A Jones Slopes Characterization of Adequate Knots

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ABSTRACT. We establish a characterization of adequate knots in terms of the degree of their colored Jones polynomial. We show that, by assuming the Strong Slope conjecture, our characterization can be reformulated in terms of "Jones slopes" of knots and the essential surfaces that realize the slopes. For alternating knots, the reformulated characterization follows by recent work of J. Greene and J. Howie.

1. INTRODUCTION

Adequate knots form a large class of knots that behave well with respect to Jones-type knot invariants and have nice topological and geometric properties [1, 2, 6–12, 24, 27]. Several well-known classes of knots are adequate; these include all alternating knots and Conway sums of strongly alternating tangles. The definition of adequate knots, much like that of alternating knots, requires the existence of a knot diagram of a particular type (see Definition 3.1). The work of Kauffman [20], Murasugi [25], and Thistlethwaite [26] that settled the Tait conjectures provided a characterization of alternating knots in terms of the degree of the Jones polynomial: it showed that a knot is alternating precisely when the degree span of its Jones polynomial determines the crossing number of the knot. In this note, we obtain a similar characterization for adequate knots in terms of the degree span of colored Jones polynomial. Roughly speaking, we show that adequate knots are characterized by the property that the degree of their colored Jones polynomial determines two basic topological invariants: the crossing number and the Turaev genus.

To state our results, recall the colored Jones polynomial of a knot K is a collection of Laurent polynomials

$$\{J_K(n) := J_K(n, t) \mid n = 1, 2, \ldots\},$$

in a variable t such that we have $J_K(1, t) = 1$, and $J_K(2, t)$ is the ordinary Jones polynomial of K. Throughout the paper, we use the normalization adopted in [19] (see Section 2). Let $d_+[J_K(n)]$ and $d_-[J_K(n)]$ denote the maximal and minimal degree of $J_K(n, t)$ in t. Garoufalidis [13] showed that, given a knot K, there is a number $n_K > 0$ such that, for $n > n_K$, we have

$$d_+[J_K(n)] - d_-[J_K(n)] = s_1(n)n^2 + s_2(n)n + s_3(n),$$

where, for $i = 1, 2, 3$, $s_i : \mathbb{N} \to \mathbb{Q}$ is a periodic function with integral period.

Observe that, given a knot diagram $D = D(K)$, one can define its Turaev genus $g_T(D)$ (see Section 4). The Turaev genus of a knot K, denoted by $g_T(K)$, is defined to be the minimum $g_T(D)$ over all knot diagrams representing K. Although the original definition of the Turaev genus is based on Kauffman states of knot diagrams [5, 28], the work of Armond, Druivenga, and Kindred [3] implies that it can be defined purely in terms of certain projections of knots on certain Heegaard surfaces of $\mathbf{S}^3$.

Our main result is the following.

Theorem 1.1. *For a knot K, let $c(K)$ and $g_T(K)$ denote the crossing number and the Turaev genus of K, respectively. The knot K is adequate if and only if, for some $n > n_K$, we have*

(1.1) $s_1(n) = c(K)/2$ and $s_2(n) = 1 - g_T(K) - c(K)/2.$

Furthermore, every diagram of K that realizes $c(K)$ is adequate, and also realizes $g_T(K)$.

Some ingredients for the proof of Theorem 1.1 are a result of Lee [21, 22] on upper bounds on the degree of the colored Jones polynomial, and of Abe [1] on the Turaev genus of adequate knots.

It is known that a knot is alternating precisely when $g_T(K) = 0$. As a corollary of Theorem 1.1 and its proof, we have the following result.

Corollary 1.2. *Let the notation and setting be as above. A knot K is alternating if and only if, for some $n > n_K$, we have*

(1.2) $2s_1(n) + 2s_2(n) = 2$ and $2s_1(n) = c(K).$

Furthermore, every diagram of K that realizes $c(K)$ is alternating.

