Jones diameter and crossing number of knots

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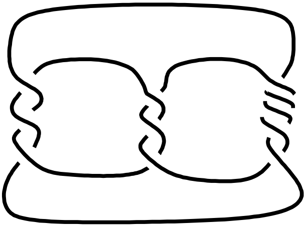
Crossing numbers.

- **Knots**: Smooth embeddings $S^1 \longrightarrow S^3$, up to ambient isotopy in S^3.
- Knots are studied through generic projections (a.k.a. *knot diagrams*) on a plane $S^2 \subset S^3$.



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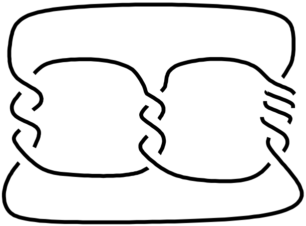
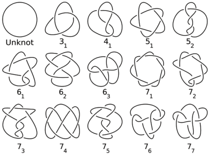


- Given a knot K, the *crossing number* $c(K)$ is the smallest number of crossings over all knot diagrams representing K.
- Hard to calculate for arbitrary knots.
- Behavior under basic topological operations (e.g. *connected sum, satellite operations*) still poorly understood.

Knot tables.

- Enumeration techniques have produced knot tables of low crossing numbers.
- E. g. To find the crossing number of a knot given by a diagram of 7 crossings: List all knot diagrams that have 7 or less crossings. Use topological methods/invariants to decide the different knot types.
- Arrive at the table of 15 *prime* knot types (up to reflection/orientation change):



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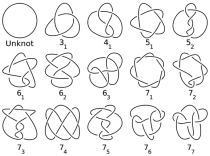
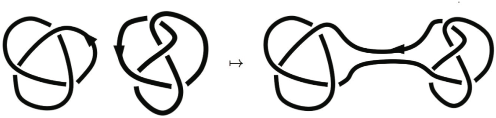


- There are 352,152,252 distinct knots up to 19 crossings!!.

Topological operations: connected sums.

- Oriented knot diagrams $D(K), D(K')$ and connected sum $D(K)\#D(K')$.



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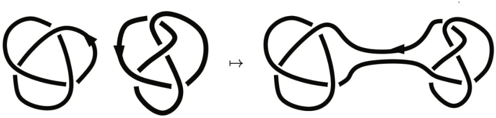
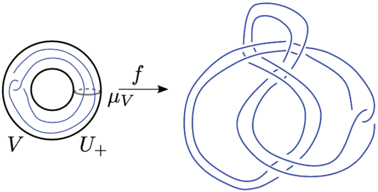


- Connected sum is well defined on knots, not just diagrams. So $D(K)\#D(K')$ is a knot diagram of the connected sum $K\#K'$.
- In the example, $D(K)$ and $D(K')$ are *minimum* (i.e. they realize the crossing number of the knots they represent).
- Does the knot $K\#K'$ admit a projection with less than 7 crossings? Not in this example, but
- **Conjecture.** (open) Crossing number is additive under connected sum : $c(K\#K') = c(K) + c(K')$.

Topological operations: Satellites.

- **Satellites.** *Satellite knot with companion K and pattern U_+*: Start with U_+ embedded "essentially" in standard solid torus $V \subset R^3$.
- Re-embed $f : V \longrightarrow V(K) \subset S^3$, where $V(K)$=neighborhood of K.



- Knot $f(U_+)$ is uniquely defined once the image of the *canonical longitude* (unique generator of $H_1(V)$ that is trivial in $H_1(S^3 \setminus V)$), is specified under

$$f_* : H_1(\partial V) \longrightarrow H_1(\partial V(K)).$$

- *Untwisted* satellite: f_* takes the canonical longitude in $H_1(V)$ to the canonical longitude in $H_1(V(K))$.
- Above Figure: *Untwisted Whitehead double* of figure-8 knot: $W(K)$, $K = 4_1$.

Crossing numbers of satellites?

- What is the crossing number of $W(4_1)$? Is the diagram below minimum?

- Yes in this case, but
- in general, the behavior of crossing numbers of satellites and relation with these of companions is not understood.
- **Question.** (open) Suppose that $S(K)$ is a satellite knot with companion K. Is it true that

$$c(S(K)) > c(K)?$$

Known results.

