Triality (after Tits)

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J.I. Hall
Department of Mathematics
Michigan State University
East Lansing, Michigan 48824, U.S.A.
jhall@math.msu.edu

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> On peut dire que le *principe de dualité* de la Géometrie projective est remplacé ici par un *principe de trialité*. E. Cartan [Car25, p. 373]

There are classical and well-studied relationships among duality of finite dimensional vector spaces, order 2 outer automorphisms of the general linear groups (Lie type A_n), and algebras with involution.

Here we discuss the more specialized relationships among triality of hyperbolic orthogonal 8-space [Stu13], order 3 outer automorphisms of orthogonal groups (Lie type D_4) [Car25], and composition algebras [Mou35, SpV00]. In particular we examine the paper by Tits [Tit58].

Other relevant references include [Dor78, Hal00, HaN01, Tay92]. We steal from, improve, and even correct some of the arguments of the original version of [Hal00].

1 Orthogonal geometry

Throughout F will be a commutative field and V will be finite dimensional vector space over F. For any subset W of V, we let $\langle W \rangle \leq V$ be the F-subspace of V spanned by W.

Let $Q\colon V \longrightarrow F$ be a quadratic form on the finite dimensional F-space V. That is,

$$Q(\alpha x) = \alpha^2 Q(x)\,,$$

for all $\alpha \in F$ and $x \in V$, and the associated form $B(\cdot,\cdot)\colon V \times V \longrightarrow F$ given by

$$B(a,b) = Q(a+b) - Q(a) - Q(b)$$

is bilinear (and symmetric). For any subspace W of V, the restriction of Q to W is a quadratic form on W.

We may call (V, Q) an *orthogonal space* or a *quadratic space*.

Always $B(a, a) = 2Q(a)$. So in characteristic other than 2, the bilinear form B determines Q. That is not the case in characteristic 2.

An important example for us will be the vector space $M_2(F)$ of 2×2 matrices over F with Q the determinant function.

For $W \subseteq V$, we let $W^\perp = \{x \in V \mid B(x, b) = 0,\ b \in W\}$, a F-subspace of V. The form Q is *nondegenerate* if $V^\perp = 0$.

(1.1) LEMMA. *Let Q be a nondegenerate quadratic form.*

(*a*) *For a subspace U, $\dim_F U + \dim_F U^\perp = \dim_F V$.*

(*b*) *If $U \cap U^\perp = 0$, then $V = U \oplus U^\perp$ (which we write as $U \perp U^\perp$).* □

A subset S of V is *singular* (or sometimes even *totally singular*) if the restriction of Q to S is identically 0. Notice that if U is a singular subspace, then Q induces a quadratic form on the quotient space $U^\perp/U$, nondegenerate if Q is nondegenerate.

(1.2) PROPOSITION. *Let Q be a quadratic form on V of dimension 2 over F.*

(*a*) *There is a quadratic extension E of F for which the extension of Q to $E \otimes_F V$ has nonzero singular vectors.*

(*b*) *If $0 \neq x \in V$ is singular with $x^\perp \neq V$, then Q is nondegenerate and there are exactly two 1-spaces in V consisting of singular vectors. These are spanned by the singular vectors x and y with $B(x, y) = 1$.*

PROOF. (*a*) Choose a basis $\{u, v\}$ of V with $Q(u) = a$, $Q(v) = c$, and $B(u, v) = b$. Then $Q(\alpha u + \beta v) = a\alpha^2 + b\alpha\beta + c\beta^2$. The polynomial $ax^2 + bx + c$ has a root in some quadratic extension E of F.

(*b*) As $Q(x) = 0$, $B(x, x) = 0$; so for $w \notin \langle x\rangle = x^\perp$, $B(x, w) \neq 0$. If necessary, replace w by a scalar multiple so that $B(x, w) = 1$. Consider $y = \alpha x + w$. Then

$$B(x, y) = B(x, w) = 1\,, \text{ and } Q(y) = Q(\alpha x) + Q(w) + B(\alpha x, w) = Q(w) + \alpha\,.$$

Therefore $\alpha = -Q(w)$ gives a second 1-space $\langle y\rangle$ of singular vectors and all other nonzero vectors are nonsingular. □

In the second part of the proposition, V is a *hyperbolic* 2-space and the pair $\{x, y\}$ is a *hyperbolic pair*. As in the proof

$$Q(x + \alpha y) = Q(x) + \alpha^2 Q(y) + \alpha B(x, y) = \alpha\,.$$

Therefore in a hyperbolic 2-space every element of the field F is realized as a Q-value.

2 Isometry and similarity groups

Let Q be a quadratic form on V. An *isometry* of (V, Q) is a $g \in \mathrm{GL}(V)$ with

$$Q(vg) = Q(v)\,, \text{ for all } v \in V\,.$$

The full *isometry group* (orthogonal group) of (V, Q) is then $\mathrm{O}(V, Q)$.

A *similarity* of (V, Q) is a $g \in \mathrm{GL}(V)$ with

$$Q(vg) = \alpha_g Q(v)\,, \text{ for all } v \in V\,,$$

for some nonzero constant $\alpha_g \in F$. The full *similarity group* (general orthogonal group) of (V, Q) is $\mathrm{GO}(V, Q)$.

An isometry g is precisely a similarity with $\alpha_g = 1$, so the isometry group is a normal subgroup of the similarity group. All nonzero scalar transformations αI are similarities, but of these only $\pm I$ are isometries.

Isometries (and similarities) of B are defined similarly. An isometry (or similarity) of Q always gives one of B. In characteristic 2 the converse is not true in general.

(2.1) Proposition. *For any nonsingular $h \in V$, consider the map*

$$s_h \colon v \mapsto v - \frac{B(v,h)}{Q(h)} h$$

(a) *s_h is an isometry of order 2 of the quadratic form Q on V (and so also of B).*

(b) *If g is an isometry of (V, Q), then $g^{-1} s_h g = s_{hg}$.*

(c) *For $W \leq V$, $W s_h = W$ if and only if $h \in W$ or $W \leq h^\perp$.*

(d) *$s_h = s_k$ if and only if $k = \alpha h$ for some nonzero $\alpha \in F$.* □

The isometry s_h is a *symmetry* of (V, Q).

3 Hyperbolic orthogonal spaces

The orthogonal space (V, Q) admits the *hyperbolic basis* $\mathcal{H} = \{\ldots, f_i, g_i, \ldots\}$ $(1 \leq i \leq m)$ provided for all i, j, l:

$$Q(f_i) = Q(g_j) = B(f_i, f_l) = B(g_j, g_l) = 0\,,\ B(f_i, g_j) = \delta_{i,j}\,.$$

Notice that the dimension $2m$ of V is even and that Q is nondegenerate.

A dimension 4 example is $M_2(F)$ with determinant form, where the four matrix units form a hyperbolic basis (up to sign).

If (V, Q) has a hyperbolic basis, then we say that Q and V are *split* or *hyperbolic*.

