Pants graph, TQFT and hyperbolic geometry

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General theme

Three phases of 3-manifold/knot theory:

1. Hyperbolic geometry
2. Quantum topology
3. “Combinatorial presentations/models” (e.g. knot diagrams, triangulations of 2 and 3-dimensional manifolds, pants decompositions...)

Relations?

- Constructions of quantum invariants use combinatorial models in low dimensional topology
- Research has shed light on relations between hyperbolic geometry and combinatorial models/descriptions in LDT
- **Question** (vague) Do these relations reflect on the “quantum side”?

General theme of talk

Talk: Relations between features of $SU(2)$-Witten-Reshetikhin-Turaev TQFT and hyperbolic geometry via the *pants graph*.

Surface pant decompositions

- **Quantum Topology:** relate to bases of the TQFT spaces
- **Geometry:** Form a metric space that “coarsely models” the WP metric geometry of the Teichmüller space

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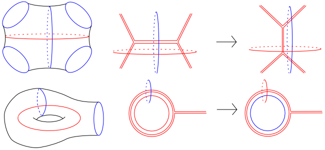

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- “Large-r” asymptotics of *curve operators* on TQFT spaces

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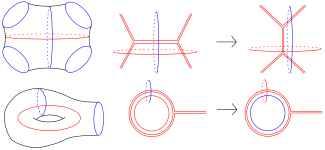

TQFT features

- “Large-r” asymptotics of *curve operators* on TQFT spaces

Hyperbolic geometry features

- WP geometry of Teichmüller space
- volumes of hyperbolic 3-manifolds

Outline

Σ =closed orientable surface of genus $g := g(\Sigma) > 1$.

1. **TQFT prelims.** Work with *skein theoretic* version of $SU(2)$-WRT-TQFT (Blanchet, Habegger, Masbaum and Vogel).
2. For $r \geq 3$, and $\zeta_r = -e^{\frac{\pi i}{2r}}$, we have a finite dimensional Hermitian vector space $V_r(\Sigma)$; and
3. *curve operators:* Hermitian operators $T_r^{\gamma} : V_r(\Sigma) \longrightarrow V_r(\Sigma)$, for any *multicurve* γ on Σ.

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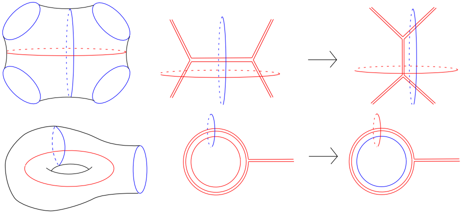
- **A pairing.** Data from "large-r" asymptotics analysis of curve operators gives a pairing, called *quantum intersection number*, on the set of pants decompositions of Σ.
- **Geometric content of quantum intersection number.**

1. Relates to geometric intersection number of curves.
2. Leads to two sided bounds of volumes of hyperbolic surface bundles.
3. leads to a metric space that is *quasi-isometric* to the Teichmüller space with its Weil-Petersson metric.
4. Relations with Nielsen-Thurston classification of mapping classes.

Prelims

- *P=pants decomposition*: 3g-3 disjointly embedded curves decomposing Σ into pants (3-holed 2-spheres).
- *Dual graph* $\Gamma = \Gamma_P$: trivalent graph; each vertex of Γ lies in single pants of P, and each edge of Γ intersects a single curve of P exactly once.



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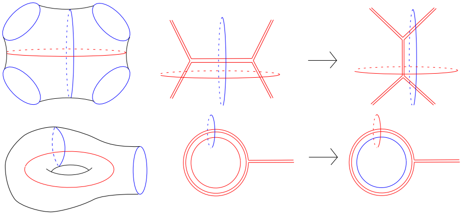

Figure: *Elementary moves*: blue curve $\gamma \longrightarrow$ red curve intersecting γ

- **Elementary moves :** Replace a single γ curve in a pants decomposition by one that intersects γ minimally.
- A-move happens on a 4-holed subsurface $\Sigma_{0,4}$ of Σ (Top row of figure).
- S-move happens on one-holed torus subsurface $\Sigma_{1,1}$ of Σ (Bottom row).

