Geometric structures of 3-manifolds and quantum invariants

Effie Kalfagianni

Michigan State University

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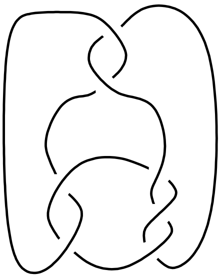
Settings and talk theme

3-manifolds: M=compact, orientable, with empty or tori boundary.

Links: Smooth embedding $K : \coprod S^1 \to M$.

Link complements: $\overline{M \setminus n(K)}$; toroidal boundary



Talk: Relations among three perspectives.

Combinatorial presentations

- knot diagrams, triangulations

3-manifold topology/geometry

- Geometric structures on M and geometric invariants (e.g. hyperbolic volume)

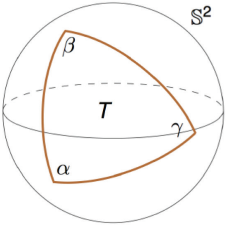
Physics originated invariants

- Quantum invariants of knots/3-manifolds

A: Warm up: 2-d Model Geometries:

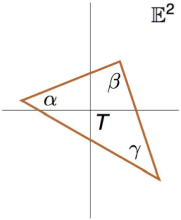
For this talk, an n-dimensional *model geometry* is a simply connected n-manifold with a "homogeneous" Riemannian metric.
In dimension 2, there are exactly three model geometries:

Spherical



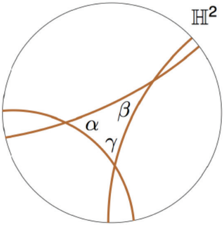
curvature $= +1$
$\text{Area}(T) = (\alpha+\beta+\gamma) - \pi$

Eucledian



curvature $= 0$
$\alpha+\beta+\gamma = \pi$

Hyperbolic

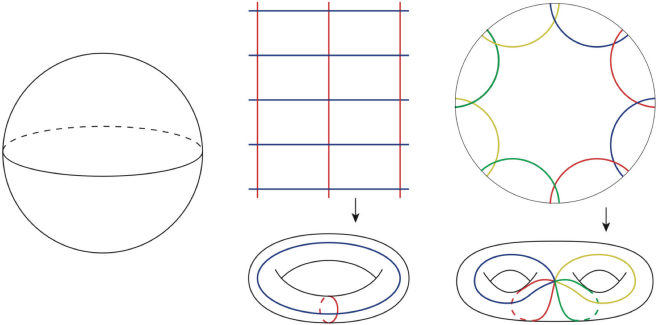


curvature $= -1$
$\text{Area}(T) = \pi - (\alpha+\beta+\gamma)$

Geometrization (a.k.a. Uniformization) in 2-d:

Every (closed, orientable) surface can be written as $S = X/G$, where X is a model geometry and G is a discrete group of isometries.

$X = \mathbf{S}^2$ $\qquad$ $X = \mathbb{E}^2$ $\qquad$ $X = \mathbb{H}^2$



- *Curvature:* $k = 1, 0, -1$
- Geometry vs topology: $k \cdot Area(S) = 2\pi\chi(S)$,

B: Geometrization in 3-d:

In dimension 3, there are eight model geometries:

$$X = \mathbf{S}^3 \quad \mathbb{E}^3 \quad \mathbb{H}^3, \ \mathbf{S}^2 \times \mathbb{R}, \quad \mathbb{H}^2 \times \mathbb{R}, \quad Sol, \quad Nil, \quad \widetilde{SL_2(\mathbb{R})}$$

Recall M= compact, oriented, ∂M=empty or tori

Theorem (Thurston 1980 + Perelman 2003)

For every 3-manifold M, there is a canonical way to cut M along spheres and tori into pieces $M_1, \ldots, M_n$, such that each piece is $M_i = X_i / G_i$, where G_i is a discrete group of isometries of the model geometry X_i .

- *Canonical* : “Unique” collection of spheres and tori.
- Poincare conjecture: $\mathbf{S}^3$ is the only compact model.

