Quantum representations and geometry of mapping class groups

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December 11, 2019

Outline

$\mathrm{Mod}(\Sigma)$ =mapping class group of surface Σ (closed or with boundary)

- **Quantum Representations.** Given odd integer $r \geqslant 3$, and a primitive $2r$-th root of unity there is a (projective) representation

$$\rho_r : \mathrm{Mod}(\Sigma) \to \mathbb{P}\mathrm{Aut}(RT_r(\Sigma)).$$

- "Large-r" behavior of ρ_r and Nielsen-Thurston Classification : Know facts and and open conjectures *(Andersen-Masbaum-Ueno Conjecture)*.
- Recall basics about TQFT underlying the quantum representations: In particular *Turaev-Viro* invariants of a mapping torus M_f, denoted $TV_r(M_f)$, are obtained from traces of ρ_r.
- **Key point.** Exponential r-growth for $TV_r(M_f)$ implies f satisfies the *AMU* conjecture.
- Manifolds with exponential r-growth for TV_r–Context/Setting (volume conjectures).
- Constructions of mapping tori with exponential r-growth of TV invariants using properties of fibered links (open book decompositions) in 3-manifolds.

Nielsen–Thurston classification

Convention. $\Sigma = \Sigma_{g,n}$= surface of genus g and n-bdry components.

Assume $3g - 3 + n > 0$.

Given a mapping class $f \in \mathrm{Mod}(\Sigma)$ there is a representative $g : \Sigma \longrightarrow \Sigma$ such that at least one holds:

1. g is *periodic*, i.e. some power of g is the identity;
2. g is *reducible*, i.e. preserves some finite union of disjoint simple closed curves Γ on Σ; or
3. g is pseudo-Anosov (never periodic or reducible)

- If $g : \Sigma \longrightarrow \Sigma$ reducible, then a power of g acts on each component of Σ cut along Γ.
- If at least one of the "pieces" is pseudo-Anosov, we say g *has non-trivial pseudo-Anosov pieces.*

Mapping tori and Nielsen–Thurston classification

For $f \in \mathrm{Mod}(\Sigma)$ a mapping class let

$$M_f = \Sigma \times [0,1]/_{(x,0)\cong(f(x),1)}$$

be the mapping torus of f. We have:

- f is *reducible* iff M_f has *incompressible* tori. In that case M_f can be cut along a canonical collection of such tori into geometric pieces (JSJ decomposition-geometric decomposition).
- Each piece of the decomposition will be either *Seifert fibered manifold* or a *hyperbolic*.
- *Gromov norm of M_f:* $v_{\mathrm{tet}}||M_f|| = \mathrm{Vol}\,(H)$, $\mathrm{Vol}\,(H)$ is the sum of the hyperbolic volumes of components of the geometric decomposition.
- f is *periodic* iff M_f is a *Seifert fibered* manifold ($||M_f|| = 0$).
- f is *pseudo-Anosov*, iff M_f has *hyperbolic structure*.
- **Summary:** $f \in \mathrm{Mod}(\Sigma)$ has non-trivial pseudo-Anosov pieces iff $||M_f|| > 0$.

Quantum representations

- *Witten-Reshetikin-Turaev, $SO(3)$-representations*:
- For each odd integer $r \geqslant 3$, let $U_r = \{0, 2, 4, \ldots, r-3\}$.
- Given a primitive $2r$-th root of unity ζ_r, a compact oriented surface Σ, and a coloring c of the components of $\partial\Sigma$ by elements of U_r,
- there is a finite dimensional $\mathbb{C}$-vector space, *$RT_r(\Sigma, c)$* and representations:

$$\rho_{r,c} : \mathrm{Mod}(\Sigma) \to \mathbb{P}\mathrm{Aut}(RT_r(\Sigma, c)).$$

- We have $\dim(RT_r(\Sigma_{g,n}, c) \cong r^{3g-3+n}$. (dimensions grow polynomially in r; Verlinde formula.)
- We will work with $\zeta_r = e^{\frac{i\pi}{r}}$. (*TQFT is not unitary*)

