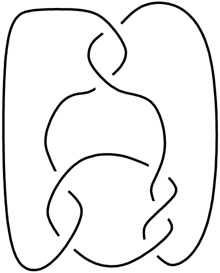
Geometric types of surfaces and Jones polynomials

joint with D. Futer and J. Purcell

Quantum Topology and Hyperbolic Geometry, Nha Trang, May 13-17, 2013

Given: Diagram of a knot or link



= 4–valent graph with over/under crossing info at each vertex.

Quantum Topology

- Colored Jones polynomials

Geometric topology

- Incompressible surfaces in knot complements
- Geometric structures and data esp. hyperbolic geometry and volume

Goal: Describe a setting to study both sides and relate them.

Outline

- **Setting:**
  - Given knot diagram construct state graphs (ribbon graphs)..
  - Build state surfaces spanned by the knot...

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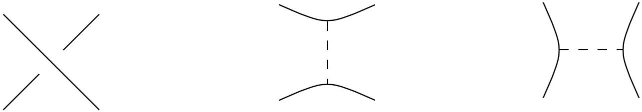
- **Method-Tools:**
  - Create ideal polyhedral decomposition of surface complements...
  - Use normal surface theory to get correspondence JSJ-decompositions $\leftrightarrow$ state graph topology

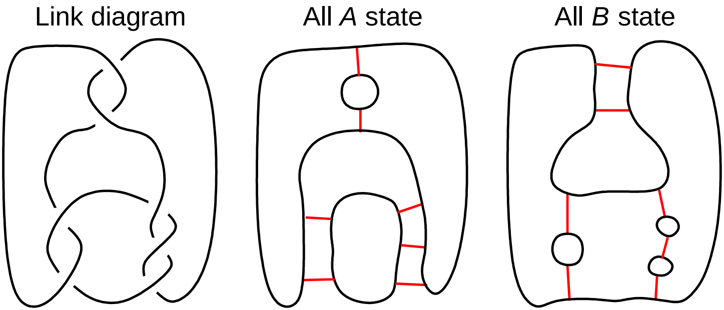
Given: Diagram of a knot or link

Two choices for each crossing, *A* or *B* resolution.



- Choice of *A* or *B* resolutions for all crossings: *state* σ.
- Result: Planar link without crossings. Components: *state circles*.
- Form a graph by adding edges at resolved crossings. Call this graph H_σ.
  (Note: n crossings $\rightarrow 2^n$ state graphs)

Example states



Above: H_A and H_B.

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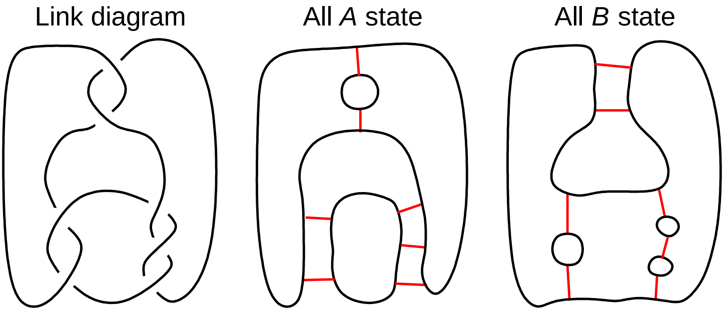

Above: H_A and H_B.

- The Jones polynomial of the knot can be calculated from H_A or H_B: *spanning graph expansion* arising from the Bollobas-Riordan *ribbon graph* polynomial (Dasbach-Futer-K-Lin-Stoltzfus, 2006).