- **General bounds:**
- (Lakenby, 2005) For any knots K_1, K_2 we have

$$c(K_1) + c(K_2) \geq c(K_1 \# K_2) \geq \frac{c(K_1) + c(K_2)}{152}.$$

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- Results support above mentioned conjectures and apply to all knots!
- General bounds are not good enough to be used for determination of crossing numbers of any knots.
- For example, for $W(4_1)$,

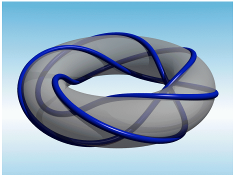
$$c(W(4_1)) \geq 10^{-13} c(4_1) = 4.10^{-13}.$$

- There are better bounds and exact determinations for important classes of knots.

Exact results for classes.

- (Murasugi) *Torus knots*: For $p, q > 0$, $T_{(p,q)} = (p, q)$-torus knots, then

$$c(T_{(p,q)}) = \min((p-1)q, (q-1)p).$$



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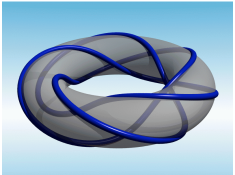
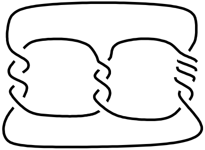


- *Alternating knots*: Diagrams w. over-under-over... crossings



- (Kauffman, Murasugi, Thistlethwaite, 1980's) A reduced alternating diagram is minimum. This was the (Tait Conjecture) formulated in 1800's.
- Additivity Conjecture holds for alternating knots (Kauffman, Murasugi, Thistlethwaite).

Exact results for classes, cont.

- (Lickosrish, Thistlethwaite, 80's) Studied *adequate knots*; a broader class than alternating knots and determined their crossing numbers.
- Adequate knots admit "special" knot diagrams; these diagrams realize the crossing number.
- The *writhe* (algebraic crossing number) of such "special" diagram $D = D(K)$ is invariant of K.
- (Lickorish-Thistlethwaite, 80's) Crossing numbers for Montesinos knots (sums of alternating tangles).
- In above cases a "special" diagram of K gives $c(K)$.

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- In above cases a "special" diagram of K gives $c(K)$.
- (K.-Lee, '21) Crossing numbers of first infinite families of prime satellites.

Theorem

Let $W(K)$=untwisted Whitehead double of a knot K. If K is adequate with crossing number $c(K)$ and writhe number zero, then $c(W(K)) = 4.c(K) + 2$.

Exact results for classes, cont.

- "Doubling" an adequate diagram $D = D(K)$, with writhe zero, produces a minimum crossing number of $W(K)$.
- Crossing number of untwisted Whitehead doubles of figure-8 is 18.

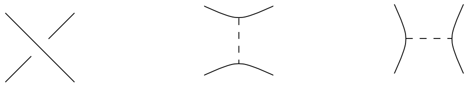
- Plenty of adequate knots with zero writhe number:

Corollary

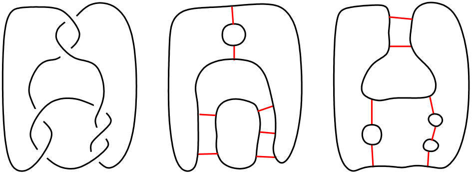
If K is adequate, with mirror image K^*, then $c(W(K\#K^*)) = 8.c(K) + 2$.

Alternating/Adequate knots.

Two choices for each crossing, of knot diagram D: A-resolution (middle) and B-resolution (right).

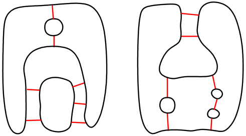


- A Kauffman *state* $\sigma(D)$ is a choice of A or B resolutions for all crossings.
- $\sigma(D)$: *state circles*.
- Form a *fat graph* H_σ by adding edges at resolved crossings.



Alternating/Adequate knots, con'd.

- K is called *A–adequate* if has a diagram $D = D(K)$ where the all-A state graph $H_A = H_A(D)$ has no 1-edge loops.
- Similarly we have B-adequate
- Left: graph from adequate state. Right: Graph from inadequate state.