Context:

- **Question.** What geometric information of $\mathrm{Mod}(\Sigma)$ do the representations $\rho_{r,c}$ detect?
- The representations $\rho_{r,c}$ are not faithful! The images of Dehn twists have finite order! However, $\rho_{r,c}$ are asymptotically faithful:

Theorem

(Andersen, Freedman-Walker-Wang, Marché-Narimannejad) Let $f \in \mathrm{Mod}(\Sigma)$. If $\rho_{r,c}(f) = 1$, for all r, c, then $f = 1$. [except in the few cases when $\mathrm{Mod}(\Sigma)$ has center and f is an involution.]

- Hence: There is n, such that

$$(\rho_{r,c}(f))^n = \lambda Id \text{ for all } r, c, \text{ iff } f^n = 1.$$

(i.e f is periodic) [*again some exceptions*].

- **Conjecture.** *(AMU, 2004)* $f \in \mathrm{Mod}(\Sigma)$ has PA pieces iff for every $r >> 0$ there a choice of colors c such that $\rho_{r,c}(f)$ has infinite order.

Remark. $f \in \mathrm{Mod}(\Sigma)$ satisfies the AMU iff at least of its PA pieces does.

What is known:

- *Andersen, Masbaum and Ueno (2004)* proved their conjecture when $\Sigma = \Sigma_{0,4}$ = the four-holed sphere.
- *Santharoubane* proved the conjecture for the one-holed torus.
- *Egsgaard and Jorgensen (2012) and Santharoubane (2015)* proved the conjecture for families for mapping classes in $\Sigma = \Sigma_{0,n}$, for all $n > 4$.
- In all above cases the quantum representations turn out to be related to previously studied braid group representations: (specializations of Burau representations, McMullen's representations related to actions on homology of branched covers of $\Sigma_{0,n}$.)
- For surfaces of genus $g > 1$ no examples known till 2016.
- Using *Birman exact sequences* of mapping class groups, one extracts representations of $\pi_1(\Sigma)$ *from the representations* $\rho_{r,c}$.
- *Marché and Santharoubane* used these representations to obtain examples of pseudo-Anosov mappings classes satisfying the AMU conjecture by exhibiting "apppropriate" elements in $\pi_1(\Sigma)$. Gave explicit curves on genus 2 surfaces (more next).

Quantum representations of surface groups

- $\chi(\Sigma) < 0$ and x_0 a marked point in the interior of Σ and $\mathrm{Mod}(\Sigma, x_0)$ group of classes preserving x_0.
- **Birman Exact Sequence.**

$$0 \longrightarrow \pi_1(\Sigma, x_0) \longrightarrow \mathrm{Mod}(\Sigma, x_0) \longrightarrow \mathrm{Mod}(\Sigma) \longrightarrow 0.$$

- **Kra's criterion.** $\gamma \in \pi_1(\Sigma, x_0)$ represents a pseudo-Anosov mapping class iff γ *fills* Σ.
- The quantum representations give representations:

$$\rho_{r,c} : \pi_1(\Sigma) \to \mathbb{P}\mathrm{Aut}(RT_r(\Sigma, c)).$$

- (*Koberda-Satharoubane*) used $\rho_{r,c}$ to answer an open question (asked by several people independently *Kent, Kisin, Marché, McMullen, ...*):
- Constructed a linear representation of $\pi_1(\Sigma, x_0)$, that has infinite image, but the image of every simple closed curve has finite order!
- Their work led to (another) algorithm that decides whether or not $\gamma \in \pi_1(\Sigma, x_0)$ is freely homotopic to a simple loop!