Skein theoretic $SU(2)$-TQFT by BHMV

- P=pants decomposition of Σ with dual (*banded*) Γ with edge set E.
- For $r \geq 3$, let $\mathcal{C}_r = \{1, 2, \ldots, r-1\}$ and let $\zeta_r = -e^{\frac{\pi i}{2r}}$.
- **TQFT spaces.** $V_r(\Sigma)$ finite v. space w. non-degenerate Hermitian pairing
$$\langle \ \rangle : V_r(\Sigma) \times V_r(\Sigma) \longrightarrow \mathbb{C}.$$
- $V_r(\Sigma)$= certain quotient of a version the Kauffman skein algebra $K(H, \zeta_r)$, where H=handlebody with $\partial H = \Sigma$

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- **Bases.** Given an *admissible coloring*, $\mathbf{c} : E \longrightarrow \mathcal{C}_r$, BHMV constructs a vector $\phi_{\mathbf{c}} \in V_r(\Sigma)$ and gets an orthonormal basis

$$\mathcal{B}_P = \{\phi_{\mathbf{c}}\}_{\mathbf{c}}.$$

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- **Curve operators.** For a *multicurve* γ on Σ there are Hermitian operators

$$T_r^{\gamma} : V_r(\Sigma) \longrightarrow V_r(\Sigma),$$

  defined by descend of the product on $K(H, \zeta_r)$.
- The map $K(\Sigma, \zeta_r) \longrightarrow \mathrm{End}(V_r(\Sigma))$ where $\gamma \longrightarrow T_r^{\gamma}$ is an algebra morphism.

Asymptotics of curve operators

Key: Given γ, P, for r "large enough" the presentation matrix of T_r^{γ} in the basis $\mathcal{B}_P = \{\phi_{\mathbf{c}}\}_{\mathbf{c}}$, has the same number of diagonals containing non-zero entries.

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- For admissible colorings $\mathbf{c} : \mathbf{E} \longrightarrow \mathcal{C}_{\mathbf{r}}$,

$$\frac{\mathbf{c}}{r} \in \mathbb{R}^E := \{\text{set of functions } E \longrightarrow [0,1]\}.$$

Theorem (Detcherry, 2015)

Given P and γ, there are analytic functions

$$G_{\mathbf{k}}^\gamma = G_{\mathbf{k}}^\gamma(\frac{\mathbf{c}}{r}, \frac{1}{r}) : \text{certain subset of } \mathbb{R}^E \times [0,1] \longrightarrow \mathbb{C},$$

indexed by functions $\mathbf{k} : E \longrightarrow \mathbb{Z}$, so that

$$T_r^\gamma(\phi_{\mathbf{c}}) = \sum_{\mathbf{k}} G_{\mathbf{k}}^\gamma(\frac{c}{r}, \frac{1}{r})\phi_{\mathbf{c}+\mathbf{k}},$$

and only finitely many of $\{G_{\mathbf{k}}^\gamma\}_{\mathbf{k}\neq \mathbf{0}}$ are non-zero functions.

Quantum intersection number

Definition. The *quantum intersection number of γ with respect to P* is

$$n(\gamma, P) := \text{the number of } \mathbf{k} \neq \mathbf{0} \text{ such that } \mathrm{G}^{\gamma}_{\mathbf{k}} \neq \mathbf{0}.$$

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- **Total geometric intersection number.** For simple curves α, β on Σ, we denote $I(\alpha, \beta)$ the minimum geometric intersection number within the isotopy classes of α, β. We have

$$I(\gamma, P) := \sum_{\alpha \in \gamma \text{ and } \beta \in P} I(\alpha, \beta).$$

Quantum vs geometric intersection

Theorem (Detcherry-K., 2021)

For any γ *and* P, *on a surface* Σ *of genus* $g > 1$, *we have*

$$\frac{I(\gamma, P)}{3g-3} \leq n(\gamma, P) \leq (I(\gamma, P)+1)^{3g-3} - 1.$$

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Corollary (Charles- Marché, 2012)

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- ....and a glimpse of hyperbolic geometry:(Why volume?)