The degree of the colored Jones polynomial is conjectured to contain information about essential surfaces in knot complements. The *Strong Slope Conjecture*, which was stated by the author and Tran in [19] and refines the *Slope Conjecture*

of Garoufalidis [14], asserts that the cluster points of the function s_1 are boundary slopes of the knot K, and that the cluster points of s_2 predict the topology of essential surfaces in the knot complement realizing these boundary slopes. The cluster points of s_1 are called *Jones slopes* of K. See the next section for more details. By assuming the Strong Slope Conjecture, Theorem 1.1 leads to a characterization of adequate knots in terms of Jones slopes and essential spanning surfaces (see Theorem 4.3). In particular, by assuming the Strong Slope Conjecture, Corollary 1.2 can be reformulated as follows. A knot K is alternating if and only if it admits Jones slopes s, s^* that are realized by essential spanning surfaces S, S^* such that

$$(1.3) \qquad (s - s^*)/2 + \chi(S) + \chi(S^*) = 2 \quad \text{and} \quad s - s^* = 2c(K).$$

The Strong Slope Conjecture is known for adequate knots [19]; the proof shows that alternating knots satisfy equations (1.3). Conversely, recent work of Howie [17] implies that knots that satisfy equation (1.3) are alternating, providing additional evidence supporting the conjecture. More specifically, Howie [17] and, independently, Greene [15] obtained intrinsic topological characterizations of alternating knots in terms of essential spanning surfaces, and gave normal surface theory algorithms to recognize the alternating property. In particular, [17] shows that a non-trivial knot K is alternating if and only if it admits essential spanning surfaces S, S^* with boundary slopes s, s^* such that

$$(1.4) \qquad\qquad\qquad (s - s^*)/2 + \chi(S) + \chi(S^*) = 2.$$

Thus, if K is a knot that satisfies equations (1.3), then it satisfies (1.4), and, therefore, K is alternating. The results of this paper, and in particular Theorem 4.3, and our discussion above, motivate the following problem.

Problem 1.3. *Show that a knot K is adequate if and only if it admits Jones slopes s, s^* that are realized by essential spanning surfaces S, S^* such that*

$$s - s^* = 2c(K) \quad \text{and} \quad \chi(S) + \chi(S^*) + c(K) = 2 - 2g_T(K).$$

Note that Theorem 1.1 and Corollary 1.2 also hold for links. On the other hand, the picture of the relations between the degree of colored Jones polynomials and boundary slopes is better developed for knots. For this reason, and for simplicity of exposition, we chose to discuss only knots throughout this note.

This paper is organized as follows: in Section 2, we recall the definition of the colored Jones polynomial and above-mentioned conjectures from [14, 19]. In Section 3, we recall definitions and background about adequate knots. In Section 4, we prove a stronger version of Theorem 4.2. Then, using the fact that the Strong Slopes Conjecture is known for adequate knots (see Theorem 3.3), we reformulate Theorem 4.2 in terms of spanning knot surfaces (Theorem 4.3). In Section 5, we discuss the special case of alternating knots and compare equations (1.2), (1.3), and (1.4), shown above.

2. THE COLORED JONES POLYNOMIAL

We briefly recall the definition of the colored Jones polynomial in terms of Chebyshev polynomials (see [23]).

For $n \geq 0$, the Chebyshev polynomials of the second kind $S_n(x)$ are defined recursively, as follows:

$$(2.1) \qquad S_{n+2}(x) = xS_{n+1}(x) - S_n(x), \quad S_1(x) = x, \quad S_0(x) = 1.$$

Let D be a diagram of a knot K. For an integer $m > 0$, let D^m denote the diagram obtained from D by taking m parallels copies of K. This is the m-cable of D using the blackboard framing; if $m = 1$, then $D^1 = D$. Let $\langle D^m \rangle$ denote the Kauffman bracket of D^m: this is a Laurent polynomial over the integers in a variable $t^{-1/4}$ normalized so that $\langle \text{unknot} \rangle = -(t^{1/2} + t^{-1/2})$. Let $c_+(D)$ and $c_-(D)$ denote the number of positive and negative crossings in D, respectively. Furthermore, let $c = c(D) = c_+(D) + c_-(D)$ denote the crossing number, and $w = w(D) = c_+(D) - c_-(D)$ denote the writhe of D.