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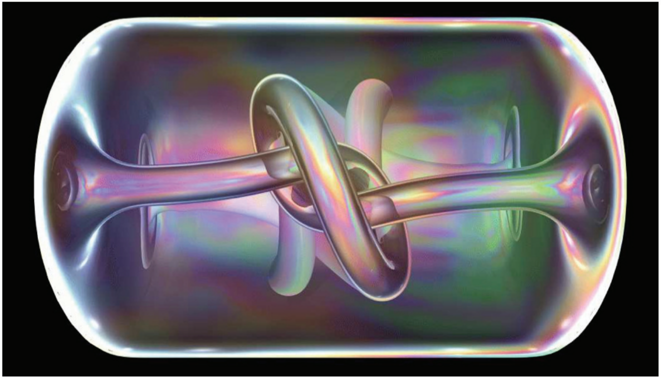
- *Canonical* : "Unique" collection of spheres and tori.
- Poincare conjecture: $\mathbf{S}^3$ is the only compact model.
- Hyperbolic 3-manifolds form a rich and very interesting class.
- Cutting along tori, manifolds with toroidal boundary will naturally arise. Knot complements fit in this class.

Knots complements; nice 3-manifolds with boundary:

Given K remove an open tube around K to obtain the *Knot complement:*
Notation. $M_K = S^3 \setminus n(K)$.



Knot complements can be visualized! (Picture credit: J. Cantarella, UGA)

Geometric decomposition picture for this talk:

Theorem (Knesser, Milnor 60's, Jaco-Shalen, Johanson 1970, Thurston 1980 + Perelman 2003)

M=oriented, compact, with empty or toroidal boundary.

1. *There is a unique collection of 2-spheres that decompose M*

$$M = M_1 \# M_2 \# \dots \# M_p \# (\# S^2 \times S^1)^k,$$

   where $M_1, \dots, M_p$ are compact orientable irreducible 3-manifolds.

2. *For M=irreducible, there is a unique collection of disjointly embedded essential tori $\mathcal{T}$ such that all the connected components of the manifold obtained by cutting M along $\mathcal{T}$, are either Seifert fibered manifolds or hyperbolic.*

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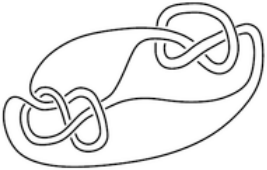
- *Seifert fibered manifolds:* For this talk, think of it as

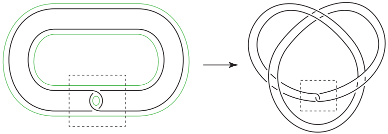
$$S^1 \times \text{surface}$$

  Complete topological classification [Seifert, 60']
- *Hyperbolic:* Interior admits complete, hyperbolic metric of finite volume.

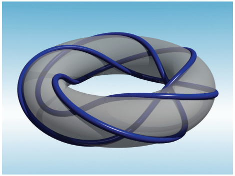
Thee types of knots:

Satellite Knots: Complement contains embedded “essential” tori; There is a *canonical* (finite) collection of such tori.





Torus knots: Knot embeds on standard torus in T in S^3 and is determined by its class in $H_1(T)$. Complement is SFM.



Hyperbolic knots: Rest of them.

Rigidity for hyperbolic 3-manifolds:

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- By rigidity, every geometric measurement of M is a *topological invariant*
- Example: *Volume* of hyperbolic manifolds (important for this talk).
- In practice M is represented by combinatorial data such as, *a triangulation*, or a *knot diagram* (in case of knot complements in S^3).

Question: How do we "see" geometry in the combinatorial descriptions of M? Can we calculate/estimate geometric invariants from combinatorial ones?

Gromov Norm/Volume highlights:

- Recall M uniquely decomposes along spheres and tori into disjoint unions of *Seifert fibered spaces S* and *hyperbolic pieces H*: So

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$$v_{\text{tet}}.||M|| = \text{Vol}\,(H), \quad \text{where}$$

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Nice Properties:

- $||M||$ is additive under glueing along *essential* 2-spheres and *essential* tori
- If M hyperbolic $v_{\text{tet}}||M|| = \text{Vol}\,(M)$=volume of hyp. metric.
- If M Seifert fibered then $||M|| = 0$.

C: Quantum invariants: Jones Polynomials

1980's:(Jones, Witten, Atiyah, Turaev, Reshetikhin.....) Ideas from physics and in representation theory led to invariants of knots and 3-manifolds. (*Quantum invariants/Quantum Topology*)

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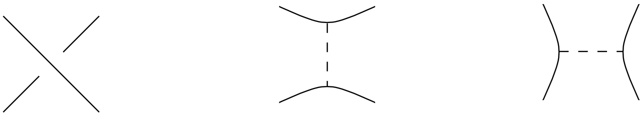
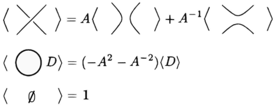
- *Jones Polynomials*: Discovered by V. Jones (1980's); using braid group representations coming from the theory of certain *operator algebras*.
- Can be calculated from any link diagram:
- Two choices of *resolution* for each crossing: *A and B*