(3.1) Proposition. *If Q is a nondegenerate quadratic form on the F-space V of finite dimension, then the following are equivalent:*

(1) *There is a singular subspace of dimension at least* $\dim_F(V)/2$.

(2) *Every maximal singular subspace has dimension* $\dim_F(V)/2$.

(3) *There are maximal singular subspaces M and N with $V = M \oplus N$.*

(4) *V is a perpendicular direct sum of hyperbolic 2-spaces.*

(5) *V has a hyperbolic basis.*

(6) *for any basis χ of singular X, V has a hyperbolic basis containing χ.*

PROOF. (4) and (5) are clearly equivalent, and both are consequences of (6). (1) is an easy consequence of all the others, and (4) easily implies (3).

Also (6) implies (2) as every singular subspace spanned by a subset of a hyperbolic basis is contained in such a subspace of dimension $\dim_F(V)/2$.

It remains to prove that (1) implies (6), which we do by induction on $\dim(V)$ with Proposition 1.2 providing the initial step. (The case of dimension 1 being trivial since nondegenerate 1-spaces contain no nonzero singular vectors.) If M is a singular subspace of dimension at least $\dim(V)/2$ and z is singular, then $z^\perp \cap M$ contains a hyperplane of M and singular $\langle z, z^\perp \cap M\rangle$ has dimension at least that of M. Thus, if necessary replacing M or enlarging χ, we may assume that $M \cap \chi$ is nonempty. Let $x \in M \cap \chi$. Then, for any y in $(\chi \setminus \{x\})^\perp$ but not its hyperplane $\chi^\perp$, the 2-space $\langle x, y\rangle$ is hyperbolic by Proposition 1.2. Nondegenerate $\langle x, y\rangle^\perp$ contains $M \cap y^\perp$ and $\chi \setminus \{x\}$. By induction $\chi \setminus \{x\}$ embeds in a hyperbolic basis of $\langle x, y\rangle^\perp$, and therefore χ is in a hyperbolic basis of V. □

(3.2) PROPOSITION. *Let Q be hyperbolic on V of dimension $2m$.*

(a) *Every singular $(m-1)$-space is contained in exactly two singular m-spaces.*

(b) *Let U be a maximal singular subspace, and let S be a singular subspace not contained in U. Then for every $s \in S \setminus U$ there is a unique maximal singular subspace T with $s \in T$ and $U \cap T$ of dimension $m-1$. The space T is $\langle s, s^\perp \cap U\rangle$, and $\dim_F(S \cap T) = 1 + \dim_F(S \cap U)$.*

PROOF. (a) If U has codimension 1 in a maximal singular subspace, then $U^\perp/U$ is a hyperbolic 2-space; so (a) follows from Proposition 1.2.

(b) As $s \notin U$, $s^\perp \cap U$ is a hyperplane of U and $T = \langle s, s^\perp \cap U\rangle$ is a singular m-space. It is unique since any T as described must contain s, whence $T \cap U \leq s^\perp \cap U$.

The hyperplane $T \cap U = s^\perp \cap U$ of T contains $S \cap U$, so the dimension of $S \cap T$ is equal to that of $S \cap U$ or exceeds it by 1. But $s \in T \setminus U$. □

(3.3) PROPOSITION. *Let the quadratic form Q be hyperbolic on the F-space V of dimension $2m$.*

The graph $(\mathcal{M}, \sim)$ on the set $\mathcal{M}$ of maximal singular subspaces, with two such adjacent when their intersection has codimension 1 in each, is connected bipartite of diameter m. In this graph, the distance between two maximal singular subspaces M and N equals the codimension of $M \cap N$ in each.

PROOF. We first claim that, for all $S \in \mathcal{M}$ and $T_1 \sim T_2$ in $\mathcal{M}$, we have

$$|\dim(S \cap T_1) - \dim(S \cap T_2)| = 1\,.$$

Let $U = T_1 \cap T_2$ of codimension 1 in each, and set $R = S \cap U$. If necessary passing to $R^\perp/R$, we may assume $R = 0$ in proving the claim. Then $U^\perp$ has dimension $m+1$ and so intersects S nontrivially. Therefore $T = \langle U, U^\perp \cap S\rangle$

is totally singular of dimension m. By the previous proposition, T is equal to exactly one of T_1 or T_2. Thus

$$\{\dim(S \cap T_1), \dim(S \cap T_2)\} = \{0, 1\},$$

giving the claim.

Let $d(M, N)$ be the distance between M, N in $(\mathcal{M}, \sim)$. Again by the previous proposition, $d(M, N) \leq m - \dim(M \cap N)$. In particular the graph is connected.

To prove $d(M, N) = m - \dim(M \cap N)$, we induct on $d(M, N)$. The result is true by definition for $d(M, N) = 0, 1$. Suppose $d(M, N) = d$, and choose a $T \in \mathcal{M}$ with $T \sim N$ and $d(T, M) = d-1$. Then by induction $d-1 = m - \dim(M \cap T)$. By the preceding paragraph and the claim $d \leq m - \dim(M \cap N) = (d-1) \pm 1 \leq d$, as desired.

It remains to prove $(\mathcal{M}, \sim)$ bipartite. Otherwise, there is a minimal cycle $\mathcal{C}$ of odd length, say $2k+1$. But for $S \in \mathcal{C}$, the two vertices T_1 and T_2 at distance k from S in $\mathcal{C}$ are adjacent with $\dim(S \cap T_1) - \dim(S \cap T_2) = 0$, contradicting the earlier claim. □

(3.4) Corollary. *The groups* $\mathrm{O}(V, Q)$ *and* $\mathrm{GO}(V, Q)$ *induce automorphisms of the bipartite graph* $(\mathcal{M}, \sim)$*. If* s_h *is a symmetry of* $\mathrm{O}(V, Q)$ *and* $M \in \mathcal{M}$ *is a singular* m*-space, then* $M \sim Ms_h$*. In particular symmetries switch the two parts and* $\mathrm{O}(V, Q)$ *and* $\mathrm{GO}(V, Q)$ *have normal subgroups of index* 2 *that globally fix the two parts of the bipartition.*

Proof. $M = M^\perp$ for every maximal singular space, so $h^\perp \cap M$ is a hyperplane of M that is equal to $M \cap Ms_h$. Thus $M \sim Ms_h$. □

The subgroup of index 2 in $\mathrm{O}(V, Q)$ will be written as $\mathrm{SO}(V, Q)$. It contains all products of an even number of symmetries (indeed, by the Cartan-Dieudonné Theorem [Tay92] is almost always equal to it). If the characteristic of F is not 2, it is the subgroup of $\mathrm{O}(V, Q)$ containing the isometries of determinant 1.

4 The oriflamme and triality geometries

Let (V, Q) be a hyperbolic orthogonal space of dimension $2m$.

4.1 Oriflamme geometries

Consider the graph $(\Gamma, \sim)$ whose vertices are the nonzero singular spaces of V. Two singular spaces are *incident* (that is, adjacent in Γ) precisely when

> one is properly contained in the other *or* they both have dimension m and intersect in a $(m-1)$-space.

