ON THE PERMUTATION MODULES FOR ORTHOGONAL GROUPS $O_m^{\pm}(3)$ ACTING ON NONSINGULAR POINTS OF THEIR STANDARD MODULES

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Abstract. We describe the structure, including composition factors and submodule lattices, of cross-characteristic permutation modules for the natural actions of the orthogonal groups $O_m^{\pm}(3)$ with $m \geq 6$ on nonsingular points of their standard modules. These actions together with those studied in [2] are all examples of primitive rank 3 actions of finite classical groups on nonsingular points.

1. Introduction

Given a group G acting on a set Ω and a field $\mathbb{F}$, the problem of determining the structure of the permutation $\mathbb{F}G$-module $\mathbb{F}\Omega$ has been studied extensively for many years. In particular, permutation modules as well as permutation representations for finite classical groups have received significant attention. We are interested in the action of a finite classical group G on points (i.e. 1-dimensional subspaces) of the standard module associated with G.

The permutation module for the natural action of G on singular points has been studied in great depth, especially in the cross-characteristic case where $\mathbb{F}$ and the underlying field of G have different characteristics (for instance, see [8, 9, 10, 11]). In defining characteristic, the study of these permutation modules seems very hard due to the fact that the number of composition factors of $\mathbb{F}G$ depends not only on the type of G but also on its dimension. This article will focus on cross-characteristic, including characteristic zero, permutation modules for finite classical groups acting on nonsingular points. We note that in the linear and symplectic groups, all points are singular.

Problem 1.1. *Let G be a finite orthogonal or unitary group and $\mathbb{F}$ an algebraically closed field of cross-characteristic. Describe the submodule structure of the permutation $\mathbb{F}G$-module for G acting naturally on the set of nonsingular points of its standard module.*

Let P^0 and P be the sets of singular and nonsingular points, respectively, of the standard module associated with G. It is well known that the action of G on P^0 is always transitive and rank 3. On the other hand, the transitivity of the action of G on P and the ranks of that action on orbits depend closely on q. We do not have an exact formula for the ranks but they are "more or less" an increasing function of q. In particular, the structure of $\mathbb{F}P$ becomes more complicated when q is large.

Date: July 5, 2011.
2010 *Mathematics Subject Classification.* Primary 20C33; Secondary 20C05, 20G40.
Key words and phrases. Permutation Modules, orthogonal groups, nonsingular points.

In [2], the authors studied the problem for orthogonal groups over a field of two elements and unitary groups over a field of four elements. These are the cases (and only cases!) where the action of G on P is transitive and rank 3.

In this article, we study the cross-characteristic permutation modules $\mathbb{F}P$ for the orthogonal groups $O^{\pm}_m(3)$ acting on P. It is not difficult to see that $O^{\pm}_m(3)$ has two orbits on P and the action on each orbit is rank 3. These actions together with those studied in [2] are all examples of primitive rank 3 actions of finite classical groups on nonsingular points, as pointed out in an important paper by Kantor and Liebler (see [7]).

Drawing upon the methods introduced in [2], as well as in [9] and [11], we prove the following:

Theorem 1.2. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $\ell \neq 3$. Let G be $O^{\pm}_m(3)$ ($m = 2n$ or $2n+1$) with $m \geq 6$ and P be the set of nonsingular points of the standard module associated with G. Then the permutation $\mathbb{F}G$-module $\mathbb{F}P$ of G acting naturally on P has the submodule structure as described in Tables* 1,2, *and* 3. *In these tables, $\delta_{i,j} = 1$ if $i \mid j$ and 0 otherwise.*

TABLE 1. Submodule structure of $\mathbb{F}O^{+}_{2n}(3)$-module $\mathbb{F}P$.

Conditions on ℓ and n	Structure of $\mathbb{F}P$
$\ell \neq 2, 3; \ell \nmid (3^n - 1)$	$\mathbb{F} \oplus X \oplus Z \bigoplus \mathbb{F} \oplus Y \oplus Z$
$\ell \neq 2, 3; \ell \mid (3^n - 1)$	$X \oplus \begin{matrix}\mathbb{F}\\ \vert \\ Z \\ \vert \\ \mathbb{F}\end{matrix} \bigoplus Y \oplus \begin{matrix}\mathbb{F}\\ \vert \\ Z \\ \vert \\ \mathbb{F}\end{matrix}$
$\ell = 2; n$ even	$\begin{tikzcd} \mathbb{F} \arrow[d, dash] & X \arrow[d, dash] \arrow[dl, dash] \\ W \arrow[d, dash] \arrow[dr, dash] & Y \arrow[d, dash] \\ \mathbb{F} & X \end{tikzcd} \bigoplus \begin{tikzcd} \mathbb{F} \arrow[d, dash] & Y \arrow[d, dash] \arrow[dl, dash] \\ W \arrow[d, dash] \arrow[dr, dash] & X \arrow[d, dash] \\ \mathbb{F} & Y \end{tikzcd}$
$\ell = 2; n$ odd	$\mathbb{F} \oplus \begin{tikzcd} & X \arrow[dl, dash] \arrow[dr, dash] & \\ Y \arrow[dr, dash] & & W \arrow[dl, dash] \\ & X & \end{tikzcd} \bigoplus \mathbb{F} \oplus \begin{tikzcd} & Y \arrow[dl, dash] \arrow[dr, dash] & \\ X \arrow[dr, dash] & & W \arrow[dl, dash] \\ & Y & \end{tikzcd}$

Here, $\dim X = \dim Y = \frac{(3^n-1)(3^{n-1}-1)}{8}$, $\dim Z = \frac{3^{2n}-9}{8} - \delta_{\ell,3^n-1}$,
and $\dim W = \frac{(3^n-1)(3^{n-1}+3)}{8} - 1 - \delta_{2,n}$.

Let V be a vector space of dimension $m \geq 6$ over the field of 3 elements $\mathbb{F}_3 = \{0, 1, -1\}$. Let Q be a non-degenerate quadratic form on V, and let $(\cdot, \cdot)$ be the non-degenerate symmetric bilinear form on V associated with Q so that $Q(au + bv) = a^2Q(u) + b^2Q(v) + ab(u, v)$ for any $a, b \in \mathbb{F}_3, u, v \in V$. Then $G = O^{\pm}_m(3)$ is the full orthogonal group consisting of all linear transformations of V preserving Q.

For $\kappa = \pm 1$, we denote

$$P^{\kappa} := P^{\kappa}(Q) := \{\langle v \rangle \in P \mid Q(v) = \kappa\}.$$

These P^{+1} and P^{-1} are often called the sets of plus points and minus points, respectively. We obtain the following isomorphism of $\mathbb{F}G$-modules:

$$\mathbb{F}P \cong \mathbb{F}P^{+1} \oplus \mathbb{F}P^{-1}.$$

TABLE 2. Submodule structure of $\mathbb{F}O_{2n}^{-}(3)$-module $\mathbb{F}P$.

Conditions on ℓ and n	Structure of $\mathbb{F}P$
$\ell \neq 2,3; \ell \nmid (3^n+1)$	$\mathbb{F} \oplus X \oplus Z \bigoplus \mathbb{F} \oplus Y \oplus Z$
$\ell \neq 2,3; \ell \mid (3^n-1)$	$X \oplus \begin{array}{c} \mathbb{F} \\ \mid \\ Z \\ \mid \\ \mathbb{F} \end{array} \bigoplus Y \oplus \begin{array}{c} \mathbb{F} \\ \mid \\ Z \\ \mid \\ \mathbb{F} \end{array}$
$\ell = 2; n$ even	$\mathbb{F} \oplus \begin{tikzcd}[column sep=small, row sep=small] & X \arrow[dr, dash] & \\ & & \mathbb{F} \arrow[d, dash] \\ W \arrow[uur, dash] \arrow[ddr, dash] & & Y \arrow[d, dash] \\ & & \mathbb{F} \arrow[dl, dash] \\ & X & \end{tikzcd} \bigoplus \mathbb{F} \oplus \begin{tikzcd}[column sep=small, row sep=small] & Y \arrow[dr, dash] & \\ & & \mathbb{F} \arrow[d, dash] \\ W \arrow[uur, dash] \arrow[ddr, dash] & & X \arrow[d, dash] \\ & & \mathbb{F} \arrow[dl, dash] \\ & Y & \end{tikzcd}$
$\ell = 2; n$ odd	$\begin{tikzcd}[column sep=small, row sep=small] \mathbb{F} \arrow[dd, dash] & X \arrow[d, dash] \\ & \mathbb{F} \arrow[d, dash] \\ Y \arrow[uur, dash] \arrow[ddr, dash] \arrow[dd, dash] & W \arrow[d, dash] \\ & \mathbb{F} \arrow[d, dash] \\ \mathbb{F} & X \end{tikzcd} \bigoplus \begin{tikzcd}[column sep=small, row sep=small] \mathbb{F} \arrow[dd, dash] & Y \arrow[d, dash] \\ & \mathbb{F} \arrow[d, dash] \\ X \arrow[uur, dash] \arrow[ddr, dash] \arrow[dd, dash] & W \arrow[d, dash] \\ & \mathbb{F} \arrow[d, dash] \\ \mathbb{F} & Y \end{tikzcd}$

Here, $\dim X = \dim Y = \frac{(3^n+1)(3^{n-1}+1)}{8} - \delta_{\ell,2}$, $\dim Z = \frac{3^{2n}-9}{8} - \delta_{\ell,3^n+1}$, and $\dim W = \frac{(3^n+1)(3^{n-1}-3)}{8} - 1 + \delta_{2,n}$.