- K is *adequate* if it admits a diagram that is both *A and B–adequate*.
- Introduced by (Lickorish–Thistlethwaite, 80's) while studying *Jones polynomials*.
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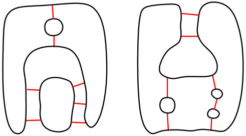

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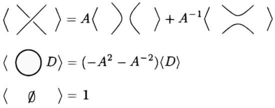
- Introduced by (Lickorish–Thistlethwaite, 80's) while studying *Jones polynomials*.
- Reduced alternating diagrams are A and B- adequate.
- (Jones, 80's) Constructed a Laurent polynomial invariant of knots $J_K(t) \in \mathbb{Z}[t, t^{-1}]$, that can be computed from any diagram $D = D(K)$.
- (KMT) The Tait conjecture is implied by: For any diagram $D = D(K)$,

$$\text{degree span of } J_K(t) \leq \text{number of crossings of D},$$

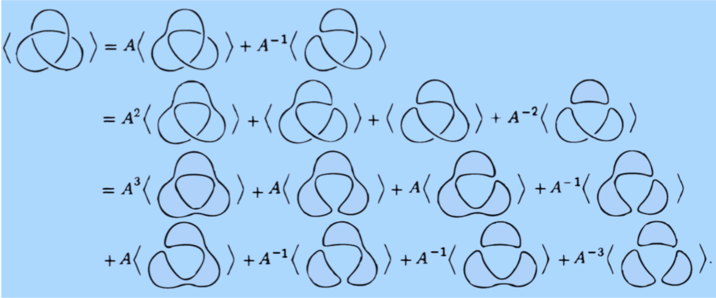
with equality if and only if $D = D(K)$ is reduced alternating.

Calculation of CJP: Example.

- Kauffman bracket: $\langle\ \rangle$: link diagrams $\longrightarrow \mathbb{Z}[A,\ A^{-1}]$ such that



- For $D = D(K)$ where $K = \text{trefoil knot}$:



- We obtain: $J_K(t) = \frac{A^{-9}}{A^2+A^{-2}}\langle D\rangle|_{t:=A^{-4}} = t + t^3 - t^4.$

A correction term: Turaev genus.

- Hence, for alternating knots we have

$$\text{degree span of } J_K(t) = c(K).$$

- For adequate knots, that are not alternating,

$$\text{degree span of } J_K(t) = c(K) - g_T(K) < c(K)$$

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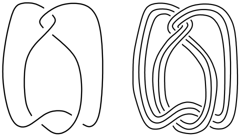
- To determine the crossing number must look at Jones polynomials of all the "parallels" of adequate knots.



- (KMT) Adequate diagrams realize the crossing number of knots they represent.

The colored Jones polynomial knot invariants.

- For non-adequate knots (with Lee) we use the *colored Jones polynomials*.
- Colored Jones function: sequence $\{J_K(n)\}_n$ of Laurent polynomials in t.
- The Jones polynomial corresponds to $n = 2$.
- (Garoufalidis - Le, 2005) $\{J_K(n)\}$ satisfies a l linear recurrence relation

$$a_d(t^{2n}, t)J_K(n+d) + \cdots + a_0(t^{2n}, t)J_K(n) = 0$$

  for all n, where $a_j(u, v) \in \mathbb{C}[u, v]$. *q-holonomicity*.
- Example: for the only crossing number three knot (a.k.a. trefoil)

$$J_K(n) = t^{-6(n^2-1)} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{24j^2+12j} \frac{t^{8j+2} - t^{-(8j+2)}}{t^2 - t^{-2}}.$$

- Recurrence relation

$$(t^{8n+12} - 1)J_K(n+2) + (t^{-4n-6} - t^{-12n-10} - t^{8n+10} + t^{-2})J_K(n+1)$$
$$-(t^{-4n+4} - t^{-12n-8})J_K(n) = 0.$$

Impact of q-holonomicity on the degree of CJP.

- Let $d_+[J_K(n)]$ and $d_-[J_K(n)]$ denote the maximal and minimal degree of $J_K(n)$ in t, and set

$$d[J_K(n)] := 4d_+[J_K(n)] - 4d_-[J_K(n)] := s_2(n)n^2 + s_1(n)n + s_0(n),$$

$$s_i : \mathbb{N} \longrightarrow \mathbb{Q}, \quad i = 0, 1, 2.$$

- " q-holonomicity" implies that the set of cluster points $\{s_2(n)\}'_{n\to\infty}$ is finite.
- Point with the largest absolute value, denoted by dj_K, is called the *Jones diameter* of K.

Theorem

(Lickorish-Thistlethwaite, 80's) For any knot we have

$$dj_K \leq 2c(K),$$

where $c(K)$ is the crossing number of K.
If K is adequate then we have equality.