This graph is $(m+1)$-partite by Proposition 3.3 above, with the collection of m-spaces falling into two parts $\mathcal{M}^\rho$ and $\mathcal{M}^\lambda$ while the remaining singular subspaces provide a part $\mathcal{S}_k$ for each dimension $1 \leq k \leq m-1$.

The associated *oriflamme* D_m geometry is this graph with the part $\mathcal{S}_{m-1}$ (the vertices of dimension $m-1$) removed.

(4.1) PROPOSITION. *The group* $\mathrm{SO}(V,Q)$ *acts on* $(\Gamma, \sim)$*, sending every part to itself and transitive on maximal cliques (of size* $m+1$*).*

If C *is a maximal such clique, then there is a symmetry* s_h *that fixes all vertices of* C *of dimension less that* m *and switches the two vertices of* C *of dimension* m*.*

PROOF. Let C be a maximal clique in $(\Gamma, \sim)$, so it consists of $m-1$ singular spaces C_k (of dimension $1 \leq k \leq m-1$) and two spaces C_m^ρ and C_m^λ of dimension m. If $i < j$ then C_i is contained in C_j (and C_m^ρ and C_m^λ) and $C_{m-1} = C_m^\rho \cap C_m^\lambda$. Start with a nonzero vector of $c_1 \in C_1$ and continue adding vectors of $c_k \in C_k \setminus C_{k-1}$ until arriving a basis $\chi = \{c_1, \ldots, c_{m-1}\}$ of C_{m-1}. Then in $C_{m-1}^\perp$ choose a hyperbolic pair $\{x, y\}$ with $\langle \chi, x\rangle = C_m^\rho$ and $\langle \chi, y\rangle = C_m^\lambda$. Call $\chi \cup \{x, y\}$ a *clique basis* for the maximal clique C.

By Proposition 3.1 there is a hyperbolic basis that contains the clique basis $\chi \cup \{x, y\}$. As $\mathrm{O}(V,Q)$ is transitive on hyperbolic bases, any clique basis, and so any maximal clique, can be mapped to any other. Indeed, given the ordering of hyperbolic bases, the map can be chosen in $\mathrm{SO}(V,Q)$. Alternatively, for $h = x + y$ the symmetry s_h fixes all vectors of $C_{m-1} \leq h^\perp$ and switches the 1-spaces spanned by x and y. □

(4.2) COROLLARY. $\mathrm{O}(V,Q)$ *is transitive on nonincident pairs of a singular* 1*-space and a singular* m*-space.*

PROOF. For such a pair, there is a clique basis that is the union of a basis of the 1-space and a basis of the m-space. □

(4.3) COROLLARY. *In the action of* $\mathrm{SO}(V,Q)$ *on* $\mathcal{R}$*, where* $\mathcal{R}$ *is any one of the parts* $\mathcal{S}_k$ *or* $\mathcal{M}^\rho$ *or* $\mathcal{M}^\lambda$*, the kernel is* $\{\pm I\}$*.*

PROOF. The intersection of all members of $\mathcal{R}$ containing the singular 1-space S is S. Therefore we need only consider $\mathcal{S}_1$. But in that case the result is clear by Propositon 1.2. □

The quotient of $\mathrm{SO}(V,Q)$ by $\{\pm I\}$ is $\mathrm{PSO}(V,Q)$.

4.2 Triality geometries

We now restrict to the case of a hyperbolic space (V,Q) of dimension 8, so $m = 4$.

Consider the tripartite subgraph $\mathcal{T}$ of Γ, consisting of the parts $\mathcal{T}_1 = \mathcal{S}_1$, $\mathcal{T}_2 = \mathcal{M}^\lambda$, and $\mathcal{T}_3 = \mathcal{M}^\rho$. This is the *triality graph* $\mathcal{T}(V,Q) = \mathcal{T}_8^+(F)$ of (V,Q).

We have immediately from Corollary 3.4 and Proposition 4.1:

(4.4) Corollary. *The group* $\mathrm{SO}(V, Q)$ *acts as automorphisms of* $\mathcal{T}$ *preserving each part* $\mathcal{T}_i$ *and is transitive on maximal cliques. Each symmetry* s_h *fixes the part* $\mathcal{T}_1$ *and switches the two parts* $\mathcal{T}_2$ *and* $\mathcal{T}_3$*. Indeed, for* $\{j, k\} = \{2, 3\}$*, if* $M \in \mathcal{T}_j$ *then* M *is incident to* $Ms_h \in \mathcal{T}_k$*.* □

The next two results give basic properties of $\mathcal{T}$ and its automorphism group. They provide motivation for the paper of Tits [Tit58]; see Section 6.

(4.5) Theorem. *Let* $\{i, j, k\} = \{1, 2, 3\}$*.*

(a) *For every nonincident pair* $p_i \in \mathcal{T}_i$ *and* $p_j \in \mathcal{T}_j$*, there is a unique* $p_k \in \mathcal{T}_k$ *that is incident to both* p_i *and* p_j*.*

(b) *If* $p_i, q_i \in \mathcal{T}_i$ *are both incident to* $p_j \in \mathcal{T}_j$*, then there are distinct* $p_k, p'_k \in T_k$ *that are incident simultaneously to* p_i *and* p_j *but not to* q_i*.*

Proof. (a) There are two distinct cases: $\{i, j\} = \{1, 2\}$ and $\{i, j\} = \{2, 3\}$.

The case $\{i, j\} = \{1, 2\}$ is contained in Proposition 3.2(b) with $p_1 = S$, $p_2 = U$, and $p_3 = T = \langle S, S^\perp \cap U\rangle$.

The case $\{i, j\} = \{2, 3\}$ comes from Proposition 3.3: as p_2 and p_3 are not incident, their intersection must have dimension 1—the unique singular 1-space $p_1 = p_2 \cap p_3 \in \mathcal{T}_1$ incident to both p_2 and p_3.

(b) There are two distinct cases: $k = 1$ and $k = 3$.

First let $k = 1$, so without loss of generality $(i, j) = (2, 3)$. As the two singular 4-spaces p_2 and q_2 both intersect the 4-space p_3 in a 3-space, their intersection $p_2 \cap q_2$ is a 2-space. We can therefore select independent singular 1-spaces p_1 and p'_1 of $\mathcal{T}_1$ from $p_2 \cap p_3 \setminus q_2 \cap p_3$. Both p_1 and p'_1 are incident simultaneously to p_2 and p_3 but not q_2, as desired.

Now consider $k = 3$. There are initially two subcases: $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$. In the first subcase, p_1 and q_1 are both incident to singular p_2, so $\langle p_1, q_1\rangle$ is a singular 2-space that is also contained in a singular 4-space q_2 of $\mathcal{T}_2$, distinct from p_1. Similarly, in the second subcase, if the singular 4-spaces p_2 and q_2 of $\mathcal{T}_2$ are both incident to p_1, then their intersection is a singular 2-space, and we can pick a second singular 1-space q_1 in that intersection.