For $n > 0$, we define $J_K(n) := ((-1)^{n-1} t^{(n^2-1)/4})^w (-1)^{n-1} \langle S_{n-1}(D) \rangle$, where $S_{n-1}(D)$ is a linear combination of blackboard cablings of D, obtained via equation (2.1), and the notation $\langle S_{n-1}(D) \rangle$ means to extend the Kauffman bracket linearly. That is, for diagrams D_1 and D_2, and scalars a_1 and a_2,

$$\langle a_1 D_1 + a_2 D_2 \rangle = a_1 \langle D_1 \rangle + a_2 \langle D_2 \rangle.$$

We have

$$J_{\text{unknot}}(n) = \frac{t^{n/2} - t^{-n/2}}{t^{1/2} - t^{-1/2}}.$$

For a knot $K \subset S^3$, let $d_+[J_K(n)]$ and $d_-[J_K(n)]$ denote the maximal and minimal degree of $J_K(n)$ in t.

Garoufalidis [13] showed that the degrees $d_+[J_K(n)]$ and $d_-[J_K(n)]$ are quadratic *quasi-polynomials*. This means, given a knot K, there is $n_K \in \mathbb{N}$ such that for all $n > n_K$ we have

$$4d_+[J_K(n)] = a(n)n^2 + b(n)n + c(n)$$

and

$$4d_-[J_K(n)] = a^*(n)n^2 + b^*(n)n + c^*(n),$$

where the coefficients are periodic functions from $\mathbb{N}$ to $\mathbb{Q}$ with integral period. By taking the least common multiple of the periods of these coefficient functions, we get a common period. This common period of the coefficient functions is called the *Jones period* of K.

For a sequence $\{x_n\}$, let $\{x_n\}'$ denote the set of its cluster points.

Definition 2.1. The elements of the sets

$$js_K := \{4n^{-2}d_+[J_K(n)]\}' \quad \text{and} \quad js_K^* := \{4n^{-2}d_-[J_K(n)]\}'$$

are called *Jones slopes* of K.

Given a knot $K \subset S^3$, let $n(K)$ denote a tubular neighborhood of K, and let $M_K := \overline{S^3 \setminus n(K)}$ denote the exterior of K. Furthermore, let $\langle \mu, \lambda \rangle$ be the canonical meridian-longitude basis of $H_1(\partial n(K))$. A properly embedded surface $(S, \partial S) \subset (M_K, \partial n(K))$ is called essential if it is π_1-injective and is not a boundary parallel annulus. An element $a/b \in \mathbb{Q} \cup \{1/0\}$ is called a *boundary slope* of K if there is an essential surface $(S, \partial S) \subset (M_K, \partial n(K))$ such that ∂S represents $a\mu + b\lambda \in H_1(\partial n(K))$. Hatcher showed that every knot $K \subset S^3$ has finitely many boundary slopes [16]. The *Slope Conjecture* [14, Conjecture 1.2] asserts that the Jones slopes of any knot K are boundary slopes.

Definition 2.2. Let $\ell d_+[J_K(n)]$ denote the linear term of $d_+[J_K(n)]$, and let

$$jx_K := \{2n^{-1}\ell d_+[J_K(n)]\}' = \{b_K(n)\}'$$

and

$$jx_K^* := \{2n^{-1}\ell d_-[J_K(n)]\}' = \{b_K^*(n)\}'.$$

The *Strong Slope Conjecture* [19, Conjecture 1.6] asserts that, given a Jones slope of K, such as $a/b \in js_K$, with $b > 0$ and $\gcd(a, b) = 1$, there is an essential surface $S \subset M_K$ with $|\partial S|$ boundary components such that each component of ∂S has slope a/b and

$$\frac{2\chi(S)}{|\partial S|b} \in jx_K.$$

Similarly, given $a^*/b^* \in js_K^*$, with $b^* > 0$ and $\gcd(a^*, b^*) = 1$, there is an essential surface $S^* \subset M_K$ with $|\partial S^*|$ boundary components such that each component of ∂S^* has slope a^*/b^* and

$$\frac{-2\chi(S^*)}{|S^*|b} \in jx_K^*.$$

Definition 2.3. With the notation as above, a *Jones surface* of K is an essential surface $S \subset M_K$ such that one of the following holds:

- Either ∂S represents a Jones slope $a/b \in js_K$, with $b > 0$ and with $\gcd(a, b) = 1$, and we have

$$\frac{2\chi(S)}{|\partial S|b} \in jx_K;$$

- Or ∂S^* represents a Jones slope $a^*/b^* \in js_K^*$, with $b^* > 0$ and with $\gcd(a^*, b^*) = 1$, and we have

$$\frac{-2\chi(S^*)}{|\partial S^*|b^*} \in jx_K^*.$$

3. JONES SURFACES OF ADEQUATE KNOTS

Let D be a link diagram, and x a crossing of D. Associated with D and x are two link diagrams called the *A-resolution* and *B-resolution* of the crossing (see Figure 3.1). A Kauffman state σ is a choice of A-resolution or B-resolution at each crossing of D. The result of applying a state σ to D is a collection s_σ of disjointly embedded circles in the projection plane. We can encode the choices that lead to the state σ in a graph G_σ, as follows. The vertices of G_σ are in 1-1 correspondence with the state circles of s_σ. Every crossing x of D corresponds to a pair of arcs that belong to circles of s_σ; this crossing gives rise to an edge in G_σ whose endpoints are the state circles containing those arcs.

Given a Kauffman state σ, we construct a surface S_σ, as follows. Each state circle of σ bounds a disk in S^3. This collection of disks can be disjointly embedded in the ball below the projection plane. At each crossing of D, we connect the pair of neighboring disks by a half-twisted band to construct a surface $S_\sigma \subset S^3$ whose boundary is K (see Figure 3.2).
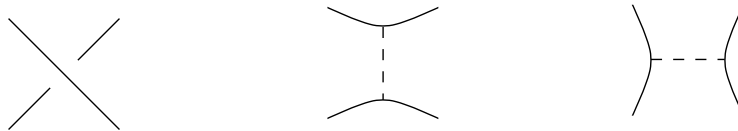

FIGURE 3.1. From left to right: *A* crossing, the *A*-resolution, and the *B*-resolution.

FIGURE 3.2. The two resolutions of a crossing, the arcs recording them, and their contribution to state surfaces.

Definition 3.1. A link diagram D is called *A-adequate* if the state graph $\mathbb{G}_A$ corresponding to the all-A state contains no 1-edge loops. Similarly, D is called *B-adequate* if the all-B graph $\mathbb{G}_B$ contains no 1-edge loops. A link diagram is *adequate* if it is both A- and B-adequate. A link that admits an adequate diagram is also called *adequate*.

It is known that the number of negative crossings $c_-(D)$ of an A-adequate knot diagram is a knot invariant. Similarly, the number of positive crossings $c_+(D)$ of a B-adequate knot diagram is a knot invariant. Observe that, in fact, the crossing number of K is realized by the adequate diagram; that is, we have

$c(K) = c(D) = c_-(D) + c_+(D)$ [23]. Let $v_A(D)$ and $v_B(D)$ be the number of state circles in the all-A (respectively, all-B) state of the knot diagram D. Also, let $S_A = S_A(D)$ and $S_B = S_B(D)$ denote the surfaces corresponding to the all-A and all-B state of D. The following theorem summarizes known results about bounds on the degree of the colored Jones polynomials. The first inequalities in both part (a) and (b) below are well-known results that can be found, for example, in Lickorish's book [23, Lemma 5.4]. Inequalities (3.1) and (3.2), which generalize and strengthen results of [18], have been more recently established by Lee (see Theorem 2.4 in [22] or [21]).

Theorem 3.2. *Let D be a diagram of a knot K.*

(a) *We have*

$$4d_-[J_K(n)] \geq -2c_-(D)n^2 + 2(c(D) - v_A(D))n + 2v_A(D) - 2c_+(D).$$

If D is A-adequate, then equality holds for all $n \geq 1$. Moreover, if D is not A-adequate, then

(3.1) $$4d_-[J_K(n)] \geq -2c_-(D)n^2 + 2(c(D) - v_A(D) + 1)n + e(n),$$

where $e(n) : \mathbb{N} \to \mathbb{Q}$ is a periodic function of n with integral period.

(b) *We have*

$$4d_+[J_K(n)] \leq 2c_+(D)n^2 + 2(v_B(D) - c(D))n + 2c_-(D) - 2v_B(D).$$

If D is B-adequate, then equality holds for all $n \geq 1$. Moreover, if D is not B-adequate, then

(3.2) $$4d_+[J_K(n)] \leq 2c_+(D)n^2 + 2(v_B(D) - c(D) - 1)n + e^*(n),$$

where $e^*(n) : \mathbb{N} \to \mathbb{Q}$ is a periodic function of n with integral period.