- With Lee we prove the converse: $dj_K = 2c(K)$, implies K is adequate.

Knots of maximal Jones diameter.

- K.-Lee, 2021:

Theorem

Let K be a knot with Jones diameter dj_K and crossing number $c(K)$. Then,

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- In fact, we show:
- Suppose a knot K admits a diagram $D = D(K)$, with $c := c(D)$, crossings and such that $dj_K = 2c(D)$. Then D must be an adequate diagram.
- So if D realizes $c(K)$ and $dj_K = 2c(D) = 2c(K)$, for some knot K, then D is adequate.

Crossing number application.

- Theorem has immediate corollary: A diagram with number of crossings "too close" to the Jones diameter gives the crossing number of the knot!!

Corollary

Suppose K is a non-adequate knot admitting a diagram $D = D(K)$ such that

$$dj_K = 2(c(D) - 1).$$

Then we have $c(D) = c(K)$.

Proof. Since K is non-adequate, Theorem gives that $2c(K) > dj_K$. Hence we get $c(D) \geq c(K) > \frac{dj_K}{2} = c(D) - 1$, giving $c(D) = c(K)$. □

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- **Example.** For $K = W(figure - 8)$, by Baker-Motegi-Takata, $dj_K = 34 = 2.17 = 2(18 - 1)$.
- Doubling the standard diagram of figure-8 produces a diagram of 18 crossings.
- The knot $K = W(figure - 8)$ is not adequate!

Doubles of amphicheiral knots.

- If K is amphicheiral adequate knot then $\mathrm{wr}(K) = 0$.

Corollary

Suppose that K is an amphicheiral adequate knot with crossing number $c(K)$. Then $c(W(K)) = 4c(K) + 2$.

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Suppose that K is an amphicheiral adequate knot with crossing number $c(K)$. Then $c(W(K)) = 4c(K) + 2$.

- For any even $n > 0$ there are alternating, amphicheiral knots $c(K) = n$.
- K= figure-8 knot is the 1st example: We have

$$c(W(\#_m K)) = 16m + 2.$$

- Prime amphicheiral adequate knots with $C(K) \leq 12$. (Knotinfo Cha-Livingston-Moore).

4_1	8_{18}	10_{43}	$12a_{435}$	$12a_{506}$	$12a_{1105}$	$12a_{1275}$
6_3	10_{17}	10_{45}	$12a_{471}$	$12a_{510}$	$12a_{1127}$	$12a_{1281}$
8_3	10_{33}	10_{99}	$12a_{477}$	$12a_{1019}$	$12a_{1202}$	$12a_{1287}$
8_9	10_{37}	10_{123}	$12a_{499}$	$12a_{1039}$	$12a_{1273}$	$12a_{1288}$

- Out of the 2977 prime knots with up to 12 crossings, 1851 are listed as adequate on Knotinfo and thus Corollary applies to $K\#K^*$.

Crossing number bounds from the CJP.

- Bounds obtained for families are much stronger than the known bounds of general knots and are compatible with conjectural bounds for general knots.
- Whitehead doubles of non-zero writhe adequate knots:

Theorem (K.-Lee)

Suppose that K is an adequate knot with crossing number $c(K)$ and writhe $\mathrm{wr}(\mathrm{K})$. Then, the crossing number $c(W(K))$, of the untwisted Whitehead double of K, satisfies the following inequalities.

$$4c(K) + 1 \leq c(W(K)) \leq 4c(K) + 2 + 2|\mathrm{wr}(K)|.$$

In particular, if $\mathrm{wr}(K) = 0$, then $W(K)$ is non-adequate we have $c(W(K) = 4c(K) + 2$

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- Whitehead doubles of torus knots (non-adequate): For the torus knot $T_{p,q}$ we have

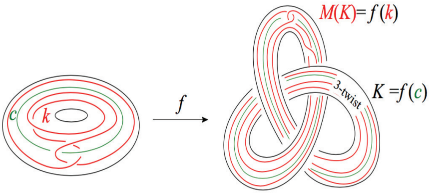
$$c(W_{\pm}(T_{p,q})) > 2c(T_{p,q}).$$

Mazur doubles.