Corollary (Detcherry-K., 2021)

There is a constant $N > 0$, only depending on the topology of Σ, so that for any pseudo-Anosov mapping class $\phi \in \mathrm{Mod}(\Sigma)$ and any $P \in C^0_P(\Sigma)$ we have

$$n(P, \phi(P)) \geq N \; Vol(M_\phi).$$

Pants graph prelims

- **Pants graph of Σ.** (Hatcher-Thurston) Abstract graph $C^1_P(\Sigma)$ with
- vertices $C^0_P(\Sigma)$ are in one-to-one correspondence with pants decompositions of Σ.
- Two vertices are connected by an edge if they are related by a single elementary move (i.e. an *A*-move or an *S*-move).

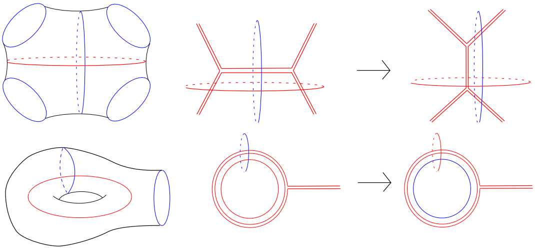

Figure: Elementary moves of pants decompositions and dual graphs.

The path metric and a model of the Teichmüller space

- **Path metric.** The metric $d_\pi : C^1_P(\Sigma) \times C^1_P(\Sigma) \to [0, \infty)$, where
- it assigns the length 1 to each edge of the pants graph; and
- $d_\pi(P, Q)$= minimum number of edges over all paths in $C^1_P(\Sigma)$ from P to Q.

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- **That is** there is a map $g : C^0_P(\Sigma) \longrightarrow \mathcal{T}(\Sigma)$, so that

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2. there are constants $B_1, B_2 > 0$, so that for all $P, Q \in C^0_P(\Sigma)$,

$$\frac{1}{B_1} d_\pi(P, Q) - B_2 \leq d_{WP}(g(P), g(Q)) \leq B_1 d_\pi(P,\ Q) + B_2,$$

A metric from TQFT on the pants graph

- TQFT properties imply that the quantum intersection number $n: C^0_P(\Sigma) \times C^0_P(\Sigma) \to [0, \infty)$, is invariant under the action of $\mathrm{Mod}(\Sigma)$:

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A closer look at the quasi-isometry

- **The Teichmüller space:** $\mathcal{T}(\Sigma)$ parametrizes finite area hyperbolic structures on Σ.
- **Points in $\mathcal{T}(\Sigma)$**: =equivalence classes of pairs (X, ϕ), of a finite area hyperbolic surface and a homomorphism $\phi : X \longrightarrow \Sigma$, up to isometry.

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- there are constants $A_1, A_2 > 0$, only depending on Σ, such that

$$\frac{1}{A_1} d_{qt}(P_X, P_Y) - A_2 \leq d_{WP}(X, Y) \leq A_1 d_{qt}(P_X,\ P_Y) + A_2,$$

for any $X, Y \in \mathcal{T}(\Sigma)$.

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2. We show that there is a constant A, only depending on the topology of the surface Σ, such that

$$\frac{A}{3g-3} d_\pi(P,Q) \leq d_{qt}(P,Q) \leq 2d_\pi(P,Q), \tag{1}$$

for any $P, Q \in C_P^0(\Sigma)$.

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$$\frac{A}{3g-3} d_\pi(P,Q) \leq d_{qt}(P,Q) \leq 2 d_\pi(P,Q), \qquad (1)$$

   for any $P, Q \in C^0_P(\Sigma)$.
3. Combine Brock's theorem that $(C^0_P(\Sigma),\ d_\pi)$ is quasi-isometric to $(\mathcal{T}(\Sigma),\ d_{WP})$ with (1) above.

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- Main Theorem implies that for any $\phi \in \mathrm{Mod}(\Sigma)$ the translation lengths $L^{d_{qt}}(\phi)$ and $L^{d_{WP}}(\phi)$ are within bounded ratios from each other, with bounds depending only on the topology of Σ.

Volumes of surface bundles, cont'

Corollary (D-K, '21)

There exist a positive constant N, depending only on Σ, so that for any pseudo-Anosov mapping class $\phi \in \mathrm{Mod}(\Sigma)$ we have

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Remark. Corollary gives relation of quantum intersection number and volume mentioned earlier. (next)

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Corollary

A mapping class $\phi \in \mathrm{Mod}(\Sigma)$ is pseudo-Anosov if and only if for any multicurve γ, we have

$$\lim_{k\to\infty} n(\phi^k(\gamma), P) = \infty,$$

for all $P \in C^0_P(\Sigma)$.