The following theorem, which shows that the Strong Slope Conjecture is true for adequate knots, was proven in [19] building on work in [8, 9].

Theorem 3.3. *Let D be an A-adequate diagram of a knot K. Then, the surface S_A is essential in the knot complement M_K, and has boundary slope $-2c_-$. Furthermore, we have*

$$-2c_- = \lim_{n \to \infty} 4n^{-2}d_-[J_K(n)] \quad \text{and} \quad 2\chi(S_A) = 2(v_A(D) - c(D)).$$

Similarly, if D is a B-adequate diagram of a knot K, then S_B is essential in the knot complement M_K, and it has boundary slope $2c_+$. Furthermore, we have

$$2c_+ = \lim_{n \to \infty} 4n^{-2}d_+[J_K(n)] \quad \text{and} \quad 2\chi(S_B) = 2(v_B(D) - c(D)).$$

In particular, if K is adequate, then it satisfies the Strong Slope Conjecture, and S_A, S_B are Jones surfaces.

4. COLORED JONES POLYNOMIALS AND ADEQUATE KNOTS

Let the notation be as in the last section. We recall that the *Turaev genus* of a knot diagram $D = D(K)$ is defined by

$$(4.1) \qquad g_T(D) = (2 - \nu_A(D) - \nu_B(D) + c(D))/2.$$

The Turaev genus of a knot K is defined by

$$g_T(K) = \min\{g_T(D) \mid D = D(K)\}.$$

The genus $g_T(D)$ is the genus of the *Turaev surface* $F(D)$, corresponding to D. This surface is constructed as follows. Let $\Gamma \subset S^2$ be the planar, 4-valent graph of the diagram D. Thicken the (compactified) projection plane to $S^2 \times [-1, 1]$, so that Γ lies in $S^2 \times \{0\}$. Outside a neighborhood of the vertices (crossings), $\Gamma \times [-1, 1]$ will be part of $F(D)$.

Observe that in the neighborhood of each vertex, we insert a saddle, which is positioned so that the boundary circles on $S^2 \times \{1\}$ are the components of the B-resolution and the boundary circles on $S^2 \times \{-1\}$ are the components of the A-resolution (see Figure 4.1). Then, we cap off each circle with a disk, obtaining a closed surface $F(D)$.
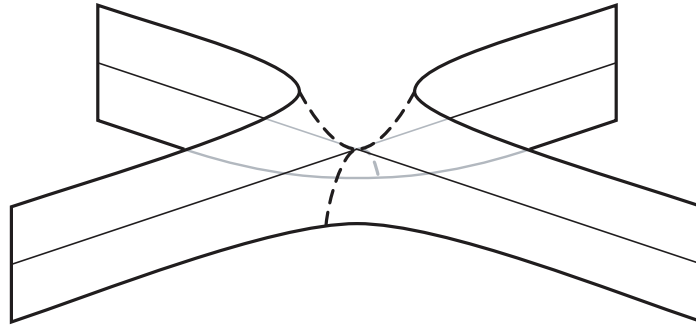

FIGURE 4.1. A saddle between components of the A-, B-resolutions near a vertex of the 4-valent graph Γ corresponding to a crossing of D. The portion of Γ on the saddle is indicated by solid lines. The dashed lines indicate the edges of the state graphs $\mathbb{G}_A$, $\mathbb{G}_B$ corresponding to the crossing.

The surface $F(D)$ has the following properties:
 (i) It is a Heegaard surface of S^3.
(ii) D is alternating on $F(D)$; in particular, D is an alternating diagram if and only if $g_T(F(D)) = 0$.

(iii) The 4-valent graph underlying D defines a cellulation of $F(D)$ for which the 2-cells can be colored in a checkerboard fashion.

We warn the reader that these properties are not enough to characterize the Turaev surface $F(D)$ (see [5] or a survey article by Champanerkar and Kofman [4]).

We will need the following result of Abe [1, Theorem 3.2].

Theorem 4.1. *Suppose that D is an adequate diagram of a knot K. Then, we have*

$$g_T(K) = g_T(D) = \frac{2 - v_A(D) - v_B(D) + c(D)}{2}.$$

We are now ready to state and prove the main result of this paper, which implies Theorem 1.1 stated in the Introduction.

