Day 17 Review of Linear Transformations (Due Wednesday Feb 19).

Sections 2.1 and 2.2 introduced linear transformations. You should memorize the definition below (this, and the definition of vector space, are the two most important definitions of the course). Let V and W be vector spaces. A *map* from V to W, written $T : A \to B$, assigns to each $\mathbf{v} \in V$ an element $T(v) \in W$. In Linear Algebra, we are interested only in those maps that behave nicely under vector addition and scalar multiplication:

Definition 0.1. *A map $T : V \to W$ between vector spaces V and W is a* linear transformation *if*

$$T(\mathbf{u}+\mathbf{v}) \;=\; T(\mathbf{u})+T(\mathbf{v}) \qquad \textit{and} \qquad T(r\mathbf{v}) \;=\; rT(\mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in V$ and $r \in \mathbb{R}$.

Everything we do with linear transformations is based on two principles:

- **Linear Transformations respect linear combinations** in the sense that

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) \;=\; c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n).$$

- **Linear Transformations are determined by what they do to basis vectors** (see Problems 1-3 below).

Here is the main new concept of Section 2.2. Let V and W be vector spaces. Fix bases

$$\alpha \;=\; \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \text{ of } V$$

$$\beta \;=\; \{\mathbf{w}_1, \ldots, \mathbf{w}_m\} \text{ of } W$$

Now suppose we are given a linear transformation $T : V \to W$. Then for each i, the image $T(\mathbf{v}_i)$ of $\mathbf{v}_i$ is an element of W, so can be written in terms of the basis β as

$$T(\mathbf{v}_i) \;=\; a_i^1\mathbf{w}_1 + \cdots + a_m^1\mathbf{w}_m.$$

The *matrix of T with respect to the basis α and β* is the matrix, denoted $[T]_\alpha^\beta$ whose ij-th entry is the number a_j^i.

In practice, $[T]_\alpha^\beta$ is found by:

Step 1. For each $\mathbf{v}_i$, consider $T(\mathbf{v}_i)$ and expand in the β basis: $T(\mathbf{v}_i) \;=\; a_i^1\mathbf{w}_1 + \cdots + a_m^1\mathbf{w}_m$.

Step 2. Write the coordinates as a column vector: $[T\mathbf{v}_i]_\beta \;=\; \begin{pmatrix} a_i^1 \\ \vdots \\ a_i^m \end{pmatrix}$

Step 3. Assemble these as the columns of a matrix whose first column is $T(\mathbf{v}_1)$, whose second column is $T(\mathbf{v}_2)$, etc.

$$[T]_\alpha^\beta \;=\; \begin{pmatrix} a_1^1 & a_1^2 & \cdots \\ \vdots & \vdots & \vdots \\ a_m^1 & a_m^2 & \cdots \end{pmatrix}$$

Homework 17. In preparation for Exam 2, reread Sections 2.1 and 2.2 and learn the major theorems (a good method is to copy them onto scratch paper several times). Then do the following problems.

1. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of a vector space V. Suppose that $S : V \to W$ and $T : V \to W$ are linear transformations for V to W that are equal for all basis vectors, that is,

$$S(\mathbf{v}_i) \;=\; T(\mathbf{v}_i) \qquad \forall i.$$

   Prove that $S = T$ (i.e show that $S(\mathbf{v}) = T(\mathbf{v})$ for every $\mathbf{v} \in V$).

2. Let $T : \mathbb{R} \to \mathbb{R}$ be a linear transformation. Set $a = T(1)$. Show that $L(x) = ax$ for all x.

3. Let $T : \mathbb{R}^n \to \mathbb{R}$ be a linear transformation. Show that T has the formula $T(x^1, \ldots, x^n) = a_1 x^1 + \cdots, a_n x^n$ for some scalars $a_i \in \mathbb{R}$.

4. Let $S : U \to V$ and $T : V \to W$ be two linear transformations. The *composition* of S and T is the map $T \circ S : U \to W$ defined by

$$T \circ S(\mathbf{u}) \;=\; T(S(\mathbf{u}))$$

   for each $\mathbf{u} \in U$. Use the definition of linear transformation above (nothing else is required!) to show that $T \circ S$ is a linear transformation. Thus *the composition of two linear transformations is a linear transformation.*

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T(x, y, z) = (x + y, y + z)$.

   (a) Find the null space $N(T)$. Write your answer in set notation: $N(T) = \{\text{something}\}$.

   (b) Use the Rank-Nullity Theorem to find $\text{rank}(T) = \dim R(T)$.

   (c) What is the range of T? *You can answer immediately using (b).*

6. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$L(x, y, z) \;=\; (x + 2y - z, y + z, x + y - 2z).$$

   (a) Find a basis and the dimension of $R(L)$. *Use the fact that $R(L)$ is spanned by the images of the basis vectors.*

   (b) Find a basis and the dimension of $N(L)$. *Use the fact that the dimension is the number of free parameters in the linear system $L(\mathbf{v}) = 0$; find $\dim N(L)$ linearly independent vectors in $N(L)$.*