The examples of Marché-Satharoubane

- Gave first examples of pseudo-Anosov mapping classes, for surfaces of genus > 1, that satisfy the following *(implied by AMU)*.
- **AMU Conjecture for surface groups.** If a non-trivial element $\gamma \in \pi_1(\Sigma, x_0)$ is not a power of a class represented by a simple loop, then $\rho_{r,c}(\gamma)$ has infinite order for $r >> 0$ and a choice of c.
- Their examples are realized by immersed curves that *fill* Σ and satisfy an additional technical condition they called *Euler incompressibility*.
- They use WRT-TQFT (at "usual" root of unity) to construct a *(Jones-type)* polynomial invariant for links in in $S^1 \times \Sigma$. Roughly speaking, non-triviality of the invariant for $\gamma \in \pi_1(\Sigma, x_0)$, viewed as link in $S^1 \times \Sigma$, implies that γ satisfies the AMU Conjecture for surface groups. *Euler incompressibility* of γ is used to derive non-triviallity.
- For fixed genus, their criterion, leads to finitely many (up to conjugation and powers) pseudo-Anosov mapping classes that satisfy the AMU Conjecture.
- Gave explicit examples in genus two. The first evidence for AMU conjecture for genus > 1.

Another approach: Growth of TV invariants and AMU

- $r = odd$, $TV_r(M) := TV_r(M, e^{\frac{2\pi i}{r}})$= Turaev-Viro invariant at level r,

$$ITV(M) := \liminf_{r\to\infty} \frac{2\pi}{r} \log |TV_r(M)|.$$

- *(Generalized) Q. Chen- T. Yang volume conjecture*:

$$ITV(M) = v_{\text{tet}}||M|| > 0$$

.

- **(Weaker) Exponential Growth Conjecture:**

$$ITV(M) > 0 \ \text{ iff } \ ||M|| > 0$$

- Relevance to AMU Conjecture: EGC imples AMU:
- **Proposition.** *(Detcherry-K., 2017)* Let $f \in \mathrm{Mod}(\Sigma)$ mapping class and let M_f be the mapping torus of f. If $ITV(M_f) > 0$, then f satisfies AMU.
- **Key point:** View $TV_r(M)$ as part of the $SO(3)$-TQFT theory rather than a combinatorial *state sum* defined on triangulations of M.

TV invariants as part of a TQFT

- *Witten-Reshetikhin-Turaev* TQFT/ *Blanchet-Habegger-Masbaum-Vogel.*
- For $r \geq 3$ and $\zeta_r = e^{\frac{i\pi}{r}}$, we have a TQFT functor RT_r:
- *M=closed*, oriented 3-manifold *$RT_r(M)$=$\mathbb{C}$-valued invariant*.
- *Σ=compact, oriented surface, w. U_r-coloring c of $\partial\Sigma$,*

$$RT_r(\Sigma, c) = f.d.\, \mathbb{C} - \text{vector space.}$$

- *M=cobordism with $\partial M = -\Sigma_0 \cup \Sigma_1$*, there is a map

$$RT_r(M) \in \mathrm{End}(RT_r(\Sigma_0), RT_r(\Sigma_1)).$$

- RT_r takes composition of cobordisms to composition of linear maps.
- We get

$$\rho_{r,c} : \mathrm{Mod}(\Sigma) \to \mathbb{P}\mathrm{Aut}(RT_r(\Sigma, c)).$$

- If $\partial\Sigma = \emptyset$, and *$C_f$=mapping cylinder of f, $\rho_r(f) = RT_r(C_f)$.*
- If $\partial\Sigma \neq \emptyset$ we color $\partial\Sigma$ with elements of U_r. To define $\rho_{r,c}$ need RT_r for cobordisms w. colored tangles.

Proof of Proposition:

- By *Beliakova, Roberts, Turaev, Walker, (Benediti-Pertronio)* and TQFT structure

$$TV_r(M_f) = \sum_c \left(\mathrm{Tr}\rho_{r,c}(f)\right)^2 .$$

where the sum ranges over all colorings of the boundary components of M_f by elements of U_r.