Therefore in both subcases we have singular 1-spaces p_1, q_1 and singular 4-spaces p_2, q_2 from $\mathcal{T}_2$, with p_1 and q_1 both incident to p_2 and q_2. We will be done when we find 4-spaces $p_3, p'_3 \in \mathcal{T}_3$ that are incident to both p_1 and p_2 but to neither q_1 nor q_2.

Let U and U' be 3-spaces in p_2 that contain p_1 but not q_1. By Propositions 3.2 and 3.3 there are unique 4-spaces p_3, p'_3 of $\mathcal{T}_3$ with $p_3 \cap p_2 = U$ and $p'_3 \cap p_2 = U'$. Thus p_3 and p'_3 are incident to p_1 and p_2 but not to q_1. If p_3 was incident to q_2, then $p_3 \cap q_2$ would have dimension 3 and $U \cap q_2$ would have dimension at least 2, whereas $q_1 \notin U \cap q_2 \leq p_2 \cap q_2 = \langle p_1, q_1\rangle$. Thus q_2 is not incident to p_3 and similarly not to p'_3. □

(4.6) Theorem. *Let* $i = 1$ *and* $\{j, k\} = \{2, 3\}$*. For each nonsingular* h*, the automorphism* $g = s_h$ *of* $\mathcal{T}$ *has the following properties.*

(a) *g fixes $\mathcal{T}_i$ and and each p_j of $\mathcal{T}_j$ is incident to $p_j g$, which belongs to $\mathcal{T}_k$.*

(b) *If $p_i \in \mathcal{T}_i$ is incident to both $p_j \in \mathcal{T}_j$ and $p_j g \in \mathcal{T}_k$, then $p_i g = p_i$.*

(c) $g^2 = 1$.

PROOF. Part (a) is contained in Corollary 4.4. Part (c) holds as all symmetries have order 2 by Proposition 2.1.

For (b), if the singular 1-space p_1 is incident to the incident pair of singular 4-spaces p_2 and $p_3 = p_2 s_h$, then it is in the hyperplane $p_2 \cap p_3$ of each. But $p_2 \cap p_3 = p_2 \cap h^\perp = p_3 \cap h^\perp$, so $p_1 \leq h^\perp$ is fixed by $g = s_h$. □

Tits [Tit58] studies $\mathcal{T}$*-geometries*—tripartite graphs $\mathcal{T}$ satisfying Theorem 4.5(a), particularly those that admit automorphisms g as in Theorem 4.6 for *all choices* of $\{i, j, k\} = \{1, 2, 3\}$. (In this case he shows that, for a $\mathcal{T}$-geometry with the nondegeneracy property Theorem 4.5(b), the properties of Theorem 4.6(b, c) follow from Theorem 4.6(a).)

So what we need to get to Tits' situation is the fundamental fact of triality for (V, Q):

> The graph $\mathcal{T}$ admits automorphisms inducing the full group Sym(3) on the index set of $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$, not just the transposition (1)(2, 3) provided by the symmetries.

This will be done in Theorem 5.18 and Proposition 5.19 below.

5 Composition algebras

An *algebra* over the field F is a F-vector space A combined with a bilinear product $\pi\colon\ A \times A \longrightarrow A$. For the purpose of these notes, we will always assume that algebras have identity elements. As A is a F-algebra, we immediately have

(5.1) LEMMA. *The maps $L_a\colon x \mapsto ax$ and $R_a\colon x \mapsto xa$ are F-linear transformations of A.* □

The algebra admits *composition* if there is defined on A a nondegenerate quadratic form $Q\colon A \longrightarrow F$ with the additional property that

$$Q(a)Q(b) = Q(ab)\,.$$

for all $a, b \in A$. Especially $Q(1) = 1$.

(5.2) THEOREM. (HURWITZ' THEOREM) *A finite dimensional composition algebra has dimension* 1, 2, 4, *or* 8 *over the field F.* □

An immediate consequence of the composition law is that all invertible elements of A are nonsingular. The converse is also true (see Corollary 5.7 below). Therefore if all nonzero elements of a composition algebra A are nonsingular, then A is a division algebra. If A is not a division algebra, then Q is actually hyperbolic (see Lemma 5.9 below). In this case, the algebra is called *split*. Proposition 1.2 and the following lemma (which we do not prove) guarantee that every division composition algebra over F tensors up to a split composition algebra over E, where E is a quadratic extension of F. In particular, it is enough to prove Hurwitz' Theorem for split composition algebras.

(5.3) LEMMA. *If E is an extension field for F, then the algebra $E \otimes_F A$ also admits composition with respect to the induced quadratic form.* □

It turns out that the split algebras are uniquely determined up to isomorphism by dimension and field. For a proof of this and Lemma 5.3, see [Hal00]. Here we are mainly interested in properties of the split composition algebras of dimension 8, although a short detour in our arguments provides a proof of Hurwitz' Theorem (in the finite dimensional, split case).

5.1 Examples

A composition algebra of dimension 1 is just a field with $Q(x) = x^2$ and is not split. (As Q is nondegenerate, the field cannot have characteristic 2.)

A composition algebra of dimension 4 is usually called a *quaternion* algebra. There is a canonical example of a split composition F-algebra of dimension 4, namely the algebra of all 2×2 matrices over F with the usual multiplication and with $Q(x) = \det(x)$:

$$\det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \alpha\delta - \beta\gamma \,.$$

The diagonal matrices give a nondegenerate split subalgebra of dimension 2, and the scalar matrices give a subalgebra of dimension 1.

A composition algebra of dimension 8 is usually called an *octonion* or *Cayley* algebra. As already mentioned, a split Cayley algebra over F is unique up to isomorphism. The following construction is essentially due to Zorn.

Consider the set of matrices

$$\mathrm{Oct}_8^+(F) = \left\{ \begin{bmatrix} x_1 & x_{234} \\ x_{567} & x_8 \end{bmatrix} \right\} ,$$

where $x_1, x_8, y_1, y_8 \in F$ and $x_{234}, x_{567}, y_{234}, y_{567} \in F^3$. Addition of such matrices is defined naturally. Multiplication is given by

$$\begin{bmatrix} x_1 & x_{234} \\ x_{567} & x_8 \end{bmatrix} \begin{bmatrix} y_1 & y_{234} \\ y_{567} & y_8 \end{bmatrix} =$$

$$\begin{bmatrix} x_1y_1 + x_{234} \cdot y_{567} & x_1y_{234} + x_{234}y_8 + x_{567} \times y_{567} \\ x_{567}y_1 + x_8y_{567} - x_{234} \times y_{234} & x_8y_8 + x_{567} \cdot y_{234} \end{bmatrix} =$$

$$\begin{bmatrix} x_1y_1 + x_{234}\cdot y_{567} & x_1y_{234} + x_{234}y_8 \\ x_{567}y_1 + x_8y_{567} & x_8y_8 + x_{567}\cdot y_{234} \end{bmatrix} + \begin{bmatrix} 0 & x_{567}\times y_{567} \\ -x_{234}\times y_{234} & 0 \end{bmatrix}.$$

Here on F^3, in addition to scalar multiplication (from both sides), the two products $\cdot$ and $\times$ are, respectively, the usual dot product

$$(a_1, a_2, a_3)\cdot(b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$$

and cross product (vector product)

$$(a_1, a_2, a_3)\times(b_1, b_2, b_3) = (a_2b_3 - a_3b_2, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2)\,.$$

The associated quadratic form is

$$x_1x_8 - x_{234}\cdot x_{567}\,.$$

(5.4) THEOREM. $\mathrm{Oct}_8^+(F)$ *with the notation and operations defined above is a split octonion algebra—a split composition algebra of dimension* 8.