Colored Jones polynomial prelims

For a knot K we write its *n-colored Jones polynomial*:

$$J_{K,n}(t) := \alpha_n t^{m_n} + \beta_n t^{m_n - 1} + \cdots + \beta'_n t^{k_n+1} + \alpha'_n t^{k_n}.$$

Some properties:

- $J_{K,n}(t)$ is determined by the Jones polynomials of certain cables of K.
- The sequence $\{J_{K,n}(t)\}_n$ is *q-holonomic*: for every knot the CJP's satisfy linear recursion relations in n (Garoufalidis-Le, 2004). Then, for every K,
- Degrees m_n, k_n are quadratic (quasi)-polynomials in n.
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- Degrees m_n, k_n are quadratic (quasi)-polynomials in n.
- Coefficients $\alpha_n, \beta_n \ldots$ satisfy recursive relations in n.
- **Note.** Properties manifest themselves in strong forms for knots with *state graphs* that have no edge with both endpoints on a single state circle!—— that is when K is *A-adequate* (next)

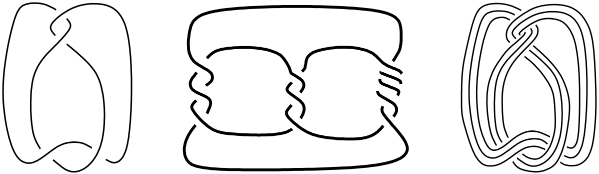
Semi-adequate links

Lickorish–Thistlethwaite 1987: Introduced *A–adequate* links (*B–adequate links*) in the context of Jones polynomials.

Definition

A link is A–adequate if has a diagram with its graph H_A has no edge with both endpoints on the same state circle. Similarly B-adequate.

Some examples:



"Who" are they?

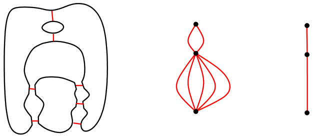
Semi-adequate (=A or B adequate) links are abundant! Thy are the "generic" class of links to the eyes of CJP.

Some familiar classes and their geometry:

- all but two of prime knots up to 11 crossings.
- all alternating knots, (prime are *torus links* or *hyperbolic*),
- all Montesinos knots (*mostly hyperbolic*),
- all positive (negative) knots (lots of *hyperbolic*),
- many arborescent knots (mostly *hyperbolic*),
- all closed 3-braids (prime are *torus knots* or *hyperbolic* (Stoimenow),
- large families of *hyperbolic* braid and plat closures (A. Giambrone),
- blackboard cables and Whitehead doubles of semi-adequate knots (*satellites*)

Colored Jones polynomials for A-adequate links

H_A=state graph from an A-adequate diagram of link K.



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Collapse each state circle of H_A to a vertex to obtain the state graph $\mathbb{G}_A$. Remove redundant edges to obtain the *reduced state graph* $\mathbb{G}'_A$.

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- Extreme Coefficients stabilize; they depend only on $\mathbb{G}'_A$!

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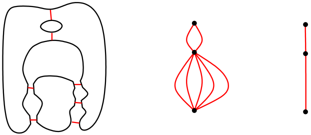

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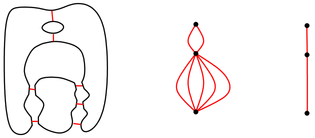

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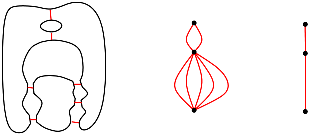

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- **Note:** In example above $\mathbb{G}'_A$ is a tree so $\beta'_K = 0$. The surface S_A is *fiber* of the knot complement (later).

CJP for A-adequate links, con't

- In fact, all coefficients of $J_{K,n}(t)$ stabilize:
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Extreme degrees of CJP

- (L-T) D any diagram of K, c_{-}=number of negative crossings in D. Then

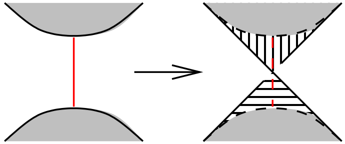
$$k_n \geq -n^2 2c_{-} + O(n),$$

$k_n :=$min deg $J_{K,n}(t)$. If D is A-adequate then we have equality!. Thus k_n is a quadratic polynomial in n; can be calculated explicitly.

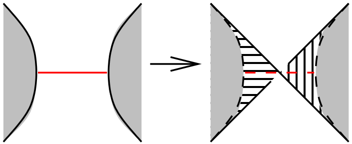
State surface

Given a state σ, using graph H_σ and link diagram, form the *state surface* S_σ.

- Each state circle bounds a disk in S_σ (nested disks drawn on top).
- At each edge (for each crossing) attach twisted band.