- *Since $ITV(M_f) > 0$, the sequence $\{TV_r(M_f)\}_r$ is bounded below by a sequence that is exponentially growing in r as $r \to \infty$.*
- The sequence $\sum_c \dim(RT_r(\Sigma, c))$ only grows polynomially in r.
- So, there will be at least one c such that $|\mathrm{Tr}\rho_{r,c}(f)| > \dim(RT_r(\Sigma, c))$.
- Then $\rho_{r,c}(f)$ must have an eigenvalue of modulus bigger than 1. Thus it has infinite order. □

More detail: Torus orthonormal basis

- RHS evaluated at $\zeta_r = e^{\frac{i\pi}{r}}$, and LHS at ζ_r^2, and $\langle .. \rangle$=Hermitian pairing of $RT_r(\Sigma, c)$.

$$TV_r(M_f) = ||RT_r(M_f)||^2 = \langle RT_r(M_f),\ RT_r(M_f)\rangle.$$

- $RT_r(\partial M_f)$ has orthonormal basis $\mathbf{e}_c$, where c runs over all n-tuples; one for each boundary component.
- $\mathbf{e}_c$ is also the RT_r-vector of the cobordism of n solid tori, with the i-th solid torus containing the core colored by c_i.
- Write $RT_r(M_f) = \sum\limits_c \lambda_c \mathbf{e}_c$. Thus

$$TV_r(M_f) = \sum_c |\lambda_c|^2 = \sum_c |\langle RT_r(M_f), \mathbf{e}_c\rangle|^2.$$

- to get $\langle RT_r(M_f), \mathbf{e}_c\rangle$: fill ∂-components of M_f; add link L= union cores colored by c_i. Thus

$$\langle RT_r(M_f), \mathbf{e}_c\rangle = RT_r(M_{\tilde{f}}, (L, c)) = \mathrm{Tr}(\rho_{r,c}(f)).$$

$ITV(M) > 0$ implies $||M|| > 0$

Theorem

(Detcherry-K., 2017) There exists a universal constant $C > 0$ such that for any compact orientable 3-manifold M with empty or toroidal boundary we have

$$ITV(M) \leq C||M||.$$

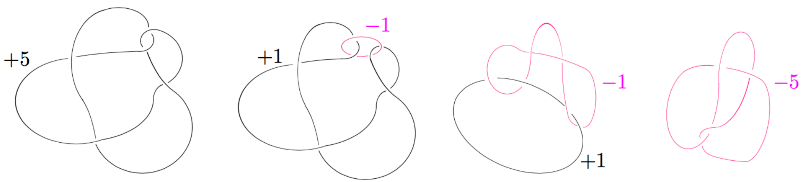
- **Remark.** if $ITV(M_f) > 0$, for some mapping class f, then f satisfies AMU.
- Computing ITV is hard! But we don't always have to compute it to decide exponential growth!
- **Limits do not increase under Dehn filling.** *(Detcherry-K)* If M is obtained by Dehn filling from M' then

$$ITV(M) \leq ITV(M')$$

- **Example.** Adding components to a link preserves exponential growth of TV invariants of link complement.

An example: Knot 5_2 and parents

- $K(p)$= 3-manifold obtained by p-surgery on M.
- $ITV(4_1(-5)) = \mathrm{Vol}\,(4_1(-5)) \simeq 0.9813688 > 0$ [Ohtsuki, 2017]
- Observe $5_2(5)$ is homeomorphic to $4_1(-5)$.



- Dehn filling result implies $ITV(S^3 \smallsetminus 5_2) \geqslant ITV(5_2(5)) = ITV(4_1(-5)) > 0$
- But Dehn filling result also implies that for any link containing 5_2 as a component we have exponential growth