PROOF. The set is closed under addition and multiplication with identity element

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The dot and cross products are bilinear, so we do have an F-algebra. There are clearly nonzero matrices that are singular with respect to the given quadratic form.

It remains to check that the form admits composition. This is not difficult and depends upon certain identities involving the dot and cross products:

Let $a, b, c, d \in F^3$. Then

(i) $a\cdot b = b\cdot a$.

(ii) $a\times b = -(b\times a)$ and $a\times a = 0$.

(iii) $a\cdot(a\times b) = 0$.

(iv) $(a\times b)\cdot(c\times d) = (a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c)$. □

Within $\mathrm{Oct}_8^+(F)$ there are numerous nondegenerate split 4-dimensional subalgebras represented as matrix algebras:

$$\left\{\begin{bmatrix} \alpha & \beta e \\ \gamma e & \delta \end{bmatrix}\right\},$$

for any $e\in F^3$ with $e\cdot e = 1$, in particular $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

5.2 Structure

We assume throughout that A is a finite dimensional algebra with identity 1 over F admitting composition with respect to nondegenerate Q.

The composition law, when written

$$Q(L_a x) = Q(R_a x) = Q(a)Q(x)\,,$$

reveals the maps L_a and R_a to be similarities for Q with respect to the scaling constant $Q(a)$ (when invertible—see Corollary 5.7 below). They are then also similiarities for the associated bilinear form $B(\cdot,\cdot)$; and we find, for $a, x, y \in A$,

$$B(xa, ya) = B(ax, ay) = Q(a)B(x, y)\,.$$

We define the operation of *conjugation* on A by $x \mapsto \bar{x} = -x + B(x, 1)1$. For instance, in the split quaternion algebra of 2×2 matrices we have the familiar

$$\overline{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}.$$

This formula then carries over to Zorn's representation of the split octonions.

(5.5) LEMMA.

(a) $\bar{\bar{x}} = x$

(b) $Q(x) = Q(\bar{x})$ *and* $B(x, y) = B(\bar{x}, \bar{y})$

PROOF. $\bar{x} = -s_1(x)$, so this follows from Proposition 2.1. □

(5.6) PROPOSITION.

(a) $\bar{x}(xy) = Q(x)y = (yx)\bar{x}$. *In particular* $\bar{x}x = Q(x)1 = x\bar{x}$.

(b) $\bar{x}(yz) + \bar{y}(xz) = B(x, y)z$ *and* $(zy)\bar{x} + (zx)\bar{y} = B(x, y)z$.

(c) $B(x, \bar{v}y) = B(vx, y)$ *and* $B(x, y\bar{v}) = B(xv, y)$.

PROOF. In each case, we only prove the first identity. We first prove (c):

$$\begin{aligned} B(x, \bar{v}y) &= B(x, (B(1, v) - v)y) \\ &= B(x, y)B(1, v) - B(x, vy) \\ &= B(x, y)(Q(1 + v) - Q(1) - Q(v)) - B(x, vy) \\ &= B((1 + v)x, (1 + v)y) - B(x, y) - B(vx, vy) - B(x, vy) \\ &= B(vx, y)\,. \end{aligned}$$

Next, for (a):

$$\begin{aligned} B(\bar{x}(xy), z) &= B(xy, xz) \\ &= Q(x)B(y, z) \\ &= B(Q(x)y, z)\,, \end{aligned}$$

for all z. Therefore by nondegeneracy $\bar{x}(xy) = Q(x)y$, giving (a).

We linearize (a) to get (b):

$$\begin{aligned}(\bar{x}+\bar{y})((x+y)z) &= Q(x+y)z\\ \bar{x}(xz)+\bar{y}(yz)+\bar{x}(yz)+\bar{y}(xz) &= Q(x)z+Q(y)z+B(x,y)z\\ \bar{x}(yz)+\bar{y}(xz) &= B(x,y)z\,.\end{aligned}\qquad\square$$

The first part of the proposition immediately gives:

(5.7) COROLLARY. *The following are equivalent:*

(1) *x is nonsingular.*

(2) *x is invertible.*

(3) *x has inverse $Q(x)^{-1}\bar{x}$.*

(4) *R_x is invertible.*

(5) *L_x is invertible.* □

(5.8) COROLLARY.

(a) $x^2 - B(x,1)x + Q(x) = 0$.

(b) $\overline{xy} = \bar{y}\bar{x}$.

PROOF. By definition $\bar{x}x = (-x + B(x,1)1)x = -x^2 + B(x,1)x$, so (a) follows directly from Proposition 5.6(a).

For (b), we use Proposition 5.6(c) many times:

$$\begin{aligned}B(\overline{xy},z) &= B(1,(xy)z) &&= B(\bar{z},xy)\\ &= B(\bar{z}\bar{y},x) &&= B(\bar{y},zx)\\ &= B(\bar{y}\bar{x},z) &&,\end{aligned}$$

for all z. Therefore, by nondegeneracy, $\overline{xy} = \bar{y}\bar{x}$. □

In particular $F1 + Fx$ is always a commutative, associative subalgebra.

(5.9) LEMMA. *If x is a nonzero singular vector in A, then there exist singular vectors y with $B(x,y) \neq 0$. Furthermore, for any such pair $\{x,y\}$, always $A = xA \oplus yA = Ax \oplus Ay$ with each xA and Ax maximal singular. In particular, (A,Q) is hyperbolic.*

PROOF. For all singular x, the subspaces xA and Ax are both singular since $Q(xA) = 0 = Q(Ax)$.

By Proposition 1.2 any nondegenerate 2-subspace containing x contains a hyperbolic pair $\{x,y\}$. Especially $x+y$ is nonsingular. Thus by Corollary 5.7

$$A = L_{x+y}A = (x+y)A \le xA + yA \le A\,.$$

That is, $A = xA + yA$. As Q is nondegenerate $xA \cap yA = 0$, and both are maximal singular. Therefore $A = xA \oplus yA$, and Q is hyperbolic by Proposition 3.1.

A similar argument proves the claims for Ax and Ay. (Here and elsewhere, lefthanded and righthanded versions of a result can be proven by similar arguments or seen to be equivalent using Corollary 5.8(b). We may only give one version.) □

From now on we will assume that the set $\mathcal{S}$ of nonzero singular vectors is not empty. By the lemma (A, Q) is hyperbolic. As before $\mathcal{S}_1$ is the set of singular 1-spaces of A, and $\mathcal{M}$ is the set of all maximal singular subspaces of A. Let m be the dimension of each member of $\mathcal{M}$, so that A has F-dimension $2m$.

