find Q).
3. Problem 3c.
4. Problem 4.
5. Problem 5.
6. Problem 6a and 6c.

Day 25 Determinants of 2×2 and 3×3 matrices

Chapter 4 of the textbook is about determinants. The determinant is a function that assigns to every square (must be square!) matrix A a number

$$\det A$$

so that

- $\det A \neq 0 \iff A$ is invertible.
- $\det(AB) \;=\; \det A \cdot \det B$
- $\det A$ is related to volume in a way explained below.

These properties make determinants extremely useful. But, as you will see, the formula for the determinant is far from obvious. Its discovery was one of the triumphs of 19th century mathematics.

Determinants of 2×2 matrices

Recall that the inverse of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Thus A is invertible if and only if the number $ad - bc$ is not 0. Accordingly, for 2×2 matrices, we define

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; ad - bc. \tag{24.2}$$

One can then verify that $\det(AB) \;=\; \det A \cdot \det B$ (HW Problem 3).

To better understand this formula, consider it as function of row vectors. For row vectors $\mathbf{v}_1 = (a, b)$ and $v_2 = (c, d)$, write

$$\det \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} \;=\; \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; ad - bc.$$

Thought if this way, the determinant has three algebraic properties:

(a) It is linear in each row vector:

$$\det \begin{pmatrix} r\mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = r \det \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} \qquad \text{and} \qquad \det \begin{pmatrix} \mathbf{v}_1 + \mathbf{u}_1 \\ \mathbf{v}_2 \end{pmatrix} = \det \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} + \det \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{v}_2 \end{pmatrix}$$

and similarly for the second row.

(b) It switches sign when the order of the rows are reversed: $\det \begin{pmatrix} \mathbf{v}_2 \\ \mathbf{v}_1 \end{pmatrix} = -\det \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}$.

(c) It is equal to 1 if the row vectors are the standard basis vectors: $\det \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} \;=\; \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;=\; 1$.

The determinant also has a geometric property that should be familiar from calculus:

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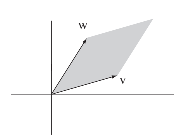

(24.3)

The *sign* of the determinant is determined by the "righthand rule": $\det\begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix}$ is positive if $\mathbf{v}_2$ points in a direction obtained by rotating $\mathbf{v}_1$ counterclockwise through an angle θ with $0 < \theta < \pi$, and is negative if the same is true for a clockwise rotation.

Example. For the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ of $\mathbb{R}^2$,

$$\det\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1 > 0 \qquad\qquad \det\begin{pmatrix}\mathbf{e}_2\\ \mathbf{e}_1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = -1 < 0.$$

Determinants of 3×3 matrices

For row vectors $\mathbf{v}_1 = (a, b, c)$, $\mathbf{v}_2 = (d, e, f)$ and $\mathbf{v}_3 = (g, h, k)$, $\det\begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\end{pmatrix}$ is defined to be

$$\det\begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\end{pmatrix} = \det\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & k\end{pmatrix} = a(ek - hf) - b(dk - gf) + c(dh - eg).$$

As in the $2 \times 2 case$, this satisfies

(a) is linear in each row vector,

(b) switches sign when the order of the rows are reversed, and

(c) It is equal to 1 if the row vectors are the standard basis vectors: $\det\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} = \det\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = 1.$

(d) The sign of the determinant is positive if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are oriented compatibly with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and the absolute value of the determinant satisfies

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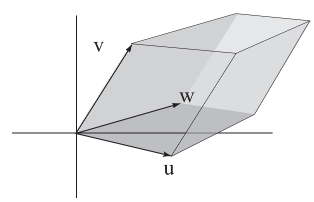

Computing 3×3 Determinants

Definition. For an $n \times n$ matrix A, the ij^{th} *cofactor* is the $(n-1) \times (n-1)$ submatrix $\tilde{A}_{ij}$ obtained by deleting the ith row and jth column of A.

Example. For the matrix A below, the cofactors $\tilde{A}_{11}$ and $\tilde{A}_{13}$ are as show.

$$A = \begin{pmatrix}3 & 5 & 9\\ 2 & 4 & 1\\ 0 & 2 & -1\end{pmatrix} \qquad \tilde{A}_{11} = \begin{pmatrix}4 & 1\\ 2 & -1\end{pmatrix} \qquad \tilde{A}_{13} = \begin{pmatrix}2 & 4\\ 0 & 2\end{pmatrix}$$

The determinant of a 3×3 matrix can be computed by "expanding along the first row" as follows.

1. First visualize the matrix filled with $\pm$ signs in the checkerboard pattern $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

2. For each entry in the top row, compute

$$\pm(\text{entry})(\text{determinant of the corresponding cofactor})$$

   where the $\pm$ sign is the one given by the checkerboard.

3. Sum these contributions for the entries in the first row. Thus

$$\det A \;=\; A_{11}\det\tilde{A}_{11} - A_{12}\det\tilde{A}_{12} + A_{13}\det\tilde{A}_{13}$$

Example. For the matrix A above,

$$\begin{aligned}\det\begin{pmatrix} 3 & 5 & 9 \\ 2 & 4 & 1 \\ 0 & 2 & -1 \end{pmatrix} &= 3\cdot\det\begin{pmatrix} 4 & 1 \\ 2 & -1 \end{pmatrix} \;-\; 5\cdot\det\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \;+\; 9\cdot\det\begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} \\ &= 3(-4-2) - 5(-2-0) + 9(4-0) \\ &= -18 + 10 + 36 = 28.\end{aligned}$$

The same determinant can be found by "expanding" along any row or column in the same manner — with the same result. For example, we could alternatively have calculated $\det A$ for the above matrix A by expanding along the first column:

$$\begin{aligned}\det\begin{pmatrix} 3 & 5 & 9 \\ 2 & 4 & 1 \\ 0 & 2 & -1 \end{pmatrix} &= 3\cdot\det\begin{pmatrix} 4 & 1 \\ 2 & -1 \end{pmatrix} \;-\; 2\cdot\det\begin{pmatrix} 5 & 9 \\ 2 & -1 \end{pmatrix} \;+\; 0\cdot(\text{something}) \\ &= 3(-4-2) - 2(-5-18) + 0 \\ &= -18 + 46 = 28.\end{aligned}$$

Usually, the calculation is easiest when one expands along a row or column with lots of zeros.

Homework 25

1. Compute the 2×2 determinants in Problems 2a, 2b, 2c on page 208.

2. Do the same for the complex matrices in Problems 3a and 3b. *Use* $i^2 = -1$.

3. Prove the formula $\det(AB) = \det A \cdot \det B$ for 2×2 matrices directly from the definition (24.2).

4. Prove the geometric formula (24.3), as follows. Take $\mathbf{v}_1$ as the base, so the area of a parallelogram is $|\mathbf{v}_1|\cdot h$ where $|\mathbf{v}_1|$ denotes the length of $\mathbf{v}_1$ and h is the height of the parallelogram. Show that $\mathbf{w} = (-b, a)$ is perpendicular to $\mathbf{v}_1$, and then express h in terms of the dot product $\mathbf{w}\cdot\mathbf{v}_2$.

5. Do Problems 4a and 4c on the same page.

6. Show that $\det A^t = \det A$ for 2×2 matrices A. (Here A^t is the transpose, obtained by "flipping" rows and columns).

The remaining problems are taken from Section 4.2, pages 221-222, of the textbook.

7. Evaluate the determinants in Problems 5, 6, 7, and 9 on page 221.

8. Evaluate the determinants in Problems 13 and 14. *Expand along a row or column with lots of zeros!*