$$ITV(S^3 \smallsetminus L) \geqslant ITV(S^3 \smallsetminus 5_2) > 0.$$

Manifolds with $ITV(M) = v_3||M|| > 0$

- *(Detcherry-K- Yang, 2016)* Figure-8 knot and Borromean rings complements.
- *(Ohtsuki, 2017)* Infinite family of closed hyperbolic 3-manifolds: Manifolds obtained by integral integer fillings of S^3 along Figure-8 knot complement.
- *(Belletti-Detcherry-K.- Yang, 2018)* Infinite family of cusped hyperbolic 3-manifolds. These are the complements of *Fundamental Shadow Links* in connected sums of copies of $S^1 \times S^2$.
- *(Constantino- D. Thurston, 2005)* Every orientable 3-manifold M with empty or toroidal boundary contains a complement of a FSL: M contains links $L \subset M$ with $ITV(M \setminus L) > 0$. *Doubles of link complements give closed 3-manifolds with $ITV > 0$*
- *Kumar, Belletti, 2019* : More octahedral link complements.
- For the AMU conjecture we need: mapping tori M_f with $ITV(M_f) > 0$.!
- There exist many fibered links in all (closed) 3-manifolds!. Look at fibered links in closed manifolds.

Pseudo-Anosov mappings

- **(Vague) Question.** For $n > 0$. How large is the class of $f \in \mathrm{Mod}(\Sigma_{g,n})$ realized as monodromies of fibered links in closed 3-manifolds we know to have *ITV* > 0? All of them?

Theorem

(Detcherry-K, 2019) For $g >> 0$, there is $f \in \mathrm{Mod}(\Sigma_{g,1})$ and a rank $\left\lfloor \frac{g}{2} \right\rfloor$ free abelian subgroup

$$H < \mathrm{Mod}(\Sigma_{g,1}),$$

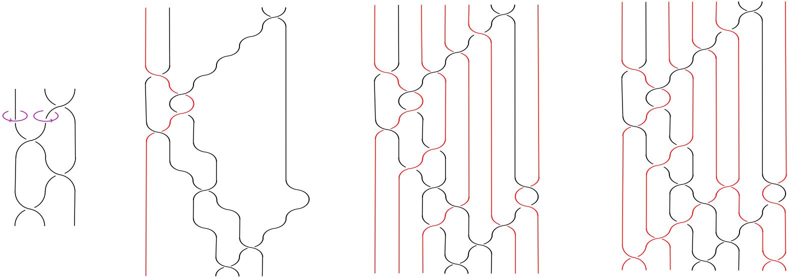
such that any class in the coset fH is PA and satisfies the AMU conjecture.
• Similarly there is $g \in \mathrm{Mod}(\Sigma_{g,1})$ there is a rank two free subgroup

$$F < \mathrm{Mod}(\Sigma_{g,1}),$$

such that any class in the coset gF is PA and satisfies the AMU.

- **Note.** No examples of PA mappings for closed surfaces of $g > 2$ that satisfy the AMU are known.

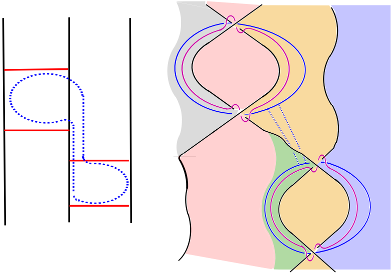
- Start with $L \subset S^3$ be a link with $ITV(S^3 \setminus L) > 0$.
- (*Stallings, 60's*) We can add a component K so that $K \cup L$ is a fibered.
- In fact, $K \cup L$ will be a closed *homogeneous braid* and fiber is a Seifert surface obtained from closed braid projection.



- Refine process so that $K \cup L$ is a hyperbolic and $ITV(S^3 \setminus (L \cup K)) > 0$.
- There are only finitely many f. m. link types in homogeneous closed braids of fixed genus! *No problem:* Use Stallings twists....“wisely”.