(5.10) Lemma.

(a) *If $x \in \mathcal{S}$, then the image of L_x is xA and its kernel is $\bar{x}A$.*

(b) *If $x \in \mathcal{S}$, then the image of R_x is Ax and its kernel is $A\bar{x}$.*

Proof. Certainly the image of L_x is xA. By Proposition 5.6(a) the m-space $\bar{x}A$ is contained in the kernel of L_x, which has dimension $2m - m = m$. □

(5.11) Lemma. *Assume $m \geq 2$. Let $x, y \in \mathcal{S}$.*

(a) *If $xy = 0$, then $xA \cap Ay = x(y^\perp) = (x^\perp)y$ of codimension 1 in each.*

(b) *$xA \neq Ay$.*

Proof. By Proposition 5.6(b), for all $a \in A$,

$$\bar{x}(ay) + \bar{a}(xy) = B(x, a)y\,,$$

and by Lemma 5.10

$$xA \cap Ay = \ker L_{\bar{x}} \cap Ay\,.$$

(*a*) If $xy = 0$ then $\bar{x}(ay) = B(x, a)y$. Thus $xA \cap Ay = (x^\perp)y$ and also $x^\perp \geq \ker R_y = A\bar{y}$. As $x^\perp$ has codimension 1 in A, the codimension of $(x^\perp)y$ in $Ay = \operatorname{im} R_y$ is 1.

(*b*) If $xA = Ay$ then $\bar{a}(xy) = B(x, a)y$, so $A(xy) \leq \langle y \rangle$ has dimension at most 1. By Corollary 5.7 and Lemma 5.10, for nonzero w the linear transformation R_w has rank m or $2m$. As we are assuming $m \geq 2$, this forces $xy = 0$ and so contradicts (*a*). □

(5.12) Lemma. *Assume $m \geq 2$.*

(a) *Let x be singular and U a maximal singular subspace with $xA \cap U$ of codimension 1 in each. Then there is a singular y with $xy = 0$, $U = Ay$, and $xA \cap U = xA \cap Ay = x(y^\perp) = (x^\perp)y$.*

(b) *Let x be singular and U a maximal singular subspace with $Ax \cap U$ of codimension 1 in each. Then there is a singular y with $yx = 0$, $U = yA$, and $Ax \cap U = yA \cap Ax = y(x^\perp) = (y^\perp)x$.*

Proof. We only prove (a). Let $U_0 = U \cap xA$, of codimension 1 in each. Let W be the preimage of U_0 under L_x, so that W has codimension 1 in A. By Lemma 5.10, $\ker(L_x) = \bar{x}A$ is contained in W. As W has codimension 1 in A, there is a y, uniquely determined up to scalar multiple, with $W = y^\perp$, hence $U_0 = L_x W = xW = x(y^\perp)$. Furthermore, $\langle y \rangle = W^\perp \subseteq (\bar{x}A)^\perp = \bar{x}A$, hence $y \in \mathcal{S}$. Also $0 = xy \in x(\bar{x}A)$, by Proposition 5.6 or Lemma 5.10.

By the previous paragraph and Lemma 5.11, we have

$$xA \cap Ay = x(y^\perp) = U_0 = xA \cap U\,.$$

Therefore $Ay = U$ by Proposition 3.2(a). □

(5.13) Proposition. *Assume $m \geq 2$. For every maximal singular subspace U, there is a singular x with U equal to one of xA or Ax. The two parts of the incidence graph $(\mathcal{M}, \sim)$ on the set $\mathcal{M}$ of maximal singular subspaces are $\mathcal{M}^\rho = \{Ax \,|\, x \in \mathcal{S}\}$ and $\mathcal{M}^\lambda = \{xA \,|\, x \in \mathcal{S}\}$.*

Proof. Consider the two sets of maximal singular subspaces $\{Ax \,|\, x \in \mathcal{S}\}$ and $\{xA \,|\, x \in \mathcal{S}\}$. They are disjoint by Lemma 5.11. By Lemma 5.12 every edge on yA in the incidence graph $(\mathcal{M}, \sim)$ goes to $\{Ax \,|\, x \in \mathcal{S}\}$, and every edge of $(\mathcal{M}, \sim)$ on Ay goes to $\{xA \,|\, x \in \mathcal{S}\}$. By Proposition 3.3 $(\mathcal{M}, \sim)$ is bipartite and connected, so these sets are the two parts of the bipartition. □

(5.14) Lemma. *Assume $m \geq 3$. Let $x, y \in \mathcal{S}$ be with $\langle x \rangle \neq \langle y \rangle$.*

(a) If $B(x,y) = 0$ then $xA \cap yA$ has codimension 2 in each and $Ax \cap Ay$ has codimension 2 in each.

(b) $xA \neq yA$ and $Ax \neq Ay$.

Proof. Let U_0 be singular of dimension $m-1\,(\geq 2)$ and containing $\langle x, y \rangle$. By Lemma 5.12 and Proposition 5.13, there are $w, z \in \mathcal{S}$ with $U_0 = wA \cap Az$. As $\langle x, y \rangle \subseteq Az$, we have $x\bar{z} = y\bar{z} = 0$ by Lemma 5.10. Therefore $xA \cap A\bar{z}$ and $yA \cap A\bar{z}$ both have dimension $m-1$ by Lemma 5.11. This implies that $xA \cap yA$ has dimension at least $m-2$. The dimension of $xA \cap yA$ cannot be $m-1$ by Lemmas 5.11 and 5.12, so (a) will follow from (b).

If $xA = yA$, then $B(x,y) = 0$; so in proving (b) we may make use of the previous paragraph. By Lemma 5.11 again $xA \cap A\bar{z} = yA \cap A\bar{z}$ equals the $m-1$ space $(x^\perp)\bar{z} = (y^\perp)\bar{z}$. Its preimage under $R_{\bar{z}}$ is then $x^\perp = y^\perp$. This forces $\langle x \rangle = \langle y \rangle$, which is not the case.

Starting again with $\bar{w}x = \bar{w}y = 0$, we find the rest of the lemma. □

(5.15) Corollary. *Assume $m \geq 3$. Then the map $\langle x \rangle \mapsto Ax$ gives a bijection of $\mathcal{S}_1$ and $\mathcal{M}^\rho$ and $\langle x \rangle \mapsto xA$ gives a bijection of $\mathcal{S}_1$ and $\mathcal{M}^\lambda$.*

Proof. This follows from Lemmas 5.11 and 5.14 and Proposition 5.13. □

We can now prove Hurwitz' Theorem (in the split, finite dimensional case).

(5.16) THEOREM. (HURWITZ' THEOREM) *A finite dimensional, split composition algebra A has dimension 2, 4, or 8.*

PROOF. Since Q is hyperbolic, the dimension $2m$ is even. We must prove that m is 1, 2, or 4. Assume that m is at least 3. Consider the part $\mathcal{M}^\lambda = \{xA \mid x \in \mathcal{S}\}$ of the graph $(\mathcal{M}, \sim)$ and distances within it.