Theorem 4.2. *For a knot K, let js_K, js_K^*, and jx_K, jx_K^* be the sets associated with $J_K(n)$ as in Definitions 2.1 and 2.2. Also, let $c(K)$ and $g_T(K)$ denote the crossing number and the Turaev genus of K, respectively. Then, K is adequate if and only if the following are true:*

(1) *There are Jones slopes $s \in js_K$ and $s^* \in js_K^*$, with $s - s^* = 2c(K)$.*
(2) *There are $x \in jx_K$ and $x^* \in jx_K^*$ with*

$$x - x^* = 2(2 - 2g_T(K) - c(K)).$$

Furthermore, any diagram of K that realizes $c(K)$ is adequate, and also realizes $g_T(K)$.

Proof. Suppose that K is a knot with an adequate diagram $D = D(K)$. We know that $c(K) = c(D) = c_+(D) + c_-(D)$. By Theorem 3.2, equation (4.1), and Theorem 4.1, we have

$4d_+[J_K(n)] - 4d_-[J_K(n)]$

$\quad = 2c(K)n^2 + 2(v_B(D) + v_A(D) - 2c(D))n + 2(c(D) - v_B(D) - v_A(D))$

$\quad = 2c(K)n^2 + 2(2 - 2g_T(D) - c(D))n + 2(c(D) - v_B(D) - v_A(D))$

$\quad = 2c(K)n^2 + 2(2 - 2g_T(K) - c(K))n + 2(2g_T(K) - 2),$

where the last equation follows from the fact that, since D is adequate, by Theorem 4.1 we have $g_T(D) = g_T(K)$. Observe that, thus, the quantities $s = 2c_+(D)$, $s^* = -2c_-(D)$, $x = 2(v_A(D) - c(D))$, and $x^* = 2(c(D) - v_B(D))$ satisfy the desired equations.

Conversely, suppose that we have s, s^*, x, x^* as in the statement above, and let $p = p(K)$ denote the common period of $4d_-[J_K(n)]$, $4d_+[J_K(n)]$.

There is $0 \leq i \leq p$ such that for infinitely many $n \gg 0$, we have

$$a(n) = s \quad \text{and} \quad a^*(n + i) = s^*.$$

Let D be a diagram of K that realizes the crossing number $c(K)$. Let $c_+(D)$ and $c_-(D)$ denote the number of positive and negative crossings in D, respectively. By applying Theorem 3.2 to D, we must have $sn^2 + o(n) \leq 2c_+(D)n^2 + o(n)$ for infinitely many $n \gg 0$; hence, we obtain $s \leq 2c_+(D)$. Similarly, we have

$$s^*(n + i)^2 + o(n) \leq 2c_-(D)(n + i)^2 + o(n),$$

and we conclude that $-s^* \leq 2c_-(D)$. Since, by assumption, $s - s^* = 2c(K)$ and $c_-(D) + c_+(D) = c(K)$, we conclude that $s = 2c_+(D)$ and $-s^* = 2c_-(D)$.

To continue, recall that, by assumption, there is $0 \leq j \leq p$ such that for infinitely many $n \gg 0$, we have

(4.2) $b(n) = x$ and $b^*(n + j) = x^*.$

Now, by equation (4.2), and using Theorem 3.2 as above, we obtain

$$x \leq 2(v_B(D) - c(D)).$$

Similarly, using $-s^* = 2c_-(D)$, we get that for infinitely many $n \gg 0$, we have

$$-x^*(n + j) + 4nc_-(D) \leq -2(c(D) - v_A(D))(n + j) + 4nc_-(D).$$

Hence, we obtain $-x^* \leq 2(c(D) - v_A(D))$. This in turn, combined with equation (4.1), gives

(4.3) $x - x^* \leq 2(v_B(D) + v_A(D) - 2c(D)) = 2(2 - 2g_T(D) - c(D)).$

On the other hand, by assumption,

(4.4) $x - x^* = 2(2 - 2g_T(K) - c(K)).$

Since $g_T(K) \leq g_T(D)$ and $c(D) = c(K)$, by equations (4.3) and (4.4), we conclude that $g_T(K) = g_T(D)$, $x = 2c(D) - 2v_B$, and $x^* = 2v_A - 2c(D)$. This in turn implies that, for infinitely many $n > n_K$, we have