Since Q is a non-degenerate quadratic form on V, $-Q$ is also a non-degenerate quadratic form on V. The two isometry groups $O(V,Q)$ and $O(V,-Q)$ are canonically isomorphic, but the corresponding sets of nonsingular points are switched: $P^{-1}(Q) = P^{+1}(-Q)$ and $P^{+1}(Q) = P^{-1}(-Q)$. When m is even the forms Q and $-Q$ have the same discriminant but when m is odd they do not. Therefore in Theorem 1.2 and its associated tables we see that for m even there are two distinct isometry groups G, but for each the modules $\mathbb{F}P^{+1}$ and $\mathbb{F}P^{-1}$ are the same up to a diagonal automorphism of G, whereas for m odd there is only one isometry group to consider, but the modules $\mathbb{F}P^{+1}$ and $\mathbb{F}P^{-1}$ are fundamentally different and indeed have different dimensions.

Remark 1.3. The three tables present the structure of each module $\mathbb{F}P = \mathbb{F}P^{+1} \oplus \mathbb{F}P^{-1}$ but also indicate the individual structures for $\mathbb{F}P^{+1}$ and $\mathbb{F}P^{-1}$ through separating them by a big direct sum symbol.

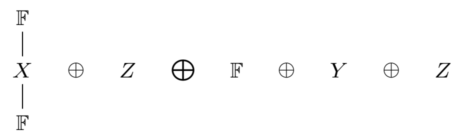
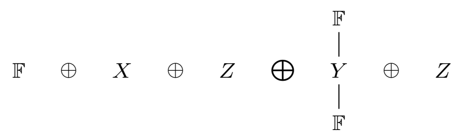
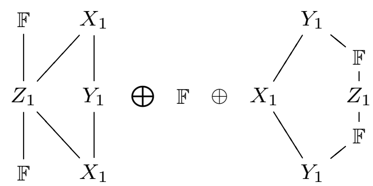
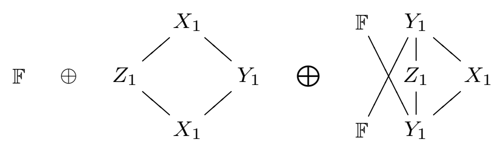
When m is odd, one can easily obtain the structures of $\mathbb{F}P^{+1}$ and $\mathbb{F}P^{-1}$ from the proofs in section 7 and Table 3 once Q is given. For instance, if $m = 2n+1$ and coordinates have be chosen so that Q has the shape

$$x_1x_{n+1} + x_2x_{n+2} + \cdots + x_nx_{2n} - x_{2n+1}^2,$$

then the structures of $\mathbb{F}P^{+1}$ and $\mathbb{F}P^{-1}$ will be, respectively, before and after the big direct sum symbol in Table 3.

When m is even, we may still say that the structures of $\mathbb{F}P^{+1}$ and $\mathbb{F}P^{-1}$, respectively, are before and after the big direct sum symbol in Tables 1 and 2. This now involves our convention that, of the two modules X and Y of the same dimension, the module named X is the one that appears more often as a composition factor of $\mathbb{F}P^{+1}$. This is not as satisfying as the odd case, since it is an implicit definition

Table 3. Submodule structure of $\mathbb{F}O_{2n+1}(3)$-module $\mathbb{F}P$.

Conditions on ℓ and n	Structure of $\mathbb{F}P$
$\ell \neq 2, 3; \ell \nmid (3^n - 1), \ell \nmid (3^n + 1)$	$\mathbb{F} \oplus X \oplus Z \bigoplus \mathbb{F} \oplus Y \oplus Z$
$\ell \neq 2, 3; \ell \mid (3^n - 1)$	
$\ell \neq 2, 3; \ell \mid (3^n + 1)$	
$\ell = 2; n$ even	
$\ell = 2; n$ odd	

Here, $\dim X = \frac{(3^n+1)(3^n-3)}{4} - \delta_{\ell,3^n-1}$, $\dim Y = \frac{(3^n-1)(3^n+3)}{4} - \delta_{\ell,3^n+1}$, $\dim Z = \frac{3^{2n}-1}{4}$, $\dim X_1 = \frac{(3^n-1)(3^n-3)}{8}$, $\dim Y_1 = \frac{(3^n+1)(3^n+3)}{8} - 1$, and $\dim Z_1 = \frac{(3^{2n}-9)}{8} - \delta_{2,n}$.

of X. If the chosen form Q is swapped with the form $-Q$, then the group stays the same but the module names X and Y are swapped since $\mathbb{F}P^{+1}$ and $\mathbb{F}P^{-1}$ are. To improve upon this, we would need explicit definitions that distinguish these two modules for the group. Then for a fixed choice of the form Q, one could decide which of the two occurs more often in $\mathbb{F}P^{+1}$ than in $\mathbb{F}P^{-1}$. This is not done here.

The paper is organized as follows. In the next section, we will outline the proof of the main theorem. Sections 3 and 4 are some preparations. Each family of groups $O^+_{2n}(3)$, $O^-_{2n}(3)$, and $O_{2n+1}(3)$ is treated respectively in sections 5, 6, and 7.

2. Notation and outline of the proof

2.1. **Preliminaries.** If the action of G on a set Ω is rank 3 then the $\mathbb{F}G$-module $\mathbb{F}\Omega$ has two special submodules, the so-called *graph submodules*. The following description of these graph submodules is due to Liebeck (see [9]).

Let G_α be the stabilizer of $\alpha \in \Omega$. Then G_α acts on Ω with 3 orbits: one of them is $\{\alpha\}$ and the other two are denoted by $\Delta(\alpha)$ and $\Phi(\alpha)$. Define the parameters: $a = |\Delta(\alpha)|$, $b = |\Phi(\alpha)|$, $r = |\Delta(\alpha) \cap \Delta(\beta)|$, and $s = |\Delta(\alpha) \cap \Delta(\gamma)|$ for $\beta \in \Delta(\alpha)$ and $\gamma \in \Phi(\alpha)$. For any subset Δ of Ω, we denote by $[\Delta]$ the element $\Sigma_{\delta\in\Delta}\delta$ of $\mathbb{F}\Omega$. For $c \in \mathbb{F}$, let U_c be the $\mathbb{F}G$-submodule of $\mathbb{F}\Omega$ generated by all elements $v_{c,\alpha} = c\alpha + [\Delta(\alpha)], \alpha \in \Omega$ and U'_c be the $\mathbb{F}G$-submodule of U_c generated by all elements $v_{c,\alpha} - v_{c,\beta} = c(\alpha - \beta) + [\Delta(\alpha)] - [\Delta(\beta)], \alpha, \beta \in \Omega$. The *graph submodules* of the permutation $\mathbb{F}G$-module $\mathbb{F}\Omega$ are defined to be U'_{c_1} and U'_{c_2} where c_1 and c_2

are the roots of the quadratic equation:

$$x^2 + (r - s)x + s - a = 0. \tag{2.1}$$

Setting $S(\mathbb{F}\Omega) = \{\sum_{\omega\in\Omega} a_\omega\omega \mid a_\omega \in \mathbb{F}, \sum a_\omega = 0\}$ and $T(\mathbb{F}\Omega) = \{c[\Omega] \mid c \in \mathbb{F}\}$. $S(\mathbb{F}\Omega)$ and $T(\mathbb{F}\Omega)$ are $\mathbb{F}G$-submodules of $\mathbb{F}\Omega$ of dimensions $|\Omega| - 1, 1$, respectively. Moreover, $T(\mathbb{F}\Omega)$ is isomorphic to the one-dimensional trivial module.

Suppose that two graph submodules are different, i.e., $c_1 \neq c_2$. Since $v_{c_1,\alpha} - v_{c_2,\alpha} = (c_1 - c_2)\alpha$ for any $\alpha \in \Omega$, we have

$$U'_{c_1} \oplus U'_{c_2} = S(\mathbb{F}\Omega). \tag{2.2}$$

The modules U'_{c_1} and U'_{c_2} are the eigenspaces of the linear transformation T : $\mathbb{F}\Omega \to \mathbb{F}\Omega$ defined by $T(\alpha) = [\Delta(\alpha)], \alpha \in \Omega$ corresponding to eigenvalues $-c_2$ and $-c_1$, respectively (see [2]). As T has trace 0 with $T(\mathbb{F}\Omega)$ an eigenspace for the eigenvalue a, we can compute the dimensions of graph submodules from the following equations:

$$\begin{cases} \dim U'_{c_1} + \dim U'_{c_2} = |\Omega| - 1, \\ c_2 \dim U'_{c_1} + c_1 \dim U'_{c_2} = a. \end{cases} \tag{2.3}$$

We note that $\mathbb{F}\Omega$ has a nonsingular and G-invariant inner product defined by $\big\langle \sum_{\omega\in\Omega} a_\omega\omega, \sum_{\omega\in\Omega} b_\omega\omega \big\rangle = \sum_{\omega\in\Omega} a_\omega b_\omega$. If U is a submodule of $\mathbb{F}\Omega$, we denote by $U^\perp$ the submodule of $\mathbb{F}\Omega$ consisting of all elements orthogonal to U. We need the following result, which is due to Liebeck and is stated as Lemma 2.1 in [2].

Lemma 2.1 ([9])**.** *If c is not a root of equation (2.1) then $U'_c = S(\mathbb{F}\Omega)$. Moreover, if c_1 and c_2 are roots of this equation then $\langle v_{c_1,\alpha}, v_{c_2,\beta}\rangle = s$ for any $\alpha, \beta \in \Omega$. Consequently, $\langle U'_{c_1}, U_{c_2}\rangle = \langle U'_{c_2}, U_{c_1}\rangle = 0$.* □

2.2. **Outline of the proof.** We first compute the roots of equation (2.1) and then determine the graph submodules of $\mathbb{F}P^{+1}$ and $\mathbb{F}P^{-1}$ by analyzing the geometry of P. As in the study of rank 3 permutation modules in cross-characteristic for finite classical groups acting on singular points in [9, 10], the graph submodules are "minimal" in an appropriate sense (see Proposition 3.1).