By Propositions 3.3 and 5.13, the distance from xA to yA in $(\mathcal{M}, \sim)$ is even and equal to the codimension of $xA \cap yA$ in each. Every even number in the range 0 to m must be realized, since $(\mathcal{M}, \sim)$ is connected of diameter m. But by Lemmas 5.9 and 5.14, the only distances realized within $\mathcal{M}^\lambda = \{xA \mid x \in \mathcal{S}\}$ are 0 (when $\langle x \rangle = \langle y \rangle$), 2 (when $B(x, y) = 0$ but $\langle x \rangle \neq \langle y \rangle$), and m (when $B(x, y) \neq 0$). This forces m to be even and $2 \geq m - 2\,(\geq 1)$. That is, $m = 4$. □

We now consider only the triality case $2m = 8$ and the associated triality geometry $\mathcal{T} = \mathcal{T}_1 \uplus \mathcal{T}_2 \uplus \mathcal{T}_3$ where $\mathcal{T}_1 = \mathcal{S}_1$, $\mathcal{T}_2 = \mathcal{M}^\lambda$, and $\mathcal{T}_3 = \mathcal{M}^\rho$.

(5.17) LEMMA. *For $\langle x \rangle, \langle y \rangle \in \mathcal{T}_1$, the following are equivalent:*

(1) $xy = 0$;
(2) $\langle y \rangle \sim \bar{x}A$;
(3) $\langle x \rangle \sim A\bar{y}$;
(4) $\langle \bar{y} \rangle \sim Ax$;
(5) $\langle \bar{x} \rangle \sim yA$;
(6) $xA \sim Ay$;
(7) $\bar{y}A \sim A\bar{x}$.

PROOF. By Lemma 5.10, $\bar{x}A$ is the kernel of L_x, so $y \in \bar{x}A$ if and only if $xy = 0$. Similarly $\langle x \rangle \in A\bar{y} = \ker(R_y)$ if and only if $xy = 0$. Also $\langle \bar{y} \rangle \in Ax$ if and only if $\bar{y}\bar{x} = 0$ if and only if $xy = 0$ by Corollary 5.8(b), and similarly for $\langle \bar{x} \rangle \in yA$.

By Lemmas 5.11, 5.12, and 5.14 the intersection $xA \cap Ay$ has codimension 1 in each if and only if $xy = 0$, and similarly $\bar{y}A \cap A\bar{x}$ has codimension 1 in each if and only if $\bar{y}\bar{x} = 0$. □

Define on $\mathcal{T}$ the map τ, for all $\langle x \rangle \in \mathcal{T}_1$:

$$\langle x \rangle \xrightarrow{\tau} \bar{x}A \xrightarrow{\tau} A\bar{x} \xrightarrow{\tau} \langle x \rangle\,.$$

The map τ is well-defined by Corollary 5.15.

(5.18) THEOREM. *The map τ is an automorphism of $\mathcal{T}$ of order 3, a triality automorphism.*

PROOF. We have τ acting on pairs:

$$(\langle y \rangle, \bar{x}A) \xrightarrow{\tau} (\bar{y}A, A\bar{x}) \xrightarrow{\tau} (A\bar{y}, \langle x \rangle) \xrightarrow{\tau} (\langle y \rangle, \bar{x}A)\,.$$

By the lemma, any one of these is an edge of $\mathcal{T}$ if and only if $xy = 0$, in which case they are all edges. Therefore τ is an automorphism of the graph $\mathcal{T}$. □

Let κ be the permutation of $\mathcal{T}$ determined by the conjugation map in A:

$$\kappa(\langle x \rangle) = \langle \bar{x} \rangle;\ \kappa(xA) = A\bar{x};\ \kappa(Ax) = \bar{x}A\,.$$

(5.19) Proposition. *κ is an automorphism of $\mathcal{T}$ of order 2 that inverts the triality automorphism τ.*

Proof. We have on pairs

$$(\langle y\rangle, \bar{x}A) \stackrel{\kappa}{\longleftrightarrow} (\langle \bar{y}\rangle, Ax) \text{ and } (\bar{y}A, A\bar{x}) \stackrel{\kappa}{\longleftrightarrow} (Ay, xA)\,.$$

Again by Lemma 5.17, any of these pairs is an edge if and only if $xy = 0$, in which case all are edges. Futhermore

$$\langle x\rangle \stackrel{\kappa}{\longrightarrow} \langle \bar{x}\rangle \stackrel{\tau}{\longrightarrow} xA \stackrel{\kappa}{\longrightarrow} A\bar{x}\,,$$

and so forth, leading to

$$\langle x\rangle \stackrel{\kappa\tau\kappa}{\longrightarrow} A\bar{x} \stackrel{\kappa\tau\kappa}{\longrightarrow} \bar{x}A \stackrel{\kappa\tau\kappa}{\longrightarrow} \langle x\rangle\,.$$

Therefore $\kappa\tau\kappa = \tau^{-1}$, as claimed. □

Of course, it should be no surprise that κ is an automorphism of $\mathcal{T}$. From Proposition 2.1, we see that κ is induced by the negative of the orthogonal symmetry s_1 on A.

6 Symmetric $\mathcal{T}$-geometries

This section is based upon §§3-4 of [Tit58]. A *$\mathcal{T}$-geometry* is a tripartite graph $\mathcal{T}$ with nonempty parts $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ and satisfying, for $\{i,j,k\} = \{1,2,3\}$:

> for every nonincident pair $p_i \in \mathcal{T}_i$ and $p_j \in \mathcal{T}_j$, there is a unique $p_k \in \mathcal{T}_k$ that is incident to both p_i and p_j.

In particular $\mathcal{T}$ is connected of diameter at most 3. There are many examples.

(6.1) Example. (Gated $\mathcal{T}$-geometries) Let $\mathcal{U}$ be a tripartite graph with parts $\mathcal{U}_1$, $\mathcal{U}_2$, and $\mathcal{U}_3$ and having the property:

> if $p_i \sim p_j \sim p_k$, for $p_i \in \mathcal{U}_i$, $p_j \in \mathcal{U}_j$, $p_k \in \mathcal{U}_k$ and $\{i,j,k\} = \{1,2,3\}$, then $p_i \sim p_k$.

For example, this will be the case for any tripartite $\mathcal{U}$ in which all connected components are complete tripartite, allowing degenerate components $K_{m,n,0}$—a complete bipartite subgraph meeting only two parts of $\mathcal{S}$—and $K_{1,0,0}$, a single vertex.

For each i let $\mathcal{T}_i = \mathcal{U}_i \cup \{\infty_i\}$, where ∞_i is a new vertex, a "gate." For $\{i,j,k\} = \{1,2,3\}$ let the gate ∞_i be incident to every vertex of $\mathcal{T}_j$ and $\mathcal{T}_k$. The tripartite graph $\mathcal{T} = \mathcal{T}_1 \uplus \mathcal{T}_2 \uplus \mathcal{T}_3$ is then a $\mathcal{T}$-geometry.