(4.5) $4d_-[J_K(n)] = -2c_-(D)n^2 + 2(c(D) - v_A(D))n + f(D),$

and

(4.6) $4d_+J_K(n)] = 2c_+(D)n^2 + 2(c(D) - v_B(D))n + f^*(D),$

where $f(D), f^*(D)$ are periodic functions of n. It follows that $f(D), f^*(D)$ can take at most finitely many distinct values, and they are bounded by a universal constant. Now, Theorem 3.2 implies that D has to be both A and B adequate, and hence adequate. Otherwise, one of the inequalities (3.1), (3.2) would have to hold, which would contradict equations (4.5) and (4.6).

To finish the proof, note that the arguments above imply that if K is a knot for which (1), (2) are satisfied and D is a diagram of K that realizes $c(K)$, then D is adequate. Thus, by Theorem 4.1, we also have $g_T(D) = g_T(K)$.   □

Now, we explain how Theorem 1.1 follows.

Proof of Theorem 1.1. First, suppose that K is a knot with an adequate diagram D. Then, $c(K) = c(D)$. The first part of the proof of Theorem 4.2 implies that equations (1.1) are satisfied for all $n > 0$. Suppose conversely that for some $n > n_K$, equations (1.1) are satisfied. Observe that, since $s_1(n), s_2(n)$ are periodic with integral period, we conclude there must be infinitely many $n \gg 0$ for which equations (1.1) are true. If we take D, a knot diagram of K that realizes $c(K)$, the argument in the second proof of Theorem 4.2 implies $g_T(D) = g_T(K)$, and equations (4.5) and (4.6) hold for D. Hence, as before, D is adequate. □

Theorem 3.3 implies that the Strong Slope Conjecture is true for adequate knots. The next result implies that, for knots that satisfy the conjecture, the characterization provided by Theorem 4.2 can be expressed in terms of properties of their spanning surfaces.

Theorem 4.3. *Given a knot K with crossing number $c(K)$ and Turaev genus $g_T(K)$, the following are equivalent:*

(1) *K is adequate.*
(2) *There are Jones surfaces S and S^* with boundary slopes s, s^* such that*

$$(4.7) \qquad s - s^* = 2c(K) \quad \text{and} \quad \frac{\chi(S)}{|\partial S|} + \frac{\chi(S^*)}{|\partial S^*|} + c(K) = 2 - 2g_T(K).$$

(3) *There are Jones surfaces S and S^* that are, in addition, spanning surfaces of K (i.e., $\partial S = \partial S^* = K$) such that*

$$s - s^* = 2c(K) \quad \text{and} \quad \chi(S) + \chi(S^*) + c(K) = 2 - 2g_T(K).$$

Proof. Suppose that K is adequate. Then, by Theorem 3.3 and the calculation in the beginning of the proof of Theorem 4.2, the state surfaces S_A and S_B obtained from any adequate diagram of K satisfy equations (4.7). In fact, in this case, we have $|\partial S| = |\partial S^*| = 1$.

Conversely, suppose there are Jones surfaces S and S^* with boundary slopes s, s^* such that $s - s^* = 2c(K)$. By the proof of Theorem 4.2, if D is a diagram realizing $c(K)$, we have $s = 2c_+(D)$ and $s^* = -2c_-(D)$. Since S and S^* have integral slopes, we have $b = b^* = 1$. Since S and S^* are Jones surfaces, we have

$$x = \frac{2\chi(S)}{|\partial S|} \in jx_K \quad \text{and} \quad x^* = \frac{-2\chi(S^*)}{|\partial S^*|} \in jx_K^*.$$

Thus, we get $x - x^* = 2(2 - 2g_T(K) - c(K))$. Therefore, by Theorem 4.2, D must be an adequate diagram of K. This shows that (1) and (2) are equivalent.

Now, (3) clearly implies (2). Finally, since, by Theorem 4.2, (2) implies that K is adequate, and (3) is true for adequate knots, we get that (2) implies (3). □

5. ALTERNATING KNOTS

Recall a knot K is alternating if and only if $g_T(K) = 0$ [5]. Combining this with Theorem 4.2, we show the following.