The problem now is divided in two cases. In the easy case when the graph submodules are different, we have seen from (2.2) that their direct sum is $S(\mathbb{F}P^\kappa)$ ($\kappa = \pm 1$), a submodule of $\mathbb{F}P^\kappa$ of codimension 1. Therefore the full structure of $\mathbb{F}P = \mathbb{F}P^{+1} \oplus \mathbb{F}P^{-1}$ can be determined without significant effort.

The difficult case is when the graph submodules are the same (i.e. $c_1 = c_2$). We will see later that the graph submodules of $\mathbb{F}P^{+1}$ are equal if and only if those of $\mathbb{F}P^{-1}$ as well as $\mathbb{F}P^0$ are equal and this happens when $\ell = \mathrm{char}(\mathbb{F}) = 2$. We handle this case by constructing some relations between $\mathbb{F}P^0$, $\mathbb{F}P^{+1}$, and $\mathbb{F}P^{-1}$ in §4.

2.3. **Further notation.** Given a finite group G, $\mathrm{Irr}(G)$ and $\mathrm{IBr}_\ell(G)$ will be the sets of irreducible complex characters and irreducible ℓ-Brauer characters, respectively, of G. For $\chi \in \mathrm{Irr}(G)$, by $\overline{\chi}$ we mean its restriction to the ℓ-regular elements of G. Following [11], we denote by $\beta(M)$ the Brauer character of G afforded by an $\mathbb{F}G$-module M. Furthermore, if $\beta(M) \in \mathrm{IBr}_\ell(G)$ is a constituent of an ℓ-Brauer character φ, we say that M is a constituent of φ. Finally, if H is a subgroup of G, we denote by $M|_H$ the restriction of M to H.

3. Minimality of the graph submodules of $\mathbb{F}P^{+1}$ and $\mathbb{F}P^{-1}$

First we fix a basis $\mathcal{B}$ of V. If Q is a quadratic form of type $+$ on V with dimension $2n$, we consider $\mathcal{B} = \{e_1, ..., e_n, f_1, ..., f_n\}$ so that $(e_i, f_j) = \delta_{ij}$ and $(e_i, e_j) = (f_i, f_j) = 0$ for $i, j = 1, ..., n$. If Q is of type $-$ on V with dimension $2n$, then $\mathcal{B} = \{e_1, ..., e_n, f_1, ..., f_n\}$ where $(e_i, f_j) = \delta_{ij}$ and $(e_i, e_n) = (f_i, f_n) = 0$ for $i, j = 1, ..., n-1$, $(e_n, f_n) = 0$, and $(e_n, e_n) = (f_n, f_n) = 1$. Finally, if $\dim V = 2n+1$, then $\mathcal{B} = \{e_1, ..., e_n, f_1, ..., f_n, g\}$ where $(e_i, f_j) = \delta_{ij}$, $(e_i, e_j) = (f_i, f_j) = (e_i, g) = (f_i, g) = 0$, and $(g, g) = 1$ for $i, j = 1, ..., n$.

Going back to the action of G on $P^\kappa, \kappa = \pm 1$, we assume from now on that $\Delta(\alpha) \subset P^\kappa \setminus \alpha$ consists of points orthogonal to α and $\Phi(\alpha) \subset P^\kappa \setminus \alpha$ consists of points not orthogonal to α. Also, we use the notation U_c^κ and $U_c^{'\kappa}$ for U_c and U_c', respectively. The graph submodules of $\mathbb{F}P^\kappa$ now are $U_{c_1}^{'\kappa}$ and $U_{c_2}^{'\kappa}$ where c_1 and c_2 are roots of the equation (2.1). At this point we understand that c_1 and c_2 depend on κ but actually they do not, as we will see later on.

As in the study of rank 3 permutation modules in cross-characteristic for finite classical groups acting on singular points in [9, 10], the graph submodules are "minimal" in the following sense:

Proposition 3.1. *Suppose that* $\mathrm{char}(\mathbb{F}) \neq 3$. *Then every nonzero* $\mathbb{F}G$*-submodule of* $\mathbb{F}P^\kappa (\kappa = \pm 1)$ *either is* $T(\mathbb{F}P^\kappa)$ *or contains a graph submodule.*

Proof. We only give here the proof for the case $G = O_{2n}^+(3)$ and $\kappa = +1$. Other cases are similar. We partly follow some ideas and notation from [2, 9, 10].

Let $\phi_1 := \langle e_2 + f_2 \rangle$, $\phi_2 := \langle e_1 + e_2 + f_2 \rangle$, and $\phi_3 := \langle -e_1 + e_2 + f_2 \rangle$. Let $\Delta_1 := \{\langle \sum_{i=1}^n (a_i e_i + b_i f_i) \rangle \in P^{+1} \mid b_1 = 1, a_2 + b_2 = 0\}$, $\Delta_2 := \{\langle \sum_{i=1}^n (a_i e_i + b_i f_i) \rangle \in P^{+1} \mid b_1 = 1, a_2 + b_2 = -1\}$, $\Delta_3 := \{\langle \sum_{i=1}^n (a_i e_i + b_i f_i) \rangle \in P^{+1} \mid b_1 = a_2 + b_2 = 1\}$, $\Delta := \Delta_1 \cup \Delta_2 \cup \Delta_3$, and $\Phi := P^{+1} \setminus \Delta$. It is clear that, for $i, j = 1, 2, 3$,

$$[\Delta(\phi_i)] - [\Delta(\phi_j)] = [\Delta_i] - [\Delta_j]. \tag{3.1}$$

Consider a subgroup $H < G$ consisting of orthogonal transformations sending elements of the basis $\{e_1, f_1, e_2, f_2, ..., e_n, f_n\}$ to those of basis $\{e_1, f_1 + \sum_{i=1}^n a_i e_i + \sum_{i=2}^n b_i f_i, e_2 - b_2 e_1, f_2 - a_2 e_1, ..., e_n - b_n e_1, f_n - a_n e_1\}$ respectively, where $a_i, b_i \in \mathbb{F}_3$ and $-a_1 = \sum_{i=2}^n a_i b_i$. In other words, H is subgroup of isometries fixing $\langle e_1 \rangle$ and acting trivially on each factor of the series $0 \leq \langle e_1 \rangle \leq \langle e_1 \rangle^\perp \leq V$. Let K be the subgroup of H consisting of transformations fixing ϕ_1. Let P_1^{+1} be the set of plus points in $V_1 = \langle e_2, f_2, ..., e_n, f_n \rangle$. For each $\langle w \rangle \in P_1^{+1}$, define $B_{\langle w \rangle} = \{\langle w \rangle, \langle e_1 + w \rangle, \langle -e_1 + w \rangle\}$. As in Propositions 2.1 and 2.2 of [9] and Lemmas 3.2 and 3.3 of [2], we have

(i) $|H| = 3^{2n-2}$, $|K| = 3^{2n-3}$, $|\Delta| = 3^{2n-2}$, and $|\Delta_1| = |\Delta_2| = |\Delta_3| = 3^{2n-3}$;
(ii) H acts regularly on Δ and K has 3 orbits $\Delta_1, \Delta_2, \Delta_3$ on Δ;
(iii) $\Phi = \bigcup_{\langle w \rangle \in P_1^{+1}} B_{\langle w \rangle}$;
(iv) K fixes B_{ϕ_1} point-wise and is transitive on B_w for every $\phi_1 \neq \langle w \rangle \in P_1^{+1}$;
(v) H acts transitively on $B_{\langle w \rangle}$ for every $\langle w \rangle \in P_1^{+1}$.

Suppose that U is a nonzero submodule of $\mathbb{F}P^{+1}$. Assume $U \neq T(\mathbb{F}P^{+1})$, so that U contains an element of the form

$$u = a\langle x \rangle + b\langle y \rangle + \sum_{\delta \in P^{+1} \setminus \{\langle x \rangle, \langle y \rangle\}} a_\delta \delta,$$

where $a, b, a_\delta \in \mathbb{F}$ and $a \neq b$. If $(x, y) = 0$, we choose an element $\langle z \rangle \in P^{+1}$ so that (x, z) and (y, z) are nonzero. Since $a \neq b$, the coefficient of $\langle z \rangle$ in u is different from either a or b. Therefore, with no loss, we may assume $(x, y) \neq 0$. Since $(e_2 + f_2, e_1 + e_2 + f_2) \neq 0$, there exists $g' \in G$ such that $\langle x \rangle g' = \phi_1$ and $\langle y \rangle g' = \phi_2$. Therefore, we can assume that $u = a\phi_1 + b\phi_2 + \sum_{\delta \in P^{+1} \setminus \{\phi_1, \phi_2\}} a_\delta \delta$.