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Corollary

Let $\phi \in \mathrm{Mod}(\Sigma)$ be a pseudo-Anosov mapping class with stretch factor λ_ϕ. Then, for any $P \in C_P^0(\Sigma)$ and any simple closed curve γ, we have

$$\limsup_{k\to\infty} \sqrt[k]{T(\phi^k(\gamma), P)} \leq \lambda_\phi^{\frac{3g-3}{2}}.$$

Proof outline of Main Theorem

- Given $P \in C^0_P(\Sigma)$ and γ= a multicurve,
- We have

$$\frac{I(\gamma, P)}{3g-3} \leq n(\gamma, P) \leq (I(\gamma, P)+1)^{3g-3} - 1.$$

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- $\Gamma := \Gamma_P$=dual graph, E=edges of Γ, $\mathbf{c}$=admissible coloring of E. Recall asymptotic expansion
$$T_r^\gamma(\phi_{\mathbf{c}}) = \sum_{\mathbf{k}} G_{\mathbf{k}}^\gamma(\frac{c}{r}, \frac{1}{r})\phi_{\mathbf{c}+\mathbf{k}},$$
where $G_{\mathbf{k}}^\gamma = G_{\mathbf{k}}^\gamma(\frac{\mathbf{c}}{r}, \frac{1}{r})$ are finitely many analytic functions indexed by functions $\mathbf{k} : E \longrightarrow \mathbb{Z}$.
- Must show: $G_{\mathbf{k}}^\gamma \neq 0$, for at least $\frac{I(\gamma,P)}{3g-3}$ functions $\mathbf{k}$.

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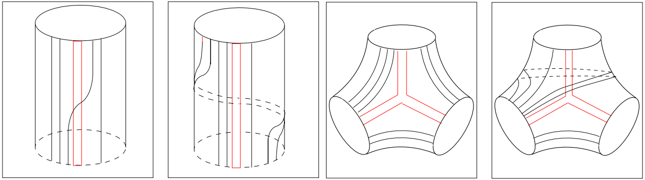
- Put γ in *Thurston-Dehn position* with respect to the decomposition system. The portions of γ on each piece look like:



  - Black : γ in Dehn-Thurston position on pieces of a decomposition system
  - Red: The dual graph Γ.
- Use Masbaum-Vogel fusion rules for $SU(2)$-skein theory to compute matrix coefficients of curve operators.

Outline:

1. For γ in Dehn-Thurston position set

$$M(\gamma,\ P) := \max_{\alpha \in P}\{I(\gamma, \alpha)\}.$$

2. Consider the limit of the functions G_k^{γ}, when $r \to \infty$, and $c_e = a$ for all edges $e \in E$, and $\frac{a}{r} \to \theta$, ($0 \leq \theta \leq 1$). This computes $G_k^{Q}(\theta, 0)$.

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6. Get $M(\gamma, P)$ non-zero funtions $G_{\mathbf{k}}^Q(\frac{\mathbf{c}}{r}, \frac{1}{r})$, giving that $M(P, Q) \leq n(Q, P)$.
7. Since $I(Q, P) \leq (3g - 3)$. $M(Q, P)$ result follows.

More detail: Fusing at a limit

- At the limit of G_k^γ, as $r \to \infty$, $c_e = a$ for all edges $e \in E$ and $\frac{a}{r} \to \theta$, we work with "limit versions" of fusion rules.

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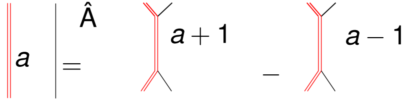
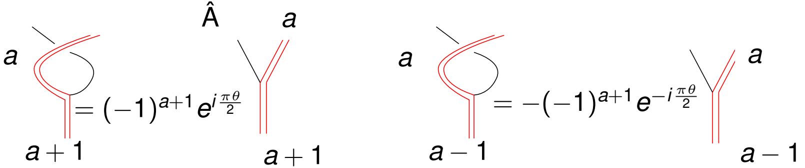
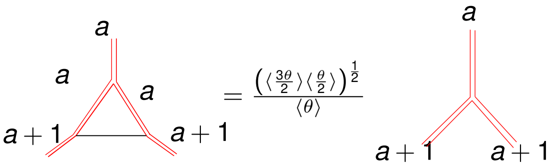
- Sample of Fusion rules at the limit ($\langle\theta\rangle = \sin(\pi\theta)$.)