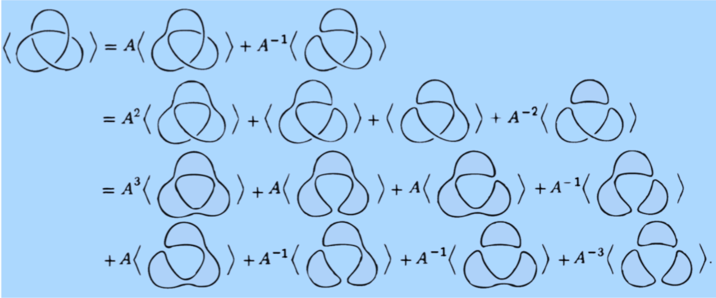
1. *state* σ*:* Choice of A or B resolutions for all crossings:
2. Assign a "*weight*" to every state.
3. JP calculated as "*state sum*" over all states of any diagram.

Calculation of CJP: Example.

- Kauffman bracket: $\langle\ \rangle$: link diagrams $\longrightarrow \mathbb{Z}[A,\ A^{-1}]$ such that



- For $D = D(K)$ where $K = \text{trefoil knot}$:



- We obtain: $J_K(t) = \frac{A^{-9}}{A^2+A^{-2}} \langle D \rangle|_{t:=A^{-4}} = t + t^3 - t^4.$

Generalization:Colored Jones Polynomials

- The *Colored Jones Polynomials*: Infinite sequence of Laurent polynomials $\{J_K^n(t)\}_n$ encoding the *Jones polynomial* of K and these of the links K^s that are the parallels of K.

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- Formulae for $J^n_K(t)$ come from representation theory of Quantum groups!: representation theory of $SU(2)$ (decomposition of tensor products of representations). For example, They look like

$$J^1_K(t) = 1, \qquad J^2_K(t) = J_K(t)\text{- } \textit{Original JP,}$$

$$J^3_K(t) = J_{K^2}(t) - 1, \quad J^4_K(t) = J_{K^3}(t) - 2J_K(t), \ \ldots$$

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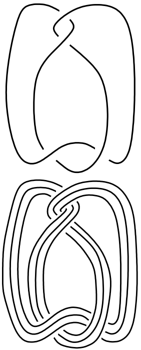
- $J_K^n(t)$ can be calculated from any knot diagram via processes such as *Skein Theory*, *State sums*, *R-matrices*, *Fusion rules*....



C: The CJP predicts Volume?

Question: How do the *CJP* relate to geometry/topology of knot complements?

Kashaev+ H. Murakami - J. Murakami (2000) proposed

Volume Conjecture. Suppose K is a knot in S^3. Then

$$2\pi \cdot \lim_{n\to\infty} \frac{\log |J_K^n(e^{2\pi i/n})|}{n} = v_{\text{tet}} ||S^3 \setminus n(K)||$$

- *Wide Open!*
- 4_1 (by Ekholm), knots up to 7 crossings (by Ohtsuki)
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Some difficulties:

- For families of links we have $J_K^n(e^{2\pi i/n}) = 0$, for all n.
- "State sum" for $J_K^n(e^{2\pi i/n})$ has oscillation/cancelation.
- No good behavior of $J_K^n(e^{2\pi i/n})$ with respect to geometric decompositions.

D. Coarse relations: Colored Jones polynomial

For a knot K, and $n = 1, 2, \ldots,$ we write its *n-colored Jones polynomial*:

$$J_K^n(t) := \alpha_n t^{m_n} + \beta_n t^{m_n - 1} + \cdots + \beta'_n t^{k_n + 1} + \alpha'_n t^{k_n} \in \mathbb{Z}[t, t^{-1}]$$

- For "nice" knots coefficients of $J_K^n(t)$ stabilize:
- (Dasbach-Lin, Armond, 2005)

$$|\alpha'_n| = |\alpha_{n-1}| = \cdots = |\alpha'_2, |$$

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- Stable coefficients control the volume of the link complement.!!

A Coarse Volume Conjecture?

Theorem (Dasbach-Lin, Futer-K.-Purcell, Giambrone, 05-'15')

Suppose that K is a nice hyperbolic link. There are universal constants $C_1, C_2 > 0$ such that for any hyperbolic link that is nice we have

$$C_1 \, B(K) \ \leq \ Vol(S^3 \setminus K) \ < C_2 \, B(K),$$

$B(K)$ =*an explicit function of stable coefficients of the colored Jones polynomials of K.*

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- **Question.** Does above theorem generalize to all hyperbolic links?

