Material for the Geometry Topology Qualifying Exam

Fall 2006 and Spring 2007

Point-set topology background:

–Topological spaces (compact, connected, Hausdorff, metric),

– Maps (homeomorphism, homotopy, isotopy, deformation retract),

– Product and quotient topologies,

– Separation axioms, pointwise and uniform convergence.

Manifolds:

–smooth manifolds; definition, coordinate charts, product manifolds

–examples; $\mathbb{R}^n$, S^n, $\mathbb{RP}^n$, T^n.

–maps, Jacobians, rank, immersions, embeddings,

–submanifolds, canonical local coordinates

– inverse and implicit function theorems, examples

– paracompact, partitions of unity, Whitney embedding theorem

The tangent bundle and vector bundles:

–the tangent and cotangent bundle,

–the differential of a smooth map,

–vector bundles; abstract definition, transition functions,

– construction techniques

– dual bundles, quotient bundles, Whitney sums, bundle maps

–local and global sections.

Tensor bundles and tensors:

–the tensor product of vector spaces,

–covariant and contravariant tensor bundles,

– multilinear maps, sections, contraction,

–the characterization of a tensor field as a map multilinear over functions

Vector fields:

–vector fields on manifolds; geometric, derivations
–integral curves, existence of integral curves, flows,
–the Lie derivative, the Lie bracket,
– tangent distributions, the Frobenius theorem
–interpretation of the Frobenius theorem; geometric, pde

Differential forms:

–the differential of a function,
– the algebra of alternating tensors,
– the wedge product,
– differential forms on manifolds,
– the exterior derivative, invariant definition
– the Frobenius theorem using differential forms
– DeRham cohomology

Integration on manifolds:

– orientations on a vector space and on a manifold,
–manifolds with boundary and boundary orientations,
–integration of differential forms on a manifold,
–Stokes's theorem, Green's theorem

Riemannian metrics:

– the definition of a Riemannian metric,
– existence,
– the Riemannian volume form,

Surface topology:

– Triangulated surfaces; Euler Characteristic,
– The classification of compact surfaces.

CW-complexes:

– definition of a CW-complex and the CW-topology
– Examples; graphs, manifolds.
– CW presentations of examples; surfaces, S^n, $\mathbb{R}P^n$, CP^n, knot

spaces.

Homotopy Theory:

– Homotopy of maps, of pointed maps,
– The homotopy category and homotopy functors; examples,
– The fundamental group $\pi_1(X, x0)$,
– Van Kampen's theorem,
– Examples; CW-complexes.

Covering Spaces:

– Definition of a covering projection and basic properties.
– Examples; coverings of S^1, S^n covering $\mathbb{R}P^n$, coverings of graphs.
– Homotopy path lifting and relation to fundamental group,
– Classification of coverings of a "reasonable" space (e.g. a CW-complex).
– Construction of all covering spaces in examples; $\mathbb{R}P^n$, S^1, $S^1 \vee S^2$

Homology Theory:

– Simplicial homology; definition, simple computations
– Singular homology; relation to simplicial homology.
– Relative homology, long exact sequence of pairs,
– Excision Property, Mayer-Vietoris sequence,
– Cellular homology; definitions.
– Axiomatic approach: Eilenberg-Steenrod Axioms for homology.
– Computations; S^n, $\mathbb{R}P^n$, CP^n, surfaces, T^n, $X \vee Y$, knot spaces.
– Relation of Homology and fundamental group.
– Degree of Maps, Euler characteristic of a CW-complex,
– Brouwer fixed point theorem, Lefschetz fixed point theorem.

Cohomology:

– Homology with coefficients.
– Singular and cellular cohomology; definitions and examples.
– Universal Coefficient Theorem.
– Cup product and the Cohomology Ring; definition and examples.