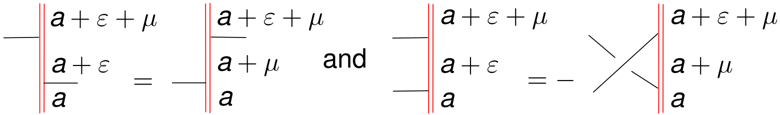




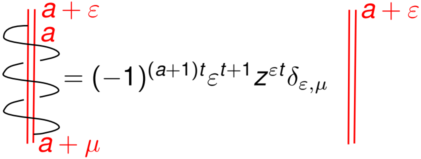


Fusing at a limit: two identities

- P=$\{\alpha_e | e \in E\}$. Use fusion for $\Gamma \cup \gamma$ rules and the following two key identities that hold at the limit to calculate the limits coefficients of matrices for curve operators.
- The color on black edges is 2.
- **First Identity:** *Sliding Lemma.* For $\epsilon, \mu \in \{\pm 1\}$, we have

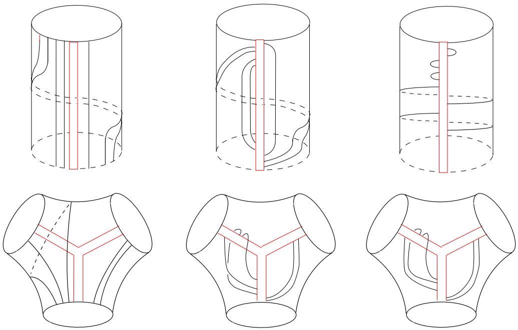


- **Second Identity:** We have ($z = e^{i\frac{\pi\theta}{2}}$, t=# of black arcs)



The process in Pictures

- Patterns $\gamma \cup \Gamma$ on pieces of a decomposition system of Σ.



1. **Top.** Patterns of γ on an annulus piece: Before fusion rules (left), after fusion rules (middle) and after Slidding Lemma (right).
2. **Bottom.** Patterns of γ on a pair of pants piece.
3. Apply more fusion rules and "second identity" on right side to complete the computation.

Restricted colorings

- Decomposition system $P \cup P' = \{\alpha_e, \alpha'_e \ |e \in E\}$ of Σ, where α'_e=parallel copy of α_e.
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- For integers δ, with $|\delta| \leq M(\gamma, P)$ consider coloring functions $\mathbf{k} : E \longrightarrow \mathbb{Z}$ given by

$$\mathbf{k}_\delta(e) = \begin{cases} \delta & \text{if} \quad e = e_0, \\ I(\gamma, \alpha_e) & \text{if} \quad e \neq e_0. \end{cases} ,$$

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  where $\alpha_e \in P$= dual to the edge $e \in \Gamma$.
- Non-vanishing coefficient functions are obtained by showing:

Theorem

For each $|\delta| \leq M(\gamma, P)$ with $\delta \equiv M(\gamma, P) (\bmod 2)$, we have $G^\gamma_{\mathbf{k}_\delta} \neq 0$.

How is done

1. Use state-sum expressions for $G^{\gamma}_{\mathbf{k}_\delta}$ to show that there is a Laurent polynomial $P_\delta(z) \in \mathbb{Q}[z^{\pm 1}]$ so that

$$G^{\gamma}_{\mathbf{k}_\delta} = A(\theta) P_\delta(e^{i\frac{\pi\theta}{2}}),$$

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Remark.

- In Goldman's picture: P induces *torus-action* on $\mathrm{SU}(2)$-moduli space.
- Trace functions f_γ admit a Fourier decomposition w.r.t. to this torus action.
- By Detcherry's work, $G^Q_k(\theta, 0)$ compute these Fourier coefficients.