Let $g \in G$ such that $e_1 g = e_1$ and $(e_2 + f_2)g = -(e_1 + e_2 + f_2)$. Then $\phi_1 g = \phi_2$, $\phi_2 g = \phi_1$, $\phi_3 g = \phi_3$, and therefore

$$u - ug = (a - b)(\phi_1 - \phi_2) + \sum_{\delta \in P^{+1} \setminus \{\phi_1, \phi_2, \phi_3\}} b_\delta \delta \in U \cap S(\mathbb{F}P^{+1}),$$

where $b_\delta \in \mathbb{F}$. Note that $u - ug \in S(\mathbb{F}P^{+1})$. Therefore, if $c_\delta = b_\delta/(a - b)$, we get

$$u_1 := (u - ug)/(a - b) = \phi_1 - \phi_2 + \sum_{\delta \in P^{+1} \setminus \{\phi_1, \phi_2, \phi_3\}} c_\delta \delta \in U \cap S(\mathbb{F}P^{+1}).$$

Hence we have $u_2 := \sum_{k \in K} u_1 k \in U \cap S(\mathbb{F}P^{+1})$. Moreover,

$$u_2 = 3^{2n-3}(\phi_1 - \phi_2) + \sum_{\delta \in \Delta} d_\delta \delta + \sum_{\langle w \rangle \in P_1^{+1}, \langle w \rangle \neq \phi_1} d_{\langle w \rangle}[B_{\langle w \rangle}],$$

where $d_\delta, d_{\langle w \rangle} \in \mathbb{F}$. Therefore $u_3 := \sum_{h \in H} u_2 h \in U \cap S(\mathbb{F}P^{+1})$ with

$$u_3 = (\sum_{\delta \in \Delta} d_\delta)[\Delta] + 3^{2n-2} \sum_{\langle w \rangle \in P_1^{+1}, \langle w \rangle \neq \phi_1} d_{\langle w \rangle}[B_{\langle w \rangle}].$$

It follows that

$$u_4 := 3^{2n-2}u_2 - u_3 = 3^{4n-5}(\phi_1 - \phi_2) + \sum_{\delta \in \Delta} f_\delta \delta \in U \cap S(\mathbb{F}P^{+1}),$$

where $f_\delta = 3^{2n-2}d_\delta - \sum_{\delta \in \Delta} d_\delta$. Hence

$$u_5 := \sum_{k \in K} u_4 k = 3^{6n-8}(\phi_1 - \phi_2) + f[\Delta_1] + f'[\Delta_2] + f''[\Delta_3] \in U \cap S(\mathbb{F}P^{+1}),$$

where $f, f', f'' \in \mathbb{F}$. In particular, $f + f' + f'' = 0$. There are two cases as follow:

Case 1: $f + f' = -f'' = 0$. Then $u_5 = 3^{6n-8}(\phi_1 - \phi_2) + f[\Delta_1] + f'[\Delta_2] = 3^{6n-8}(\phi_1 - \phi_2) + f([\Delta_1] - [\Delta_2]) = 3^{6n-8}(\phi_1 - \phi_2) + f([\Delta(\phi_1)] - [\Delta(\phi_2)]) \in U$ by (3.1). Assume that $f = 0$. Then $u_5 = 3^{6n-8}(\phi_1 - \phi_2) \in U$. It follows that $\phi_1 - \phi_2 \in U$. Hence $\alpha - \beta \in U$ for every $\alpha, \beta \in P^{+1}$ and therefore $U \supseteq S(\mathbb{F}P^{+1})$, which implies that U contains a graph submodule.

It remains to consider $f \neq 0$. Then we have $(3^{6n-8}/f)(\phi_1 - \phi_2) + [\Delta(\phi_1)] - [\Delta(\phi_2)] \in U$. It follows that $(3^{6n-8}/f)(\alpha - \beta) + [\Delta(\alpha)] - [\Delta(\beta)] \in U$ for every $\alpha, \beta \in P^{+1}$ and hence $U \supseteq U'_{3^{6n-8}/f}$, which implies that U contains a graph submodule by Lemma 2.1.

Case 2: $f + f' \neq 0$. Define an element $g \in G$ which sends elements of the basis $\{e_1, f_1, e_2, f_2, ..., e_n, f_n\}$ to those of basis $\{e_1, f_1 - f_2, -e_2 - e_1, -f_2, e_3, f_3, ..., e_n, f_n\}$ respectively. It is easy to check that $\phi_1 g = \phi_2$, $\phi_2 g = \phi_1$, and $\phi_3 g = \phi_3$. Also, $\Delta_1 g = \Delta_2$, $\Delta_2 g = \Delta_1$, and $\Delta_3 g = \Delta_3$. So we have

$$u_6 := u_5 - u_5 g = 2 \cdot 3^{6n-8}(\phi_1 - \phi_2) + (f + f')([\Delta_1] - [\Delta_2]) \in U.$$

As above, we obtain $U \supseteq U'_{2\cdot 3^{6n-8}/(f+f')}$, which again implies that U contains a graph submodule, as desired. □

We will see later for $\ell = 2$ that $c_1 = c_2$ and hence $\mathbb{F}P^\kappa$ has a unique graph submodule. In this case we set $U^\kappa := U^\kappa_{c_1}$ and $U'^\kappa := U'^\kappa_{c_1}$. The following lemma is an important property of the graph submodule and is useful in determining the modulo 2 structure of the permutation module.

Lemma 3.2. *For $\ell = 2$ and $\kappa = \pm 1$,*

(i) $U^\kappa = T(\mathbb{F}P^\kappa) \oplus U'^\kappa$. *In particular, by Proposition 3.1, U'^κ is simple and U^κ is the socle of $\mathbb{F}P^\kappa$;*
(ii) *U^κ and U'^κ are self-dual. Furthermore, U'^κ appears at least twice as a composition factor of $\mathbb{F}P^\kappa$.*

Proof. (i) If $G = O^-_{2n}(3)$ and $\kappa = -1$, let $S \subset P^\kappa$ be the set of points of the form $\langle e_n + v\rangle$ where $v \in \langle e_1, ..., e_{n-1}\rangle$. In all other cases, let $S \subset P^\kappa$ be the set of points of the form $\langle e_n + \kappa f_n + v\rangle$ where $v \in \langle e_1, ..., e_{n-1}\rangle$. We then have

$$\sum_{\alpha\in S} v_{1,\alpha} = \sum_{\alpha\in S}(\alpha + [\Delta(\alpha)]) = \sum_{\alpha\in S}\alpha + 3^{i(\beta)}\sum_{\beta\in P^\kappa\setminus S}\beta = [P^\kappa],$$

where $i(\beta) = n-2$ or $n-1$. Hence $[P^\kappa] \in U^\kappa$ or equivalently $T(\mathbb{F}P^\kappa) \subset U^\kappa$. As $|S| = 3^{n-1} \neq 0$ in $\mathbb{F}$, $T(\mathbb{F}P^\kappa) \nsubseteq U'^\kappa$. Since U'^κ is a submodule of U^κ of codimension at most 1, $U^\kappa = T(\mathbb{F}P^\kappa) \oplus U'^\kappa$. This and Proposition 3.1 show that U'^κ is simple and U^κ is the socle of $\mathbb{F}P^\kappa$.

(ii) Recall that the submodule U^κ consists of $\mathbb{F}$-linear combinations of $v_{c,\alpha}$ where $\alpha \in P^\kappa$ and $c = c_1 = c_2$. Define a bilinear form $[\cdot,\cdot]$ on U^κ by $[v_{c,\alpha}, v_{c,\beta}] = \langle v_{c,\alpha}, \beta\rangle$. It is clear that this form is symmetric, non-singular, and G-invariant. Hence, U^κ is self-dual. It then follows that U'^κ is also self-dual by (i). Therefore $\mathbb{F}P^\kappa/U'^{\kappa\perp} \cong \mathrm{Hom}_{\mathbb{F}}(U'^{\kappa\perp}, \mathbb{F}) \cong U'^{\kappa\perp}$. Combining this with the inclusion $U'^\kappa \subseteq U'^{\kappa\perp}$ (by Lemma 2.1), we have that U'^κ appears at least twice as a composition factor of $\mathbb{F}P^\kappa$. □

4. Relations between $\mathbb{F}P^0$, $\mathbb{F}P^{+1}$ and $\mathbb{F}P^{-1}$

In this section, we establish some relations between the structures of $\mathbb{F}P^0$, $\mathbb{F}P^{+1}$ and $\mathbb{F}P^{-1}$. This helps us to understand $\mathbb{F}P^{+1}$ and $\mathbb{F}P^{-1}$ from the known results of $\mathbb{F}P^0$ in [10, 11].

For $i, j \in \{0, +1, -1\}$, define

$$\begin{array}{rccc} Q_{i,j}: & \mathbb{F}P^i & \to & \mathbb{F}P^j \\ & \alpha & \mapsto & [\{\beta \in P^j \mid \beta \perp \alpha\}]. \end{array} \tag{4.1}$$

It is obvious that $Q_{i,j}$ is an $\mathbb{F}G$-homomorphism. Also, $Q_{i,j}(S(\mathbb{F}P^i)) \subseteq S(\mathbb{F}P^j)$. The following lemma is easy to check.

Lemma 4.1. $\mathrm{Im}(Q_{i,j})$ *and* $\mathrm{Im}(Q_{i,j}|_{S(\mathbb{F}P^i)})$ *are nonzero and different from* $T(\mathbb{F}P^j)$. *Also,* $\mathbb{F}P^i/\mathrm{Ker}(Q_{i,j}) \cong \mathrm{Im}(Q_{i,j})$ *and* $S(\mathbb{F}P^i)/\mathrm{Ker}(Q_{i,j}|_{S(\mathbb{F}P^i)}) \cong \mathrm{Im}(Q_{i,j}|_{S(\mathbb{F}P^i)})$. □

Let ρ^0, ρ^{+1}, and ρ^{-1} be the complex permutation characters of G afforded by permutation modules $\mathbb{C}P^0$, $\mathbb{C}P^{+1}$, and $\mathbb{C}P^{-1}$, respectively. Recall that G acts with rank 3 on each P^i with $i = 0, +1, -1$. Hence ρ^i has 3 constituents, all of multiplicity 1 and exactly one of them is the trivial character.

Lemma 4.2. *For $i,j \in \{0,+1,-1\}$, ρ^i and ρ^j have a common nontrivial constituent.*

Proof. This is an immediate consequence of Lemma 4.1. $\square$

5. The orthogonal groups $O^+_{2n}(3)$

In this section, we always assume $G = O^+_{2n}(3)$. For $\kappa = \pm 1$, we have $P^\kappa = \{\langle \sum_{i=1}^n (a_i e_i + b_i f_i)\rangle \mid a_i, b_i \in \mathbb{F}_3, \sum_{i=1}^n a_i b_i = \kappa\}$ and $|P^\kappa| = 3^{n-1}(3^n-1)/2$. The parameters of the action of G on P^κ are:

$$a = \frac{3^{n-1}(3^{n-1}-1)}{2}, b = 3^{2n-2}-1, r = \frac{3^{n-2}(3^{n-1}+1)}{2}, s = \frac{3^{n-1}(3^{n-2}-1)}{2}.$$

The equation (2.1) now has two roots 3^{n-2} and -3^{n-1}. Therefore, $\mathbb{F}P^\kappa$ has graph submodules $U'^{\kappa}_{3^{n-2}}$ and $U'^{\kappa}_{-3^{n-1}}$.