In particular any complete tripartite graph $K_{m,n,p}$ is a $\mathcal{T}$-geometry [Tit58, §4.1].

(6.2) Example. The 6-cycle C_6 is a $\mathcal{T}$-geometry.

Among the above examples of $\mathcal{T}$-geometries, the complete graphs $K_{m,m,m}$ and the cycle C_6 have large automorphism groups.

Specifically, consider the subgroup Sym(3) of Aut(C_6) whose three elements of order 2 are the reflections of the 6-cycle that fix none of its vertices. If a is one such element, then a fixes $\mathcal{T}_i$, switches $\mathcal{T}_j$ and $\mathcal{T}_k$ (for an appropriate numbering of the three parts of C_6), and has the following three properties:

(i) for all $p_i \in \mathcal{T}_i$, p_i and p_i^a are incident;

(ii) if $p_i \in \mathcal{T}_i$ is incident simultaneously to $p_j\,(\in \mathcal{T}_j)$ and $p_j^a\,(\in \mathcal{T}_k)$, then $p_i^a = p_i$;

(iii) $a^2 = 1$.

Of course for C_6, the second property holds trivially.

Similarly, consider the complete graph $K_{m,m,m}$. Here the wreath product $\mathrm{Sym}(m)^3 \rtimes \mathrm{Sym}(3)$ acts on the associated $\mathcal{T}$-geometry with each involution a of the wreathing quotient Sym(3) having the three properties above. In this example, the first property is essentially trivial but the second is very strong, saying that a fixes each vertex of the part it leaves invariant.

We call an automorphism a of the $\mathcal{T}$-geometry $\mathcal{T}$ acting as above a *symmetry* of $\mathcal{T}$. For a given automorphism group G of $\mathcal{T}$, we denote by D_i the set of symmetries of $\mathcal{T}$ that leave part $\mathcal{T}_i$ fixed and switch $\mathcal{T}_j$ and $\mathcal{T}_k$. Further set $\Delta = D_1 \cup D_2 \cup D_3$. In Aut($\mathcal{T}$) a conjugate of a symmetry is again a symmetry, so Δ is a normal set of elements of order 2.

The automorphisms of D_i induce the permutation $(i)(j,k)$ on the parts of $\mathcal{T}$. Tits [Tit58, §3.2] calls $\mathcal{T}$ a *symmetric $\mathcal{T}$-geometry* provided all permutations of $\{1,2,3\}$ are induced by Aut($\mathcal{T}$). Thus C_6 and $K_{m,m,m}$ are symmetric.

We next have Tits' "Fundamental Lemma" [Tit58, §3.3]:

(6.3) LEMMA. *Let $\{i,j,k\} = \{1,2,3\}$. If $a \in D_i$ and $b \in D_j$, then*

(a) *$aba = bab \in D_k$;*

(b) $(ab)^3 = 1$.

PROOF. As a conjugate of a symmetry is a symmetry, both $a^{-1}ba = aba$ and bab are in D_k, inducing the permutation $(k)(i,j)$. It remains to prove

$$1 = (aba)(bab) = (ab)^3\,.$$

First let $p \in \mathcal{T}_3$. Then $p \sim p^b$ by (i), hence $p^{ab} \sim p^{bab}$. Similarly $p^{ba} \sim (p^{ba})^b = p^{bab}$ as $p^{ba} \in \mathcal{T}_i$. That is,

$$p^{ab} \sim p^{bab} \sim p^{ba}\,,$$

and by symmetry

$$p^{ba} \sim p^{aba} \sim p^{ab}\,.$$

If $p^{ab} \not\sim p^{ba}$, then by the defining axiom for $\mathcal{T}$-spaces $p^{bab} = p^{aba}$ and $p^{(ab)^3} = p$. On the other hand, if $p^{ab} \sim p^{ba}$ this would combine with $p^{ab} \sim p^{bab} = (p^{ba})^b$ (from above) to give $p^{ab} = (p^{ab})^b = p^a$ by (ii). That is, $p^{aba} = p$ and by symmetry $p^{bab} = p$; again $p^{bab} = p^{aba}$ and $p^{(ab)^3} = p$. Therefore for $p \in \mathcal{T}_k$ we always have $p^{(ab)^3} = p$.

This in turn implies that

$$(p^{ab})^{(ab)^3} = p^{(ab)^4} = (p^{(ab)^3})^{ab} = p^{ab}$$

and

$$(p^{(ab)^2})^{(ab)^3} = p^{(ab)^5} = (p^{(ab)^3})^{(ab)^2} = p^{(ab)^2}\,.$$

We conclude that $(ab)^3$ is trivial on $\mathcal{T}_k$ and additionally on $\mathcal{T}_k^{ab} \cup \mathcal{T}_k^{(ab)^2} = \mathcal{T}_i \cup \mathcal{T}_j$. That is, $(ab)^3 = 1$, as desired. □

(6.4) COROLLARY. *Let $G \leq \mathrm{Aut}(\mathcal{T})$ with $D = G \cap \Delta$ meeting at least two of D_1, D_2, and D_3. Then D is a conjugacy class in G and in $\langle D \rangle$ such that, for arbitrary $a \in D_i \cap D$ and $b \in D_j \cap D$ (with $i \neq j$) we have $(ab)^3 = 1$.*

In particular $\langle a, b \rangle \simeq \mathrm{Sym}(3)$, and $\mathcal{T}$ is symmetric. □

(6.5) REMARKS.

(a) *The consequences of the previous lemma and corollary for $\mathcal{T} = K_{m,m,m}$, where the automorphism group is the wreath product $\mathrm{Aut}(\mathcal{T}) = \mathrm{Sym}(m)^3 \rtimes \mathrm{Sym}(3)$, were detailed by Tits [Tit58, §4.1] and later (and independently) rediscovered by Doro [Dor78] and Zara [Zar85].*

(b) *Tits [Tit58, §3.4] also observes that, provided a certain nondegeneracy condition (namely that of Theorem 4.5(b)) holds in the $\mathcal{T}$-geometry $\mathcal{T}$, the two defining properties* (ii) *and* (iii) *of symmetries are consequences of the property* (i). *The nondegeneracy condition holds in $\mathcal{T}_8^+(F)$ by Theorem 4.5(b) but not in C_6 or $K_{m,m,m}$.*

One of Tits' motivating observations is:

(6.6) THEOREM. (TITS [TIT58, §4.2].) *Each $\mathcal{T}_8^+(F)$ is a symmetric $\mathcal{T}$-geometry with D_1 containing all orthogonal symmetries.*

PROOF. This is immediate from Theorems 4.6 and 5.18. □

Thus by Tits' Fundamental Lemma 6.3, for the class D of $\mathrm{PSO}_8^+(F) \rtimes \mathrm{Sym}(3)$ containing the orthogonal symmetries:

> for arbitrary $a, b \in D$ with a and b mapping to different involutions of the quotient $\mathrm{Sym}(3)$, we have $(ab)^3 = 1$.

This is the motivating example for *abstract triality* in groups [Dor78, HaN01].

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