Theorem

(Baker-Motegi-Takata, 2022) Suppose that K is an adequate knot with crossing number $c(K)$ and writhe $\mathrm{wr}(\mathrm{K})$. Then, the crossing number $c(W(K))$, of a Mazur double $M(K)$, satisfies the following inequalities.

$$9c(K) + 2 \leq c(W(K)) \leq 9c(K) + 3 + 6|\mathrm{wr}(K)|.$$



Mazur double of K

Questions:

- *wrapping number* $\omega = \omega(S(K))$ of a of satellite knot $S(K)$, is geometric the intersection number of $S(K)$ with a meridian disk of neighborhood of the companion.
- For Whitehead doubles $\omega = 2$ and for Mazur double $\omega = 3$.
- **Speculation.** The lower bound on crossing numbers given by the degree of the CJP should give the following: If $S(K)$ is a satellite of an adequate knot then $c(S(K)) \geq \omega^2.\ c(K)$.

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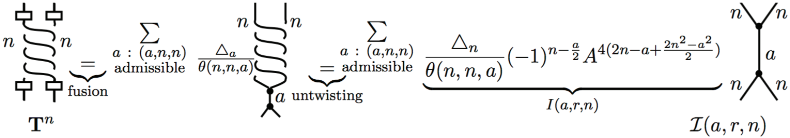
- **Speculation.** The lower bound on crossing numbers given by the degree of the CJP should give the following: If $S(K)$ is a satellite of an adequate knot then $c(S(K)) \geq \omega^2.\ c(K)$.
- *winding number* $\omega_h = \omega_h(S(K))$ of a of satellite knot $S(K)$, is the algebraic intersection number of $S(K)$ with a meridian disk of neighborhood of the companion.
- For Whitehead doubles $\omega_h = 0$ and for Mazur double $\omega_h = 1$.
- For Whitehead doubles we determined the crossing numbers; for Mazur doubles the method restricts the crossing number two possible values.
- **Question.** Suppose that K is an adequate knot with $\mathrm{wr(K)} = 0$. For what zero winding number satellites of K, can we determine the crossing number using the method used for $W(K)$?

How do you compute the CJP?

- Computing the *n*-th CJP of *K* involves computing Jones polynomials of all *i*-parallels of *K*, for $0 < i \leq n$.
- Process is facilitated by viewed knot diagrams as elements in a certain "Temperlie-Lieb" algebra, decorated by "Jones-Wenzl" idempotents and using "fusion algebra" underlying the combinatorics of the representation theory of the "$SU(2)$-quantum group".
- " Fusion Rules" also involve *colored trivalent graphs* that enter the picture through: (Variable $t = A^{-4}$).



- Additional rules allow to reduce complexity of resulting colored trivalent graphs and reduce the calculation to "basic blocks"

What is the function $\theta(a, b, c)$?

- To illustrate the complexity involved we furthers discuss the function $\theta(a, b, c)$.
- For a, b, c integers, with $a + b + c$ is even, $a \leq b + c$, $b \leq a + c$, and $c \leq a + b$, we have

$$\theta(a, b, c) = \frac{\triangle_{x+y+z}!\triangle_{x-1}!\triangle_{y-1}!\triangle_{z-1}!}{\triangle_{y+z-1}!\triangle_{z+x-1}!\triangle_{x+y-1}!},$$

- where,

1. $x = \frac{a+c-b}{2}, y = \frac{b+c-a}{2}, z = \frac{a+b-c}{2}$,
2. $\triangle_n! := \triangle_n \triangle_{n-1} \triangle_{n-2} \cdots \triangle_1$ and $\triangle_{-1} = \triangle_0 := 1$.
3. $$\triangle_c = (-1)^c \frac{A^{2(c+1)} - A^{-2(c+1)}}{A^2 - A^{-2}}.$$
4. Degree span $d[J_K(n)]$, easy to compute for adequate knots.. hard in general

Why compute the CJP?

- There are open conjectures about the degrees $d[J_K(n)]$.
- The degrees $d[J_K(n)]$ encodes important information about π_1-injective surfaces in the complement of K (*Slopes Conjectures*).
- Slopes conjectures predict:
- the degree $d[J_K(n)]$ detects the trivial knot and torus knots
- the degree $d[J_K(n)]$ characterizes alternating knots
- K is alternating if and only if

$$2d_+[J_K^n] - 2d_-[J_K^n] = cn^2 + (2-c)n - 2, \qquad (*)$$

  for some integer $c \geq 0$.
- The CJP conjecturally is related to character varieties of knots (*AJ-Conjecture*).
- *Volume Conjecture*: The colored Jones polynomials of a hyperbolic knot determine the volume of the knot complement.