We note that if $\ell = \mathrm{char}(\mathbb{F}) \neq 2,3$ then $3^{n-2} \neq -3^{n-1}$ and therefore two graph submodules are different.

Lemma 5.1. *If $\ell = \mathrm{char}(\mathbb{F}) \neq 2,3$, then*

$$\dim U'^{\kappa}_{3^{n-2}} = \frac{(3^n-1)(3^{n-1}-1)}{8}, \dim U'^{\kappa}_{-3^{n-1}} = \frac{3^{2n}-9}{8} \text{ and } U'^{+1}_{-3^{n-1}} \cong U'^{-1}_{-3^{n-1}}.$$

Proof. The dimensions of $U'^{\kappa}_{3^{n-2}}$ and $U'^{\kappa}_{-3^{n-1}}$ follow from (2.3). Using results about the permutation module for G acting on P^0 in [10], we see that $\mathbb{F}P^0$ has two graph submodules of dimensions $(3^n-1)(3^{n-1}+3)/8$ and $(3^{2n}-9)/8$, which we temporarily denote by U'_c and U'_d, respectively. By Theorem 2.1 of [10], U'_c and U'_d are minimal in the same sense as in Proposition 3.1. Applying Lemma 4.1 and Proposition 3.1, we deduce that $U'^{+1}_{-3^{n-1}} \cong U'_d$ and $U'^{-1}_{-3^{n-1}} \cong U'_d$ and the lemma follows. $\square$

Proposition 5.2. *Theorem 1.2 holds when $G = O^+_{2n}(3), n \geq 3$ and $\ell \neq 2,3$.*

Proof. First, consider $\ell \nmid (3^n-1)$. Then $[P^\kappa] \notin S(\mathbb{F}P^\kappa)$ and we have

$$\begin{aligned}
\mathbb{F}P &\cong \mathbb{F}P^{+1} \oplus \mathbb{F}P^{-1} = T(\mathbb{F}P^{+1}) \oplus S(\mathbb{F}P^{+1}) \oplus T(\mathbb{F}P^{-1}) \oplus S(\mathbb{F}P^{-1}) \\
&\cong^{(2.2)} T(\mathbb{F}P^{+1}) \oplus U'^{+1}_{3^{n-2}} \oplus U'^{+1}_{-3^{n-1}} \oplus T(\mathbb{F}P^{-1}) \oplus U'^{-1}_{3^{n-2}} \oplus U'^{-1}_{-3^{n-1}} \\
&\cong 2\mathbb{F} \oplus X \oplus Y \oplus 2Z,
\end{aligned}$$

where $X := U'^{+1}_{3^{n-2}}$, $Y := U'^{-1}_{3^{n-2}}$, and $Z := U'^{+1}_{-3^{n-1}} \cong U'^{-1}_{-3^{n-1}}$ (Lemma 5.1). The modules X, Y, and Z are simple by Proposition 3.1.

Next we consider $\ell \mid (3^n-1)$. It is easy to see that $T(\mathbb{F}P^\kappa)$ is contained in $U'^{\kappa}_{-3^{n-1}}$ but not in $U'^{\kappa}_{3^{n-2}}$. We then have $\mathbb{F}P^\kappa = U'^{\kappa}_{3^{n-2}} \oplus U^{\kappa}_{-3^{n-1}}$, where $U'^{\kappa}_{3^{n-2}}$ is simple and $U^{\kappa}_{-3^{n-1}}$ is uniserial (by Proposition 3.1) with composition series

$$0 \subset T(\mathbb{F}P^\kappa) \subset U'^{\kappa}_{-3^{n-1}} \subset U^{\kappa}_{-3^{n-1}}.$$

Putting $X := U'^{+1}_{3^{n-2}}$, $Y := U'^{-1}_{3^{n-2}}$, and $Z := U'^{\kappa}_{-3^{n-1}}/T(\mathbb{F}P^\kappa)$, we get

$$\mathbb{F}P \cong X \oplus Y \oplus 2(\mathbb{F} - Z - \mathbb{F}),$$

as described in Table 1. $\square$

For the rest of this section, we consider the case $\ell = \mathrm{char}(\mathbb{F}) = 2$, where the two graph submodules are the same. We write $U^\kappa := U^\kappa_1 = U^\kappa_{3^{n-2}} = U^\kappa_{-3^{n-1}}$ and $U'^{\kappa} := U'^{\kappa}_1 = U'^{\kappa}_{3^{n-2}} = U'^{\kappa}_{-3^{n-1}}$.

Proposition 5.3. *Theorem 1.2 holds when $G = O^+_{2n}(3), n \geq 3$ and $\ell = 2$.*

Proof. Recall that, for $i = 0, \pm 1$, ρ^i is the permutation character of G afforded by $\mathbb{C}P^i$. Since G acts with rank 3 on each P^i, we have $\rho^i = 1 + \varphi^i + \psi^i$, where $\varphi^i, \psi^i \in \mathrm{Irr}(G)$ and the $\psi^i s$ have the same degree $(3^{2n} - 9)/3$ by Lemma 4.2 and the proof of Lemma 5.1. We set $\psi := \psi^0 = \psi^{+1} = \psi^{-1}$. Note that $\varphi^{+1}(1) = \varphi^{-1}(1) = (3^n - 1)(3^{n-1} - 1)/8$ from Lemma 5.1. Since the smallest degree of a nonlinear irreducible 2-Brauer characters of G is $(3^n - 1)(3^{n-1} - 1)/8$ (see Theorem 1 of [4]), $\overline{\varphi^{+1}}$ and $\overline{\varphi^{-1}}$ must be irreducible.

First we give the proof for n odd. From the study of $\overline{\rho^0}$ in Corollary 6.5 of [11], we have $\overline{\psi} = \beta(W) + \beta(X) + \beta(Y)$, where W, X, and Y are simple G-modules of dimensions $(3^n - 1)(3^{n-1} + 3)/8 - 1$, $(3^n - 1)(3^{n-1} - 1)/8$, and $(3^n - 1)(3^{n-1} - 1)/8$, respectively. Furthermore, X and Y are not isomorphic. Now using Proposition 3.2(ii) together with the conclusion of the previous paragraph, we deduce that U'^{κ} is isomorphic to either X or Y and $\overline{\varphi^\kappa} = \beta(U'^{\kappa})$.

Let $U'^{+1} \cong X$. We wish to show that $U'^{-1} \cong Y$. Assuming the contrary, we then have $U'^{+1} \cong U'^{-1} \cong X$ and therefore $\overline{\varphi^{+1}} = \overline{\varphi^{-1}}$. Now we temporarily add subscript n to the standard notations. Then $\overline{\varphi^{+1}_n} = \overline{\varphi^{-1}_n}$ and hence $\overline{\rho^{+1}_n} = \overline{\rho^{-1}_n}$. Since $\mathbb{F}P^\kappa_n \cong 5\mathbb{F}P^\kappa_{n-1} \oplus 2\mathbb{F}P^{-\kappa}_{n-1} \oplus 2\mathbb{F}P^0_{n-1} \oplus 2\mathbb{F}$ as $\mathbb{F}G_{n-1}$-modules, it follows that $\overline{\rho^{+1}_{n-1}} = \overline{\rho^{-1}_{n-1}}$. By downward induction, we get $\overline{\varphi^{+1}_3} = \overline{\varphi^{-1}_3}$, which is a contradiction by checking the complex and 2-Brauer character tables of $O^+_6(3)$ (see [1, 6]).

We have shown that $U'^{+1} \cong X$ and $U'^{-1} \cong Y$. Notice that, for $\kappa = \pm 1$, $|P^\kappa| \neq 0$ (in $\mathbb{F}$) and hence $\mathbb{F}P^\kappa = T(\mathbb{F}P^\kappa) \oplus S(\mathbb{F}P^\kappa)$ and the composition factors of $S(\mathbb{F}P^\kappa)$ are X (twice), Y, and W. By Proposition 3.1 and the self-duality of $S(\mathbb{F}P^\kappa)$, the socle series of $S(\mathbb{F}P^{+1})$ and $S(\mathbb{F}P^{-1})$ are $X - (Y \oplus W) - X$ and $Y - (X \oplus W) - Y$, respectively, as described in Table 1.

Now we consider n even. In this case, $\overline{\psi} = 1 + \beta(W) + \beta(X) + \beta(Y)$, where W, X, and Y are simple G-modules of dimensions $(3^n - 1)(3^{n-1} + 3)/8 - 2$, $(3^n - 1)(3^{n-1} - 1)/8$, and $(3^n - 1)(3^{n-1} - 1)/8$, respectively. Repeating the above arguments, we see that $U'^{+1} \cong X$ and $U'^{-1} \cong Y$.

Notice that $T(\mathbb{F}P^{+1}) \subset S(\mathbb{F}P^{+1})$, $T(\mathbb{F}P^{+1})^\perp = S(\mathbb{F}P^{+1})$, and $S(\mathbb{F}P^{+1})/T(\mathbb{F}P^{+1})$ is self-dual and has composition factors: X (twice), Y, and W. Again, Proposition 3.1 gives the socle series of $S(\mathbb{F}P^{+1})/T(\mathbb{F}P^{+1})$: $X - (W \oplus Y) - X$. The submodule structure of $\mathbb{F}P^{+1}$ will be completely determined if we know that of $U'^{+1\perp}/U'^{+1}$. Note that $U'^{+1\perp}/U'^{+1}$ has composition factors: $\mathbb{F}$ (twice), W, and Y. Using Lemma 4.1 and inspecting the structure of $\mathbb{F}P^0$, we see that Y must be a submodule of $U'^{+1\perp}/U'^{+1}$ but W is not. Therefore, the structure of $U'^{+1\perp}/U'^{+1}$ is $Y \oplus (\mathbb{F} - W - \mathbb{F})$.