A–resolution



B–resolution

Example state surfaces

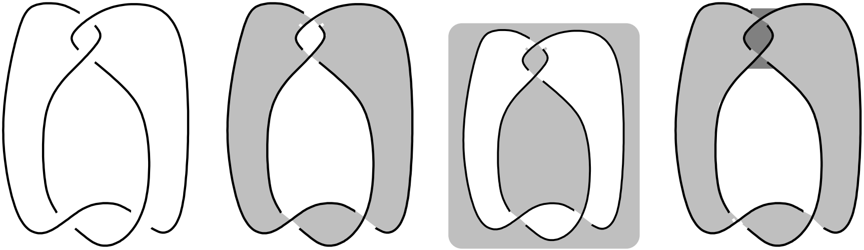


Fig-8 knot

S_A

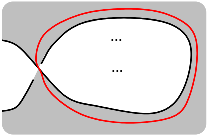
S_B

Seifert surface

- For alternating diagrams: S_A and S_B are checkerboard surfaces.

When are state surfaces essential?

Essential=π_1*-injective*
(May define as incompressible, ∂-incompressible. Warning.If S_A is non-orientabe use double: $\overline{N(S_A) - N(\partial S_A)}$.)
The only trouble. If H_A has edge with both endpoints on a single state circle (boundary compression disk for double):



Theorem

(Ozawa, Futer-K-Purcell)The following are equivalent:

- *D(K) is A–adequate (no edge of $\mathbb{G}_A$ is a loop)*
- *S_A is essential.*

Ozawa proof was first.— We see a lot more about S_A.

Boundary slopes and CJP

S_A= state surface with $K = \partial S_A$ an A-adequate knot (one component). The class $[K]$ in $H_1(\partial(S^3 \smallsetminus K))$ is determined by an element in $\mathbf{Q} \cup \{\infty\}$, called *the boundary slope of S_A*.

Theorem (FKP)

For an A–adequate diagram,

$$\textit{bdry slope of } S_A = \lim_{n\to\infty} \frac{-4}{n^2} k_n,$$

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What's known

- For knots that are A and B-adequate (a.k.a. *adequate*), slopes conjecture is know for "both sides".
- (Garoufalidis) torus knots, certain 3-string pretzel knots (these are semi-adequate but not adequate)
  For pretzel knots the boundary slopes are all known./ For torus knots CJP has been calculated.
- (Dunfield–Garoufalidis) Verified conjecture for the class of *2-fusion knots*.— (normal surface theory+character variety techniques to get the incompressible surface).
- (van der Veen) Formulated a Slopes conjecture for the *multi-colored* CP of links. Showed that S_A verifies it A-adequate links.

CJP detects fibers in state surfaces

Recall the *reduced state graph* $\mathbb{G}'_A$ corresponding to an A-adequate diagram and β'_K the (abs. value of) stabilized penultimate coefficient of the CJP of the corresponding knot K. They are related via $\beta'_K = 1 - \chi(\mathbb{G}'_A)$.

Theorem (FKP)

The following are equivalent:

- *The complement $S^3 \setminus K$ fibers over S^1 with fiber S_A.*
- *The reduced graph $\mathbb{G}'_A$ is a tree.*
- *We have $\beta'_K = 0$.*

Remark If $\beta'_K \neq 0$ then S_A doesn't lift to a fiber in any finite cover!.

Next. If K is hyperbolic, β'_K determines the geometric type of S_A in the Thurston trichotomy.

Essential surfaces in cusped hyperbolic 3-manifolds

M=a *hyperbolic* link complement in 3-sphere: M admits complete, Riemannian metric of constant curvature -1.

$\mathbb{H}^3$ hyperbolic 3-space; $\mathbb{C}$=plane at infinity.

