Day 21 Matrix algebra, compositions and Inverses

Today and next time (Day 22) will will cover material from Sections 2.3 and 3.2 of the textbook.

Matrix Operations.

1. To multiply a matrix by $r \in \mathbb{R}$, multiply every entry by r. $5\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 15 & 20 \end{pmatrix}$.
2. To add matrices, add the corresponding entries. $\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 6 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 8 & 3 \end{pmatrix}$.
3. The product AB of two matrices is found by:

   The ij entry of AB = the dot product (i^{th} row of A) $\cdot$ (j^{th} column of B).

   The product is only defined if the vectors in this dot product have the same length.

These operations satisfy:

- Addition and scalar multiplication are commutative: $A + B = B + A$ and $rA = Ar$.
- The $n \times n$ identity matrix I_n is the multiplicative identity: $I_n A = A$ for all $n \times m$ matrices A.
- Multiplication is associative: $(AB)C = A(BC)$ whenever both sides are defined.
- The distributive property $A(rB + sC) = rAB + sAC$ holds whenever the left side is defined.
- Multiplication is NOT commutative: it often happens that $AB \neq BA$.

Compositions of linear transformations.

Definition. The *composition* of transformations $S : U \to V$ and $T : V \to W$ is the transformation $T \circ S : U \to W$ defined by

$$(T \circ S)(\mathbf{u}) = T(S(\mathbf{u})).$$

Caution – read backwards! $T \circ S$ means "do S first, then do T". To keep things straight, write down this diagram:

$$U \xrightarrow{S} V \xrightarrow{T} W$$

Composing maps in this manner keeps us in the world of linear transformations:

Lemma. If S and T are linear transformations, then so is $T \circ S$.

Proof. If $\mathbf{u}, \mathbf{v} \in U$ and $r, s \in \mathbb{R}$ then

$$\begin{aligned} (T \circ S)(r\mathbf{u} + s\mathbf{v}) &= T(S(r\mathbf{u} + s\mathbf{v})) && \text{Def. of } T \circ S \\ &= T(rS(\mathbf{u} + sS(\mathbf{v})) && S \text{ is a LT} \\ &= (rT(S\mathbf{u})) + sT(S(\mathbf{v})) && T \text{ is a LT} \\ &= r(T \circ S)(\mathbf{u}) + s(T \circ S)(\mathbf{v})) && \text{Def. of } T \circ S \end{aligned}$$

□

After we choose bases of the vector spaces U, V and W, then we can write $S : U \to V$ and $T : V \to W$ as matrices. What is the matrix of $T \circ S$?

Theorem. Fix bases of the vector spaces U, V and W. If the matrix of $S : U \to V$ is A and the matrix of $T : V \to W$ is B, then the matrix of $T \circ S$ is the matrix product BA.

Example. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the dilation by a factor of 2 in the x direction and 4 in the y direction, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the line $x = y$. In terms of matrices,

$$S = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T \circ S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}$$

Homework 21

1. Sketch the image of the unit square under the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose matrix is $\begin{pmatrix} -2 & -3 \\ 0 & 4 \end{pmatrix}$.
   This image will be a parallelogram — see Example 1 in the handout for Day 20.
2. Write down the matrix for the linear transformation $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6x - 9y + z \\ 5x + 8y - 2z \\ 4x - 3y + 7z \end{pmatrix}$
3. Construct the 2×2 matrix for the following linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ defined by the following compositions. *In each case, write down the matrix of each transformation, then multiply the matrices in the correct order.*
   (a) A dilation by a factor of 4, then a reflection across the x-axis.
   (b) A counterclockwise rotation through $\pi/2$, then a dilation by a factor of $\frac{1}{2}$.
   (c) A reflection about the line $x = y$, then a rotation though an angle of π.
4. (a) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the composition: A scaling factor of 6 in the x direction and 2 in the y direction, then a *clockwise* rotation of 45° (= a counterclockwise rotation of -45°).
   (a) Write down the matrix for L.
   (b) Find the images under L of each of the four corners of the unit square.
   (c) Plot the points found in (b) and sketch the image of the unit square under L.
5. (a) Write down the matrix R for a counterclockwise rotation through $\pi/4$ radians.
   (b) Compute R^2, R^4 (square R^2), and show that $R^8 = Id$.
   (c) Give a geometric explanation why $R^8 = Id$.
6. Verify the associative property of matrix multiplication in one example by computing $A(BC)$ and $(AB)C$ for the matrices
$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 4 \\ -2 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
7. Consider the matrices
$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 5 & 4 \\ -2 & 1 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 3 \\ 6 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix}$$
   Calculate, if possible, (a) AB and BA, (b) AC and CA, (c) AD and DA.
   Observe that $AB \neq BA$ since one of these does not exist, $AC \neq CA$, and $AD = DA$, illustrating all possibilities when the order of multiplication is reversed.

8. Let $A = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}$ and let $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the 2×2 identity matrix. Calculate A^2 and use your answer to find the matrices

$$\text{(a) } A^2 + 2A - 5I_2 \qquad \text{and} \qquad \text{(b) } A^2 - 7A + 13I_2.$$

9. The matrix $S = \begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ is called *strictly upper triangular* for the obvious reason.