Similarly, the structure of $S(\mathbb{F}P^{-1})/T(\mathbb{F}P^{-1})$ is $Y - (W \oplus X) - Y$ and that of $U'^{-1\perp}/U'^{-1}$ is $X \oplus (\mathbb{F} - W - \mathbb{F})$. Now $\mathbb{F}P$ is determined completely as described in Table 1. □

Propositions 5.2 and 5.3 complete the proof of Theorem 1.2 for the type "+" orthogonal groups in even dimension.

6. The orthogonal groups $O^-_{2n}(3)$

In this section, we always assume $G = O^-_{2n}(3)$. For $\kappa = \pm 1$, we have $P^\kappa = \{\langle \sum_{i=1}^n (a_i e_i + b_i f_i) \rangle \mid \sum_{i=1}^{n-1} a_i b_i - a_n^2 - b_n^2 = \kappa, a_i, b_i \in \mathbb{F}_3\}$ and $|P^\kappa| = 3^{n-1}(3^n +$

1)/2. The parameters of the action of G on P^κ are:

$$a = \frac{3^{n-1}(3^{n-1}+1)}{2}, b = 3^{2n-2}-1, r = \frac{3^{n-2}(3^{n-1}-1)}{2}, s = \frac{3^{n-1}(3^{n-2}+1)}{2}.$$

The equation (2.1) now has two roots -3^{n-2} and 3^{n-1}. Therefore, $\mathbb{F}P^\kappa$ has graph submodules $U'^{\kappa}_{-3^{n-2}}$ and $U'^{\kappa}_{3^{n-1}}$.

Lemma 6.1. *If $\ell = \mathrm{char}(\mathbb{F}) \neq 2, 3$, then*

$$\dim U'^{\kappa}_{-3^{n-2}} = \frac{(3^n+1)(3^{n-1}+1)}{8}, \dim U'^{\kappa}_{3^{n-1}} = \frac{3^{2n}-9}{8} \text{ and } U'^{1}_{3^{n-1}} \cong U'^{-1}_{3^{n-1}}.$$

Proof. As in the proof of Lemma 5.1. □

Proposition 6.2. *Theorem 1.2 holds when $G = O^-_{2n}(3), n \geq 3$ and $\ell \neq 2, 3$.*

Proof. First we consider $\ell \nmid (3^n+1)$. As in §3, $\mathbb{F}P \cong 2\mathbb{F} \oplus X \oplus Y \oplus 2Z$, where $X := U'^{+1}_{-3^{n-2}}$, $Y := U'^{-1}_{-3^{n-2}}$, $Z := U'^{+1}_{3^{n-1}} \cong U'^{-1}_{3^{n-1}}$ and X, Y, Z are simple by Proposition 3.1.

Second we consider $\ell \mid (3^n+1)$. We have $\mathbb{F}P^\kappa = U'^{\kappa}_{-3^{n-2}} \oplus U^{\kappa}_{3^{n-1}}$, where $U'^{\kappa}_{-3^{n-2}}$ is simple and $U^{\kappa}_{3^{n-1}}$ is uniserial with composition series $0 \subset T(\mathbb{F}P^\kappa) \subset U'^{\kappa}_{3^{n-1}} \subset U^{\kappa}_{3^{n-1}}$. Putting $X := U'^{+1}_{-3^{n-2}}$, $Y := U'^{-1}_{-3^{n-2}}$, and $Z := U'^{\kappa}_{3^{n-1}}/T(\mathbb{F}P^\kappa)$, we obtain $\mathbb{F}P \cong X \oplus Y \oplus 2(\mathbb{F} - Z - \mathbb{F})$, as stated. □

For the rest of this section, we consider the case $\ell = 2$. As in §5, we have $\rho^i = 1 + \varphi^i + \psi$ for $i = 0, \pm 1$, where $\varphi^i, \psi \in \mathrm{Irr}(G)$, $\psi(1) = (3^{2n}-9)/3$, and $\varphi^{+1}(1) = \varphi^{-1}(1) = (3^n+1)(3^{n-1}+1)/8$. From Corollary 8.10 of [11], $\overline{\psi} = 1 + \beta(W) + \beta(X) + \beta(Y)$ when n is even and $\overline{\psi} = 1 + 1 + \beta(W) + \beta(X) + \beta(Y)$ when n is odd. Here, X, Y, and W are simple $\mathbb{F}G$-modules of dimensions $(3^n+1)(3^{n-1}+1)/8 - 1$, $(3^n+1)(3^{n-1}+1)/8 - 1$, and $(3^n+1)(3^{n-1}-3)/8 - 1 + \delta_{2,n}$, respectively. Moreover, $X \ncong Y$ and W has smallest dimension among simple $\mathbb{F}G$-modules of dimensions greater than 1.

Lemma 6.3. *With the above notation,*

(i) *For $\kappa = \pm 1$, U'^{κ} is isomorphic to either X or Y and $\overline{\varphi^\kappa} = 1 + \beta(U'^{\kappa})$.*
(ii) *If we let $U'^{+1} \cong X$, then $U'^{-1} \cong Y$.*

Proof. (i) We only give here the proof for n even and $\kappa = 1$. Other cases are similar.

Assume the contrary: U'^{+1} is not isomorphic to both X and Y. Lemma 3.2 then implies that $U'^{+1} \cong W$ and $\beta(W)$ is a constituent of $\overline{\varphi^{+1}}$. Hence all other constituents of $\overline{\varphi^{+1}}$ have degrees at most $\varphi^{+1}(1) - \dim W = (3^n+1)/2$, which imply that they are linear since $(3^n+1)(3^{n-1}-3)/8 - 1 + \delta_{2,n}$ is the smallest dimension of nonlinear irreducible 2-Brauer characters of G.

Let P_1^0, P_1^{+1}, and P_1^{-1} be the sets of singular points, plus points, and minus points, respectively, in $V_1 := \langle e_1, f_1, ..., e_{n-1}, f_{n-1}\rangle$. Note that V_1 equipped with Q is an orthogonal space of type $+$. Let $G_1 := O^+_{2n-2}(3) \leq G$. Then we obtain an $\mathbb{F}G_1$-isomorphism:

$$\mathbb{F}P^{+1} \cong 2\mathbb{F} \oplus \mathbb{F}P_1^{+1} \oplus 4\mathbb{F}P_1^{-1} \oplus 4\mathbb{F}P_1^0. \tag{6.1}$$

Inspecting the structures of $\mathbb{F}P_1^0$ in Figure 5 of [11] and of $\mathbb{F}P_1^{+1}$ as well as $\mathbb{F}P_1^{-1}$ in Table 1, we see that $\mathbb{F}P^{+1}$, when considered as $\mathbb{F}G_1$-module, has 15 composition factors (counting multiplicities) of dimension 1 (actually all of them are isomorphic to $\mathbb{F}$). This contradicts the conclusion of the previous paragraph.

We have shown that U'^{+1} is isomorphic to either X or Y. Notice from Lemma 3.2 that U'^{+1} appears at least twice as a composition factor of $\mathbb{F}P^{+1}$. The second statement of (i) now follows by comparing the degrees and using (6.1).

(ii) Assuming the contrary that $U'^{-1} \ncong Y$, then $U'^{+1} \cong U'^{-1} \cong X$. It follows that $\overline{\varphi^{+1}} = \overline{\varphi^{-1}}$ and hence $\overline{\rho^{+1}} = \overline{\rho^{-1}}$. Therefore, the isomorphism (6.1) together with $\mathbb{F}P^{-1} \cong 2\mathbb{F} \oplus \mathbb{F}P_1^{-1} \oplus 4\mathbb{F}P_1^{+1} \oplus 4\mathbb{F}P_1^{0}$ imply that the modulo 2 permutation characters afforded by $\mathbb{F}P_1^{+1}$ and $\mathbb{F}P_1^{-1}$ are the same. This is a contradiction as seen in the proof of Proposition 5.3. $\square$

Since we will use an induction argument to determine the structure of $\mathbb{F}P$, we temporarily add the subscript n to our standard notations. Notice that $G_2 = O_4^-(3)$ acts with rank 3 on both P_2^{+1} and P_2^{-1}. Everything we have proved for $n \geq 3$ works exactly the same in the case $n = 2$ except that $W_2 = 0$.

Lemma 6.4. *The structures of $\mathbb{F}P_2^{+1}$ and $\mathbb{F}P_2^{-1}$ are given as follows:*

$$\mathbb{F}P_2^{+1}: \quad \begin{array}{ccc} & & X_2 \\ & & | \\ & & \mathbb{F} \\ & & | \\ \mathbb{F} & \oplus & Y_2 \\ & & | \\ & & \mathbb{F} \\ & & | \\ & & X_2 \end{array} \qquad\qquad \mathbb{F}P_2^{-1}: \quad \begin{array}{ccc} & & Y_2 \\ & & | \\ & & \mathbb{F} \\ & & | \\ \mathbb{F} & \oplus & X_2 \\ & & | \\ & & \mathbb{F} \\ & & | \\ & & Y_2 \end{array}$$

Proof. By Lemma 3.2 the module U'^{κ} is simple and self-dual. The module $\mathbb{F}P_2^{\kappa}$ has dimension $15 = 3^{2-1}(3^2+1)/2$, and by Lemma 6.3 its submodule U'^{κ} has dimension $4 = (3^2+1)(3^{2-1}+1)/8 - 1$. Therefore by Lemmas 3.2 and 4.1 the module U'^{κ} appears twice as a composition factor of $\mathbb{F}P_2^{\kappa}$ and once as a composition factor of $\mathbb{F}P_2^{-\kappa}$.