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Let S be a properly embedded essential surface in M. We have a faithful representation

$$\rho : \pi_1(S) \hookrightarrow \pi_1(M) \subset PSL(2, \mathbb{C}).$$

Through this representation $\pi_1(S)$ acts on $\mathbb{H}^3$. According to properties of ρ and the corresponding action, essential surfaces are divided into three mutually disjoint classes:

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Definitions of types

Under $\rho : \pi_1(S) \hookrightarrow \pi_1(M) \subset PSL(2,\mathbb{C})$, the conjugacy class of $[\partial S]$ is mapped to a *parabolic* element of $PSL(2,\mathbb{C})$. (parabolics fix a horosphere).

- An *accidental parabolic* is a *non-peripheral* element of $\pi_1(S)$ (i.e. a non-conjugate of $[\partial S]$) that is is mapped by ρ to a parabolic in $\pi_1(M)$.

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- The surface S is a *semi-fiber* if it is a fiber in M or covered by a fiber in a two-fold cover of M. If S is a semi-fiber but not a fiber, we call it a *strict semi-fiber*.

Definitions of types

Under $\rho : \pi_1(S) \hookrightarrow \pi_1(M) \subset PSL(2,\mathbb{C})$, the conjugacy class of $[\partial S]$ is mapped to a *parabolic* element of $PSL(2,\mathbb{C})$. (parabolics fix a horosphere).

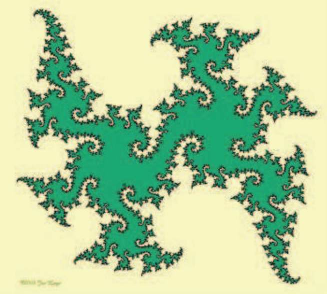
- An *accidental parabolic* is a *non-peripheral* element of $\pi_1(S)$ (i.e. a non-conjugate of $[\partial S]$) that is is mapped by ρ to a parabolic in $\pi_1(M)$.

  Equivalently, *accidental parabolic* on S: a free homotopy class of a closed curve not ∂–parallel on S but can be homotoped to the boundary of M.
- The surface S is a *semi-fiber* if it is a fiber in M or covered by a fiber in a two-fold cover of M. If S is a semi-fiber but not a fiber, we call it a *strict semi-fiber*.
- S is called *quasifuchsian* if the embedding $S \hookrightarrow M$ lifts to a topological plane in $\mathbb{H}^3$ whose *limit set* $\Lambda \subset \partial\mathbb{H}^3$ is a Jordan curve (topological circle).

Limit sets of surface groups:

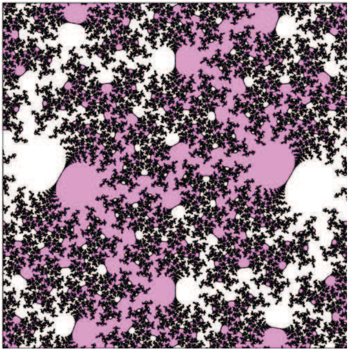
Recall $\pi_1(S)$ identified with discrete infinite subgroup of $PSL(2, \mathbb{C})$. The orbits of the action on $\mathbb{H}^3$ accumulate on plane at infinity to give the *limit set*.

- Limit sets of quasifuchsian surface groups are Jordan curves:



Note. Jordan curve is fixed by by isometries corresponding to $\pi_1(S)$. If S non-orientable the two disks bounded by the Jordan curve will be be interchanged by these isometries.

- Limit sets of semi-(fiber) surface groups is all of $\mathbb{C} = \partial\mathbb{H}^3$ (*space filling* curve).
  Example. The limit set of the fiber group of the complement of figure-8 knot is the entire plane at infinity.



Picture above and the one on previous slide are borrowed from Jos Leys Kleinian Groups art gallery http://www.josleys.com/galleries.php

Types of surfaces: A trichotomy

Every properly embedded, essential surface S in a hyperbolic cusped 3–manifold M falls into exactly one of the three types above (Marden, Thurston, Bonahon...)

Theorem

Let S be an essential, properly embedded surface in a hyperbolic link complement in S^3. Then exactly one of the following is true:

- *contains accidental parabolics*
- *semi-fibers*
- *quasifuchsian*

We will apply this to state surfaces of A-adequate knots. But first...

Bibliographical Remarks

Types of surfaces in hyperbolic link complements.

