Day 20 Linear transformation and matrix multiplication

Today and next time we will cover Section 2.3 in the textbook. Recall that for a linear transformation $T : V \to W$

- T is determined by what it does to the basis elements.
- After we choose a basis for V and a basis for W, T is described by a matrix A.

The matrix of a linear transformation $T : V \to W$ is built as follows:

1. Fix a basis $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V and a basis $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ of W.
2. For each i, write $T(\mathbf{v}_i)$ as a linear combination of the $\mathbf{w}_j$ and put the coordinates into a column vector:
$$T(\mathbf{v}_i) = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}$$
3. Build the matrix A whose columns are $T(\mathbf{v}_1), T(\mathbf{v}_2)$, etc. :
$$A \;=\; [T]_\alpha^\beta \;=\; \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$
This is called the *matrix of n T with respect to the bases α and β.*

The point of the matrix is that it gives a simple way of computing, as described in this theorem.

Theorem 0.1. *Suppose that is a linear transformation with matrix A with respect to bases α and β. Then the image of*
$$\mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$$
is
$$T\mathbf{v} = y_1\mathbf{w}_1 + \cdots + y_m\mathbf{w}_m$$
where the y_j's are obtained by matrix multiplication $\mathbf{y} = A\mathbf{x}$:
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

For linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$, we can use the standard bases. The procedure for finding the matrix is then easy:

1. Write $T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ as a column vector.
2. Do the same for the other basis vectors.
3. Assemble these column vectors into a matrix A.

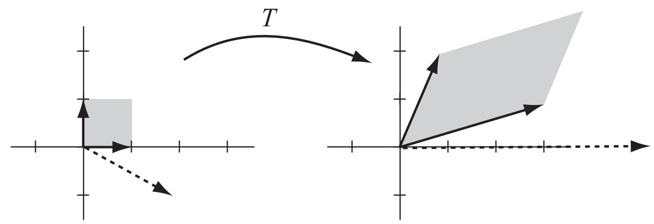
Then the transformation is given by matrix multiplication by A.

Example 1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that takes $(1,0)$ to $(3,1)$ and $(0,1)$ to $(1,2)$. In column form

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} \qquad \text{and} \qquad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

so the matrix of T is $A = \begin{pmatrix}3 & 1\\1 & 2\end{pmatrix}$. To find where T takes the vector $\begin{pmatrix}2\\-1\end{pmatrix}$ we calculate

$$T\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}3 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}5\\0\end{pmatrix}.$$



Example 2. What is the matrix of the linear transformation $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}4x-2y\\-x+7y\end{pmatrix}$?

Solution: taking $x = 1$ and $y = 0$, then $x = 0$ and $y = 1$ we obtain:

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}4\\-1\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-2\\7\end{pmatrix} \qquad \text{so the matrix is} \qquad A = \begin{pmatrix}4 & -2\\-1 & 7\end{pmatrix}.$$

Examples of six types of linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$.

1. Dilation by factor of 3 horizontally, factor of 5 vertically: $D = \begin{pmatrix}3 & 0\\0 & 5\end{pmatrix}$.
2. Rotation counterclockwise by 90°: $R_{90} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ and by angle θ: $R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$.
3. The reflection across the x-axis: $R_x = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$.
4. A *shear* that fixes the first basis vector $\mathbf{e}_1$ and moves $\mathbf{e}_2$ to $\mathbf{e}_1 + \mathbf{e}_2$: $S = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$.
5. The orthogonal projection onto the x-axis: $P = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$.
6. The *embedding* $E : \mathbb{R}^2 \to \mathbb{R}^3$ that takes $\mathbb{R}^2$ into the xy-plane: $E : \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix}$.

Homework 20

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that takes $\mathbf{e}_1 = (1,0)$ to $3\mathbf{e}_1 + 4\mathbf{e}_2$ and takes $\mathbf{e}_2 = (0,1)$ to $-2\mathbf{e}_1 + 5\mathbf{e}_2$. What is the matrix for T with respect to the (standard) basis $\{\mathbf{e}_1, \mathbf{e}_2\}$?
2. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation about the origin by 60°.

   (a) What is $R(\mathbf{e}_1)$? What is $R(\mathbf{e}_2)$?

(b) What is the matrix of R with respect to the standard basis?

(c) What is the image $R(\mathbf{v})$ of the vector $\mathbf{v} = (1,1)$?

3. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the yz plane, given by $P(x,y,z) = (0,y,z)$.

   (a) What are $P(\mathbf{e}_1)$, $P(\mathbf{e}_2)$ and $P(\mathbf{e}_3)$?

   (b) What is the matrix of L with respect to the standard basis?

   (c) What is the image $P(\mathbf{v})$ of the vector $\mathbf{v} = (1,2,-1)$?

4. Do the following matrix multiplications:

$$(a)\quad \begin{pmatrix} 2 & -3 & 8 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

$$(c)\quad \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} \qquad (d)\quad \begin{pmatrix} -1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ -1 & 0 & 2 & 4 \\ 0 & -2 & 0 & 7 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

5. Sketch the image of the unit square under the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose matrix is
$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

6. Find the matrix of the orthogonal projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ onto the z-axis.

7. (a) Using the formula given in Example 2 at the top of this page, find the matrix of the rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$ about the origin by $30°$.

   (b) This rotation takes $\begin{pmatrix} 4 \\ 9 \end{pmatrix}$ to what vector?

8. Let $L = \{y = -x\}$ be the "anti-diagonal" line in $\mathbb{R}^2$.

   (a) Find the matrix of the reflection R through L.

   (b) Where does R take the vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$?

9. Find a non-zero 2×2 matrix A such that $A\mathbf{v}$ is perpendicular to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for all $\mathbf{v} \in \mathbb{R}^2$. *Hint: Find one perpendicular vector* $\mathbf{w}$*, write down the formula for the projection onto* $\mathbf{w}$*, and apply this formula to the basis vectors.* Check by calculating $A\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

10. Find the matrices of the following transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$.

    (a) The reflection across the xz-plane.

    (b) The rotation about the z-axis trhough an angle θ counterclockwise as viewed from the positive z-axis.

    (c) Reflection across the plane $y = z$.

11. Let D be the dilation in Example 1 at the top of this page. Show that D takes the unit circle $x^2+y^2 = 1$ to an ellipse. Sketch the ellipse. *Hint: What is the image of* $\begin{pmatrix} x \\ y \end{pmatrix}$?

12. Let $T : P_3 \to P_3$ be the linear transformation that takes a polynomial $p(x) = ax^3 + bx^2 + cx + d$ to $p(x-2)$. What is the matrix for T with respect to the basis $\{x^3, x^2, x, 1\}$ of P_3? (Be sure to keep the basis elements in this order.)