As 15 is odd, $\mathbb{F}P_2^{\kappa} = T(\mathbb{F}P_2^{\kappa}) \oplus S(\mathbb{F}P_2^{\kappa})$. Here U'^{κ} is the unique minimal submodule of the dimension 14 module $S(\mathbb{F}P_2^{\kappa})$, and $M = (U'^{\kappa})^{\perp} \cap S(\mathbb{F}P_2^{\kappa})$ its unique maximal submodule. The submodule M has dimension 10 and quotient $S(\mathbb{F}P_2^{\kappa})/M$ isomorphic to U'^{κ}. The quotient $Q = M/U'^{\kappa}$ is thus self-dual of dimension 6, possessing two trivial composition factors in addition to the factor $U'^{-\kappa}$. There are only three possibilities:

$$Q = \mathbb{F} - U'^{-\kappa} - \mathbb{F} \quad \text{or} \quad Q = \mathbb{F} \oplus U'^{-\kappa} \oplus \mathbb{F} \quad \text{or} \quad Q = (\mathbb{F} - \mathbb{F}) \oplus U'^{-\kappa}.$$

The first gives the lemma, so we must eliminate the second and third.

The usual dot product on the natural $\mathbb{F}_3$-permutation module for $Sym(6)$ is an invariant bilinear form with radical spanned by the vector of 1's. The action of $Sym(6)$ on the unique nontrivial composition factor thus gives an injection of $Sym(6)$ into $O_4^-(3)$. More specifically, $O_4^-(3) \cong 2 \times Sym(6)$. With this in mind, we can choose notation so that the module $\mathbb{F}P_2^{\kappa}$ is the usual permutation module $M^{(4,2)}$ for $Sym(6)$ acting on the 15 unordered pairs from a set of size six.

As the representation theory of symmetric groups is highly developed (see, for instance, the elegant treatment in James's book [5]), the lemma is presumably well known. Indeed the needed calculations can be done easily, following Example 5.2 of [5]. We give a short proof, using only some of the elementary theory.

By an easy calculation and [5, Cor. 8.5] the Specht submodule $S = S^{(4,2)}$ of $\mathbb{F}P_2^{\kappa} = M^{(4,2)}$ has dimension 9, so it must have codimension 1 in M with S/U'^{κ} of dimension 5 in Q. Assume (for a contradiction) that $Q = \mathbb{F} \oplus U'^{-\kappa} \oplus \mathbb{F}$ or $Q = (\mathbb{F} - \mathbb{F}) \oplus U'^{-\kappa}$. Then S/U'^{κ} must have shape $\mathbb{F} \oplus U'^{-\kappa}$. In particular, the Specht module S has two maximal submodules, one of codimension 1 and the other of codimension 4. But by [5, Theorem 4.9], Specht modules have unique maximal submodules. This contradiction proves the lemma. □

Lemma 6.5. *For any $n \geq 2$, $\mathbb{F}P_n^{+1}$ does not have any submodule of structure $X_n - Y_n$. Similarly, $\mathbb{F}P_n^{-1}$ does not have any submodule of structure $Y_n - X_n$.*

Proof. Case $n = 2$ is clear from Lemma 6.4. So we assume that $n \geq 3$. Let Q be the parabolic subgroup of G_n fixing $\langle e_1 \rangle$. Then $Q = O.L$ where $O = O_3(Q)$, the maximal normal 3-subgroup of Q and $L \cong G_{n-1} \times \mathbb{Z}_2$, a Levi subgroup of G_n. Set $V_1 := \langle e_2, ..., e_n, f_2, ..., f_n \rangle$. Let P_1 be the set of plus points in V of the form $\langle xe_1 + u \rangle$ and P_2 the set of plus points in V of the form $\langle f_1 + xe_1 + u \rangle$ with $x \in \mathbb{F}_3$ and $u \in V_1$. Then P_n^{+1} is the disjoint union of P_1 and P_2.

It is clear that $|P_2| = |O| = 3^{2n-2}$ and the stabilizer of $\langle f_1 + e_1 \rangle$ in O is trivial. Therefore O acts transitively on P_2. This O-orbit is fixed under the action of G_{n-1} on the set of O-orbits on P^{+1}. For any plus point $\langle u \rangle$ in V_1, the O-orbit of $\langle u \rangle$ consists of three points: $\langle u \rangle$, $\langle u + e_1 \rangle$, and $\langle u - e_1 \rangle$. Hence the action of G_{n-1} on the set of O-orbits in P_1 is equivalent to that on the set of plus points in V_1. We have proved the following $\mathbb{C}G_{n-1}$-isomorphism:

$$C_{\mathbb{C}P_n^{+1}}(O) \cong \mathbb{C}P_{n-1}^{+1} \oplus \mathbb{C}, \tag{6.2}$$

where $C_{\mathbb{C}P_n^{+1}}(O)$ is the centralizer of O in $\mathbb{C}P_n^{+1}$. If χ is the character of G_n afforded by a module M, we denote by $C_\chi(O)$ the character of G_{n-1} afforded by $C_M(O)$. The isomorphism (6.2) then implies

$$C_{\varphi_n^{+1}}(O) + C_{\psi_n}(O) = \varphi_{n-1}^{+1} + \psi_{n-1} + 1_{G_{n-1}}. \tag{6.3}$$

From Frobenius reciprocity,

$$(\psi_n|_Q, 1_Q)_Q = (\psi_n, 1_Q^{G_n})_{G_n} = (\psi_n, \rho_n^0)_{G_n} > 0.$$

Hence, 1_Q is a constituent of $\psi_n|_Q$. It follows that $1_{G_{n-1}}$ is a constituent of $C_{\psi_n}(O)$.

Now we will show that ψ_{n-1} is also a constituent of $C_{\psi_n}(O)$. Assume not. Then ψ_{n-1} would be a constituent of $C_{\varphi_n^{+1}}(O)$ by (6.3). It follows that $\overline{\psi_{n-1}}$ is contained in $C_{\overline{\varphi_n^{+1}}}(O)$. As $\overline{\varphi_n^{+1}}$ is always contained in $\overline{\psi_n}$, we find that $2\overline{\psi_{n-1}}$ is contained in $C_{\overline{\varphi_n^{+1}}}(O) + C_{\overline{\psi_n}}(O)$. The formula (6.3) then implies that $\overline{\psi_{n-1}}$ is contained in $\overline{\varphi_{n-1}^{+1}} + 1$, a contradiction.

We have shown that both $1_{G_{n-1}}$ and ψ_{n-1} are constituents of $C_{\psi_n}(O)$. Therefore $C_{\varphi_n^{+1}}(O) = \varphi_{n-1}^{+1}$. It follows by Lemma 6.3 that $C_{X_n}(O) \cong X_{n-1}$. Similarly, $C_{Y_n}(O) \cong Y_{n-1}$.

Now we prove the lemma by induction. Assuming that the lemma is true for $n-1$ and supposing the contrary that $\mathbb{F}P_n^{+1}$ has a submodule of structure $X_n - Y_n$. Proposition 3.1 and the previous paragraph then show that $C_{(X_n - Y_n)}(O) \cong X_{n-1} - Y_{n-1}$ is a submodule of $C_{\mathbb{F}P_n^{+1}}(O) \cong \mathbb{F}P_{n-1}^{+1} \oplus \mathbb{F}$. We deduce that $X_{n-1} - Y_{n-1}$ is a submodule of $\mathbb{F}P_{n-1}^{+1}$, contradicting the induction hypothesis. □

Proposition 6.6. *Theorem 1.2 holds when $G = O_{2n}^-(3), n \geq 3$ and $\ell = 2$.*

Proof. Using Proposition 3.1 and Lemma 3.2, we see that the structure of $\mathbb{F}P^\kappa$ will be determined if we know that of ${U'}^{\kappa\perp}/{U'}^{\kappa}$. We study ${U'}^{+1\perp}/{U'}^{+1}$ first.

Consider the case n even. Then, for $\kappa = \pm 1$, $|P^\kappa| \neq 0$ and therefore $\mathbb{F}P^\kappa = T(\mathbb{F}P^\kappa) \oplus S(\mathbb{F}P^\kappa)$.

From the constituents of $\overline{\varphi^{+1}}$ and $\overline{\psi}$, the composition factors of ${U'}^{+1\perp}/{U'}^{+1}$ are: $\mathbb{F}$ (3 times), Y, and W. Since $\mathbb{F}P^\kappa = T(\mathbb{F}P^\kappa) \oplus S(\mathbb{F}P^\kappa)$, we know that ${U'}^{+1\perp}/{U'}^{+1}$ has a direct summand $\mathbb{F}$. Furthermore, $X \cong {U'}^{+1}$ is the socle of $S(\mathbb{F}P^{+1})$ by Proposition 3.1. Lemma 4.1 now implies that $\operatorname{Im}(Q_{0,+1}|_{S(\mathbb{F}P^0)}) \cong S(\mathbb{F}P^0)/\operatorname{Ker}(Q_{0,+1}|_{S(\mathbb{F}P^0)})$ also has socle X. Inspecting the structure of $\mathbb{F}P^0$ given in Figure 8 of [11], we see that the only quotient of $S(\mathbb{F}P^0)$ having X as the socle is $X - W$. This means that $S(\mathbb{F}P^{+1})$ has submodule of structure $X - W$ and therefore W is a submodule of ${U'}^{+1\perp}/{U'}^{+1}$. By self-duality, W must be a direct summand of ${U'}^{+1\perp}/{U'}^{+1}$.

By Lemma 6.5, Y is not a submodule of ${U'}^{+1\perp}/{U'}^{+1}$. Combining this with the previous paragraph, we conclude that the structure of ${U'}^{+1\perp}/{U'}^{+1}$ is $\mathbb{F} \oplus W \oplus (\mathbb{F} - Y - \mathbb{F})$.