1. No alternating link complement contains embedded quasifuchsian closed surfaces. (Menasco- Reid).
2. There are closed, immersed quasifuchsian surfaces in any hyperbolic link complement (Masters-Zhang).
3. There are closed accidental surfaces in a wide range of link complements (Finkelstein- Moriah, Wu...)
4. Minimum genus Seifert surfaces are not accidental (Cooper-Long, Fenley).
5. There are hyperbolic knots with accidental Seifert surfaces (Tsutsumi...).
6. Checkerboard surfaces in alternating link complements are not virtual fibers (Wise); they are always quasifuchsian (Adams, Futer-K.-Purcell).

CJP determines the geometric type of state surfaces

Recall the *reduced state graph* $\mathbb{G}'_A$ corresponding to an A-adequate diagram and β'_K the stabilized penultimate coefficient of the CJP of the corresponding knot K.

Theorem (FKP)

Suppose K is hyperbolic. Then, β'_K determines the geometric type of the all–A surface S_A, as follows:

- *S_A is a fiber in $S^3 \setminus K$ iff $\beta'_K = 0$ or equivalently $\mathbb{G}'_A$ is a tree.*
- *S_A is quasifuchsian iff $\beta'_K \neq 0$ or equivalently $\mathbb{G}'_A$ is not a tree.*

Remark. We have large families of quasifuchsian surfaces that fit nicely with recent work of Thistlethwaite-Tsvietkova, who proposed an algorithm to construct the hyperbolic structure on a link complement directly from a diagram. Their algorithm works whenever a link diagram admits a non-accidental state surface, which is exactly what our results ensure

Is there more in β'_K? How about in the whole tail?

- In general, β'_K measures the "size" (in the sense of Guts) of the hyperbolic part in Jaco-Shalen-Johannson decomposition S_A. This, combined with work of Agol- W. Thurston- Storm, large β'_K implies large hyperbolic volume for $S^3 \setminus K$ (See next talk).
- **What about the tail?**
- Recall $T_K(t) = 1 \pm \beta'_K t + O(t^2)$.

Theorem (Armond-Dasbach)

Suppose K A-adequate. Then, $T_K(t) = 1$ if and only if $\beta'_K = 0$.

Note: if $\beta'_K = 0$ then $\mathbb{G}'_A$ is a tree

Thus (in particular): $T_K(t) = 1$ if and only if S_A is a fiber in $S^3 \setminus K$.

- **Question.** Does $T_K(t)$ contain more information about the complement of S_A and the geometry of K than β'_K?

The method: A “nice” polyhedral decomposition.

$M_A = S^3 \backslash\backslash S_A$ is obtained by removing a neighborhood of S_A from S^3. On ∂M_A we have the *parabolic locus* (what remains from $\partial(S^3 \setminus K)$ after cutting along S_A).

Result:

Starting with an A-adequate diagram, we obtain an *ideal* polyhedral decomposition of $S^3 \backslash\backslash S_A$.

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Use these polyhedra and *normal surface theory* to study geometry of S_A.

Tools: A word on the proofs

By the Thurston-Bonahon trichotomy,

- S_A is never a semi-fiber (FKP, 2010).
- S_A doesn't contain any *accidental parabolics* (FKP, 2012).

A word on ruling out parabolics:

- Existence of an accidental parabolic implies the existence of an essential, embedded annulus $\mathcal{A}$ in $M_A = S^3 \backslash\backslash S_A$, with one ∂-component on S_A and one on the parabolic locus.
- Put the embedded annulus in *normal form* and analyze its intersections with the polyhedral decomposition of M_A.—- Get contradiction– annulus cannot exist if link to start with K, is hyperbolic.

Example. Suppose K is a knot (one component): On any *white face* W, of the decomposition, Σ intersect in arcs such that

- each arc in $\mathcal{A} \cap \mathcal{W}$ runs from an *ideal vertex* of W to a side of W; and
- $\mathcal{A} \cap \mathcal{W}$ passes through every *ideal vertex* of W.

Then an arc in $\mathcal{A} \cap \mathcal{W}$ runs from an ideal vertex to an adjacent side of W. Contradiction to normal form!