Stallings twists

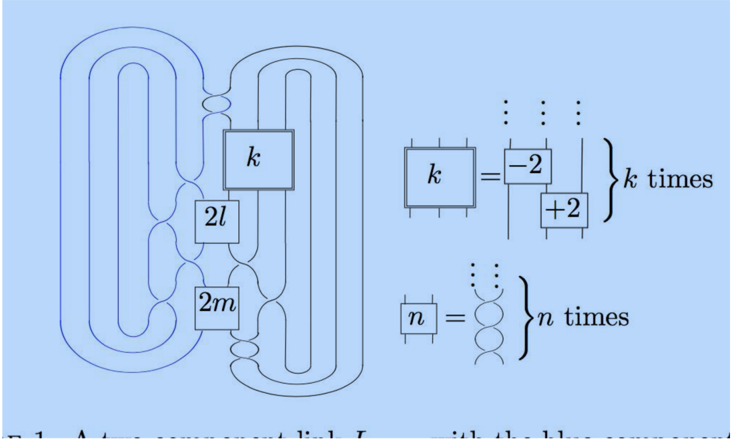
- L fibered link with fiber F and monodromy f.
- c= a non-trivial s.c.c on the fiber with $lk(c, c^+) = 0$, c^+ is the curve c pushed along the positive normal of F. *Need c not parallel to ∂F that bound a disc in $D \subset S^3$:*



- **A Stallings twist of order** m**:** *A full twist of order m along D.*
- Gives fibered links L_m with fiber F and monodromy $f \circ \tau_c^m$,, where τ_c= Dehn-twist on F along c.
- *(Long-Morton, Fathi)* If f pseudo-Anosov, for all $m >> 0$, $f \circ \tau_c^m$ is pseudo-Anosov.

Concrete examples: start with $ITV(S^3 \setminus 4_1) > 0$.

- 4_1=closure of the alternating braid $\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$.
- Fibered, hyperbolic, monodromies in $\mathrm{Mod}(\Sigma_{7+2k,2})$, for $m = 4, l = 1$.
- Fiber supports k-Stallings twists.



Concrete examples Cont'n

- Take a monodromy f for $L(4,1,k)$. Dehn twists on Stallings curves generate Rank k abelian subgroup of $H < \mathrm{Mod}(\Sigma_{7+2k,2})$.
- elements in coset fH satisfy AMU. For large powers of Dehn twists maps are all pseudo-Anosov.
- Dehn (-5)-surgery on the figure-8 component with produces examples in $N = 4_1(-5)$. This manifold is hyperbolic.
- The result of $L(4,1,k)$ in the closed manifold is a knot, $K(k)$ that fibers with fiber the fiber of $L(4,1,k)$ with one boundary component capped-off. Monodromies are in $\mathrm{Mod}(\Sigma_{7+2k,1})$.
- The Stallings twists will survive the surgery along 4_1. We get abelian subgroup of rank k in $H_k < \mathrm{Mod}(\Sigma_{7+2k,1})$.
- We have $ITV(N) > 0$ by Ohtsuki's result! Hence, all link complements in N have same property.
- For k take f a monodromy so that $N \setminus K(k) = M_f$. Now all mappings of the form fH_k are realized in $N = 4_1(-5)$.

Mapping Tori: Integer and non integer values of TV_r

- *(D-K)* Let M_f be the mapping torus of a periodic mapping class $f \in \mathrm{Mod}(\Sigma)$ of order N. Then, for any odd integer $r \geqslant 3$, with $\gcd(r, N) = 1$, we have $TV_r(M_f) \in \mathbb{Z}$, for any choice of root of unity.
- **Corollary.** For co-prime integers p, q let $T_{p,q}$ denote the (p, q)-torus link. Then, for any odd r co-prime with p and q, we have $TV_r(S^3 \smallsetminus T_{p,q}) \in \mathbb{Z}$.
- In particular: $TV_r(M_f) \in \mathbb{Z}$, for infinitely many r.
- If $lTV(M_f) > 0$ at some root of unity, then there can be at most finitely many values r for which $TV_r(M_f) \in \mathbb{Z}$.
- **Conjecture.** Suppose that $f \in \mathrm{Mod}(\Sigma)$ contains a PA part. Then, there can be at most finitely many odd integers r such that $TV_r(M_f) \in \mathbb{Z}$.

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