   (a) Compute S^2 and S^3.

   (b) Formulate and prove a theorem about general strictly upper triangular 3×3 matrices.

Day 22 Inverse matrices

For any vector space V, the *identity transformation* $I : V \to V$ is given by $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$. The matrix of the identity transformation $I_n : \mathbb{R}^n \to \mathbb{R}^n$ is the $n \times n$ matrix

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Definition *An $n \times n$ matrix A is called* **non-singular** *or* **invertible** *if there is a matrix B such that*

$$AB = I_n \qquad \text{and} \qquad BA = I_n. \tag{0.1}$$

If so, B is called the multiplicative inverse of A and written as A^{-1}. If no such matrix exists, we say that A is **singular***.*

Examples. (a) The inverse of $A = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$ is $B = \begin{pmatrix} 2 & -1 \\ \frac{5}{2} & \frac{3}{2} \end{pmatrix}$ because

$$AB = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ \frac{5}{2} & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad BA = \begin{pmatrix} 2 & -1 \\ \frac{5}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) The matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is singular because $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, so there is no matrix B so that $AB = I_2$.

Because matrix multiplication is not commutative, one has to be careful about the meaning of inverses — that is why the definition (**??**) requires two equations (if we knew that $AB = BA$, these two equations would be the same). The following theorem list some basic, reassuring facts about inverses.

Theorem on Inverses. (a) If A is non-singular, then it has a unique inverse.

(b) If an $n \times n$ matrix B satisfies one of the equations (**??**), then it satisfies both, so is A^{-1}.

(c) If A is invertible, then so in A^{-1} and the inverse of $(A)^{-1}$ is A.

(d) If A and B are invertible, then so in AB and $(AB)^{-1} = B^{-1}A^{-1}$. *The order reverses!*

There is a direct way to find the inverse of a given matrix using row operations:

Algorithm for finding the inverse of an $n \times n$ matrix A.

1. Write down the augmented matrix $(A \,|\, I_n)$.
2. Do row operations to put A in reduced row echelon form, doing the same operations on the I_n side.
3. Is the reduced row echelon form of A is the identity?
   - If yes, the end result is $(I_n \,|\, B)$ where B is the matrix for A^{-1}.
   - If not, then A^{-1} does not exist, i.e. A is singular.

Examples. To see how this algorithm works in practice, see Examples 5 and 6 on pages 162-164 of the textbook.

Definitions *The* **rank** *of a matrix A, denoted rank A, is the number of pivots in the row echelon form on A.*

Invertibility Theorem. *An $n \times n$ matrix A is invertible if and only if rank $A = n$.*

This theorem is an easy consequence of the algorithm, and we will soon have a nice way to interpret it geometrically.

Solving linear systems: What's the point of finding the inverse? Well, we can use inverses to immediately solve linear systems. To see this, write a linear system

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

as a matrix equation

$$A\mathbf{x} = \mathbf{y}$$

Here $\mathbf{y}$ and the matrix A are given, and we want to find the column vector $\mathbf{x}$. But applying A^{-1} to both sides shows that

$$\mathbf{w} = A^{-1}\mathbf{y}$$

so if A^{-1} is known, the system is solve by a single matrix multiplication!

Homework 22

1. Prove part (d) of the Theorem on Inverses: if A and B are invertible $n \times n$ matrices, then AB is also invertible with $(AB)^{-1} = B^{-1}A^{-1}$. *You need only to verify the two equations (**??**) in the definition of inverse.*

2. For each of the following matrices, determine the rank and state whether the matrix is invertible. *Use the Invertibility Theorem above. For (d), answer in terms of the entries a, e, h, j.*

$$A = \begin{pmatrix} 3 & -2 & 4 & 1 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 0 & 3 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & -4 \end{pmatrix} \qquad C = \begin{pmatrix} 7 & 2 & 1 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{pmatrix}$$

3. Use the algorithm described on the box above to find the inverses for each of the following matrices *If the inverse does not exist, write "no inverse".*

$$A = \begin{pmatrix} 3 & 4 \\ 7 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 8 & 2 \\ -2 & 3 & -1 \\ 3 & -1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 2 & -1 \\ 2 & 3 & -6 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 4 & 1 & 2 \end{pmatrix}$$

4. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that if $ad - bc \neq 0$ then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

No need to use the algorithm – just check by multiplying.

5. Recall that the counterclockwise rotation $\mathbb{R}^2 \to \mathbb{R}^2$ by an angle θ is given by the matrix

$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Use the formula in Problem 4 to show that the inverse of R_θ is $R_{-\theta}$.

6. Given

$$A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

use the formula in Problem 4 to compute A^{-1}; then use it to:

(a) Find a 2×2 matrix P such that $AP = B$.

(b) Find a 2×2 matrix Q such that $QA = B$. *Remember, matrix multiplication is not commutative!*

7. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$.

(a) Use matrix multiplication to verify that $A^{-1} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}$.

(b) Use the matrix A^{-1} to solve the linear system $A\mathbf{x} = \mathbf{b}$ for the following three choices of $\mathbf{b}$:

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathbf{b}_3 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$