Corollary 5.1. *For a knot K, let js_K, js_K^*, and jx_K, jx_K^* be the sets associated with $J_K(n)$, as in Definitions 2.1 and 2.2. Then, K is alternating if and only if the following are true:*

    (1) *There are Jones slopes $s \in js_K$ and $s^* \in js_K^*$, with $s - s^* = 2c(K)$.*
    (2) *There are $x \in jx_K$ and $x^* \in jx_K^*$ with $x - x^* = 4 - 2c(K)$.*

Proof. If K is alternating, then Theorem 4.2 and the fact that $g_T(K) = 0$ imply that (1) and (2) hold. Conversely, suppose we have s, s^*, x, x^*, as in the statement above, and let D be a diagram of K such that $c(D) = c(K)$. Let $g_T(D)$ denote the Turaev genus of D. The argument in the proof of Theorem 4.2 implies that $x - x^* = 4 - 2c(K) \leq 4 - 2g_T(D) - 2c(K)$, which can only hold if $g_T(D) = 0$, and hence D is alternating. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Recently, Howie [17] and, independently, Greene [15] gave characterizations of alternating knots in terms of properties of spanning surfaces. In particular, [17, Theorem 2] states that a non-trivial knot K is alternating if and only if it admits spanning surfaces S and S^* such that the following hold:

$$\chi(S) + \chi(S^*) + i(\partial S, \partial S^*)/2 = 2 \quad \text{and} \quad i(\partial S, \partial S^*)/2 = c(K),$$

where $i(\partial S, \partial S^*)$ denotes the geometric intersection number of $\partial S, \partial S^*$ on the ∂M_K. Note, we assume that this intersection number is minimal in the isotopy classes of $\partial S, \partial S^*$.

Combining this with Theorem 4.3, we obtain the following result.

Corollary 5.2. *Given a non-trivial knot K with crossing number $c(K)$, the following are equivalent:*

    (1) *K is alternating.*
    (2) *There are Jones surfaces S and S^* that are spanning surfaces of K with boundary slopes s, s^* such that*

$$\chi(S) + \chi(S^*) + (s - s^*)/2 = 2 \quad \text{and} \quad s - s^* = 2c(K).$$

    (3) *There are spanning surfaces S and S^* of K such that*

$$\chi(S) + \chi(S^*) + i(\partial S, \partial S^*)/2 = 2 \quad \text{and} \quad i(\partial S, \partial S^*) = 2c(K).$$

Proof. Suppose D is a reduced alternating diagram for a knot K. Then, D is both A- and B-adequate and $g_T(D) = g_T(K) = 0$. Now, the checkerboard surfaces S, S^* of D are the all-A and all-B state surface, which are Jones surfaces. Thus, they satisfy the desired properties.

Conversely, suppose we have Jones surfaces S and S^* as above, and let D be a diagram of K that realizes $c(K)$. Set

$$x = 2\chi(S) \quad \text{and} \quad x^* = -2\chi(S^*).$$

By hypothesis,

(5.1) $$x - x^* = 4 - 2c(K).$$

By Theorem 3.2, as in the proof of Theorem 4.2, we get

(5.2) $$x - x^* \leq 2(v_B(D) + v_A(D) - 2c(D)) = 2(2 - 2g_T(D) - c(D)).$$

Since $c(D) = c(K)$, combining equations (5.1) and (5.2), we have

$$4 - 2c(K) \leq 2(2 - 2g_T(D) - c(K)),$$

which gives $g_T(D) = 0$. This in turn implies that D is alternating [5]. Thus, we show that (1) and (2) are equivalent.

Now, we explain how (2) implies (3). Let D be a diagram of K that realizes $c(K)$; as above, D is alternating. The argument in the proof of Theorem 4.2 shows that if we have Jones slopes s, s^* as above such that $2c(K) = s - s^*$, then $s = 2c_+(D)$ and $s^* = -2c_-(D)$. Suppose that the simple closed curves $\partial S, \partial S^*$ have been isotoped on the torus $\partial n(K)$ to minimize their intersection number. Then, we have

$$i(\partial S, \partial S^*) = 2c_+(D) - (-2c_-(D)) = 2c(K) = s - s^*.$$

Thus (3) follows. Hence, (1) above implies (3).

Finally, by [17, Theorem 2], (3) implies (1). $\qquad\qquad\square$

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