E. A Volume Conjecture for all 3-manifolds

- (Turaev-Viro, 1990): For odd integer r and $q = e^{\frac{2\pi i}{r}}$

$$TV_r(M) := TV_r(M, q),$$

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- $TV_r(M, q)$ are combinatorially defined invariants and can be computed from triangulations of M by a *state sum* formula. Sums involve *quantum 6j-sympols*.
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- Terms are highly “oscillating” and there is term canellation. Combinatorics have roots in representation theory of quantum groups.
- **Volume Conjecture**(Q. Chen- T. Yang, 2015) For M compact, orientable

$$\lim_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M, e^{\frac{2\pi i}{r}})) = v_{\mathrm{tet}}||M||.$$

What do we know?:

Quite a bit.....

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- *(Detcherrry-K.-Yang, 2016)* (First examples) of hyperbolic links in S^3: The complement of 4_1 knot and of the Borromean rings.
- *(Ohtsuki, 2017)* Infinite families of closed hyperbolic 3-manifolds.
- *(Belletti-Detcherry-K- Yang, 2018)* Infinite family of cusped hyperbolic 3-manifolds that are universal: They produce all M by a "standard" topological operation (*Dehn filling*).
- *(Detcherry-K, 2017)* All links zero Gromov norm links in S^3 and in connected sums of copies of $S^1 \times S^2$.
- *(Detcherry, Detcherry-K, 2017)* Several families of 3-manifolds with non-zero Gromov, with or with or without boundary.
- *(Kumar, 2019, Wong-Yang)* Infinite families of hyperbolic links in S^3.
- *(Kumar-Melby, 2021)*: infinite families of closed manifolds with arbitrarily large number of hyperbolic pieces...
- More, *Kumar-Melby (2022), Belletti (2019), Wong, Yang-Wong...*

Links complements in S^3?

For link complements $TV_r(S^3 \setminus K, e^{\frac{2\pi i}{r}})$ are obtained from colored Jones link polynomial.

Theorem (Detcherry-K., 2017)

For $K \subset S^3$ and $r = 2m+1$ there is a constant η_r independent of K so that

$$TV_r(S^3 \setminus K, e^{\frac{2\pi i}{r}}) = \eta_r^2 \sum_{n=1}^{m} |J_K^n(e^{\frac{4\pi i}{r}})|^2.$$

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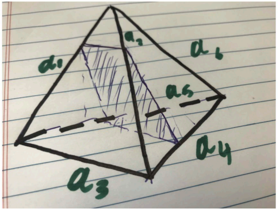
- **Good news:** no technical difficulties as in original Volume Conjecture (of KMM)
- $TV_r((S^3 \setminus K)$ are not identically zero for any link in S^3!
- The quantity $\log(TV_r((S^3 \setminus K))$ is always well defined.
- This version of VC behaves nicely under certain topological operationas

F. Building blocks of TV invariants relate to volumes!!

- Color the edges of a triangulation with certain “quantum ” data



- Colored tetrahedra get “6j-symbol” $\mathbf{Q} := Q(a_1, a_2, a_3, a_4, a_5, a_6)$= function of the a_i and r. $TV_r(M)$ is a weighted sum over all tetrahedra of triangulation (*State sum*).
- *(BDKY)* Asympotics of **Q** relate to volumes of geometric polyhedra:

$$\frac{2\pi}{r} \log(\mathbf{Q}) \leqslant v_{\mathrm{oct}} + O\left(\frac{\log r}{r}\right).$$

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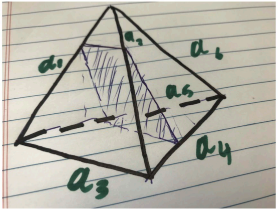

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- In general, hard to control term cancellation in state sum.

A more robust statement?:

Consider

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Conjecture: There exists universal constants B, C such that for any compact orientable 3-manifold M with empty or toroidal boundary we have

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- Half is done!:

Theorem (Detcherry-K., 2017)

There exists a universal constant $C > 0$ such that for any compact orientable 3-manifold M with empty or toroidal boundary we have

$$LTV(M) \leqslant C||M||,$$

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- TV invariants are defined on triangulations of 3-manifolds: For hyperbolic 3-manifolds the (hyperbolic) volume can be estimated/calculated from appropriate triangulations.
- TV invariants are part of a Topological Quantum Field Theory (TQFT) and they can be computed by cutting and gluing 3-manifolds along surfaces. The TQFT behaves particularly well when cutting along spheres and tori. In particular it behaves well with respect to prime and JSJ decompositions.