Now we consider the case n odd. By Lemma 4.1, $\operatorname{Im}(Q_{0,+1})$ is nonzero and different from $T(\mathbb{F}P^{+1})$. Hence it has the socle either X or $\mathbb{F} \oplus X$ by Proposition 3.1. Notice that $\operatorname{Im}(Q_{0,+1}) \cong \mathbb{F}P^0/\operatorname{Ker}(Q_{0,+1})$. Inspecting the structure of $\mathbb{F}P^0$ again, we learn that the structure of $\operatorname{Im}(Q_{0,+1})$ must be $X - \mathbb{F} - W - \mathbb{F}$. It follows that ${U'}^{+1\perp}/{U'}^{+1}$ has a submodule of structure $\mathbb{F} - W - \mathbb{F}$. Recall that ${U'}^{+1\perp}/{U'}^{+1}$ has composition factors: $\mathbb{F}$ (4 times), Y, and W and Y is not its submodule by Lemma 6.5. By self-duality, the structure of ${U'}^{+1\perp}/{U'}^{+1}$ is $(\mathbb{F} - Y - \mathbb{F}) \oplus (\mathbb{F} - W - \mathbb{F})$.

Arguing similarly for $\kappa = -1$, the structure of ${U'}^{-1\perp}/{U'}^{-1}$ is $\mathbb{F} \oplus W \oplus (\mathbb{F} - X - \mathbb{F})$ when n even and $(\mathbb{F} - X - \mathbb{F}) \oplus (\mathbb{F} - W - \mathbb{F})$ when n odd. $\square$

Propositions 6.2 and 6.6 complete the proof of Theorem 1.2 for the type " $-$ " orthogonal groups in even dimension.

7. The orthogonal groups $O_{2n+1}(3)$

In this section, we assume $G = O_{2n+1}(3)$. Recall that we fix a basis $\mathcal{B} = \{e_1, ..., e_n, f_1, ..., f_n, g\}$ of V where $(e_i, f_j) = \delta_{ij}$, $(e_i, e_j) = (f_i, f_j) = (e_i, g) = (f_i, g) = 0$, and $(g, g) = 1$ for $i, j = 1, ..., n$. For $\kappa = \pm 1$, we have $P^\kappa = \{\langle cg + \sum_{i=1}^n (a_i e_i + b_i f_i)\rangle \mid a_i, b_i \in \mathbb{F}_3, \sum_{i=1}^n a_i b_i - c^2 = \kappa\}$ and $|P^\kappa| = 3^n(3^n - \kappa)/2$. The parameters of the action of G on P^κ are:

$$a = \frac{3^{n-1}(3^n + \kappa)}{2}, b = (3^n + \kappa)(3^{n-1} - \kappa), r = s = \frac{3^{n-1}(3^{n-1} + \kappa)}{2}.$$

Equation (2.1) now has two roots -3^{n-1} and 3^{n-1}. Therefore, for $\kappa = \pm 1$, $\mathbb{F}P^\kappa$ has graph submodules ${U'}^{\kappa}_{-3^{n-1}}$ and ${U'}^{\kappa}_{3^{n-1}}$.

Lemma 7.1. *If $\ell = \operatorname{char}(\mathbb{F}) \neq 2, 3$, then*

$${U'}^{+1}_{-3^{n-1}} \cong {U'}^{-1}_{3^{n-1}}, \dim {U'}^{+1}_{-3^{n-1}} = \dim {U'}^{-1}_{3^{n-1}} = \frac{3^{2n} - 1}{4},$$

$$\dim {U'}^{-1}_{-3^{n-1}} = \frac{(3^n - 1)(3^n + 3)}{4}, \ \textit{and} \ \dim {U'}^{+1}_{3^{n-1}} = \frac{(3^n + 1)(3^n - 3)}{4}.$$

Proof. This is similar to the proof of Lemma 5.1. We remark in this case that ${U'}^{-1}_{-3^{n-1}}$ and ${U'}^{+1}_{3^{n-1}}$ are isomorphic to graph submodules of $\mathbb{F}P^0$. $\square$

Proposition 7.2. *Theorem 1.2 holds when $G = O_{2n+1}(3), n \geq 3$ and $\ell \neq 2, 3$.*

Proof. **Case 1**: $\ell \nmid (3^n - 1), \ell \nmid (3^n + 1)$. In this case, $\mathbb{F}P \cong 2\mathbb{F} \oplus X \oplus Y \oplus 2Z$, where $X := U^{'+1}_{3^{n-1}}$, $Y := U^{'-1}_{-3^{n-1}}$, and $Z := U^{'+1}_{-3^{n-1}} \cong U^{'-1}_{3^{n-1}}$. By Proposition 3.1, X, Y, and Z are simple.

Case 2: $\ell \mid (3^n - 1)$. We have

$$\mathbb{F}P^{-1} = T(\mathbb{F}P^{-1}) \oplus U^{'-1}_{-3^{n-1}} \oplus U^{'-1}_{3^{n-1}} \cong \mathbb{F} \oplus Y \oplus Z,$$

where $Y := U^{'-1}_{-3^{n-1}}$, $Z := U^{'1}_{-3^{n-1}} \cong U^{'-1}_{3^{n-1}}$ and

$$\mathbb{F}P^{+1} = U^{+1}_{3^{n-1}} \oplus U^{'+1}_{-3^{n-1}} \cong U^{+1}_{3^{n-1}} \oplus Z,$$

where $U^{+1}_{3^{n-1}}$ is uniserial with composition series $0 \subset T(\mathbb{F}P^{+1}) \subset U^{'+1}_{3^{n-1}} \subset U^{+1}_{3^{n-1}}$. Setting $X := U^{'+1}_{3^{n-1}}/T(\mathbb{F}P^{+1})$, we get

$$\mathbb{F}P \cong \mathbb{F} \oplus (\mathbb{F} - X - \mathbb{F}) \oplus Y \oplus 2Z.$$

Case 3: $\ell \mid (3^n + 1)$. As in Case 2,

$$\mathbb{F}P \cong \mathbb{F} \oplus X \oplus (\mathbb{F} - Y - \mathbb{F}) \oplus 2Z,$$

where $X := U^{'+1}_{3^{n-1}}$, $Y := U^{'-1}_{-3^{n-1}}/T(\mathbb{F}P^{-1})$, and $Z := U^{'+1}_{-3^{n-1}} \cong U^{'-1}_{3^{n-1}}$. □

Now we consider the case $\ell = 2$. Following Lemma 7.1, we assume that $\rho^\kappa = 1 + \varphi^\kappa + \psi$ for $\kappa = \pm 1$, where $\varphi^\kappa, \psi \in \mathrm{Irr}(G)$, $\varphi^\kappa(1) = (3^n + \kappa)(3^n - \kappa)/4$, and $\psi(1) = (3^{2n} - 1)/4$. Then $\rho^0 = 1 + \varphi^{+1} + \varphi^{-1}$. From Corollary 7.5 of [11], we have $\overline{\varphi^{+1}} = \chi + \beta(X_1)$ and $\overline{\varphi^{-1}} = 1 + \chi + \beta(Y_1)$, where χ is a 2-Brauer character of G and X_1, Y_1 are simple G-modules of dimensions $(3^n - 1)(3^n - 3)/8$, $(3^n + 1)(3^n + 3)/8 - 1$, respectively. Furthermore, $\chi = \beta(Z_1)$ if n is odd and $\chi = 1 + \beta(Z_1)$ if n is even, where Z_1 is a simple module of dimension $(3^{2n} - 9)/8 - \delta_{2,n}$. The following lemma gives the decomposition of $\overline{\psi}$ into irreducible 2-Brauer characters of G.

Lemma 7.3. *With the above notation, $U^{'1} \cong X_1$ and $U^{'-1} \cong Y_1$. Consequently, $\overline{\psi} = \beta(X_1) + \beta(Y_1)$.*

Proof. By Proposition 3.1 and Lemma 4.1, the submodule $\mathrm{Im}(Q_{0,+1})$ of $\mathbb{F}P^{+1}$ has $U^{'+1}$ as a composition factor. It follows that $U^{'+1} \in \{X_1, Y_1, Z_1\}$ since $\mathrm{Im}(Q_{0,+1}) \cong \mathbb{F}P^0/\mathrm{Ker}(Q_{0,+1})$ and the composition factors of $\mathbb{F}P^0$ are: $\mathbb{F}$ (twice or four times), X_1, Y_1, and Z_1 (twice) (see Figure 6 of [11]).

Set $G_1 := O^+_{2n}(3) \leq G$. Let P^{+1}_1 and P^{-1}_1 be the sets of plus points and minus points in $\langle e_1, ..., e_n, f_1, ..., f_n\rangle$. Since $P^{+1} = P^{+1}_1 \cup \{\langle v + g\rangle \mid \langle v\rangle \in P^{-1}_1\} \cup \{\langle v - g\rangle \mid \langle v\rangle \in P^{-1}_1\}$, we have $\mathbb{F}P^{+1}|_{G_1} \cong \mathbb{F}P^{+1}_1 \oplus 2\mathbb{F}P^{-1}_1$. Moreover, if $P^{-1}_1 + g := \{\langle v + g\rangle \mid \langle v\rangle \in P^{-1}_1\}$ and $P^{-1}_1 - g := \{\langle v - g\rangle \mid \langle v\rangle \in P^{-1}_1\}$ then we get an $\mathbb{F}G_1$-isomorphism:

$$U^{+1}|_{G_1} \cong \langle v_{1,\alpha} \mid \alpha \in P^{+1}_1\rangle \oplus \langle v_{1,\alpha} \mid \alpha \in P^{-1}_1 + g\rangle \oplus \langle v_{1,\alpha} \mid \alpha \in P^{-1}_1 - g\rangle,$$

where all summands are clearly nonzero and nontrivial $\mathbb{F}G_1$-modules. These summands are submodules of $\mathbb{F}P^{+1}_1 \oplus 2\mathbb{F}P^{-1}_1$. It follows that, by Proposition 3.1, each of them contains a graph submodule of $\mathbb{F}P^{+1}_1$ or $\mathbb{F}P^{-1}_1$. Notice that the dimensions of the graph submodules of $\mathbb{F}P^1_1$ as well as $\mathbb{F}P^{-1}_1$ are $(3^n - 1)(3^{n-1} - 1)/8$.

We have shown that $U^{+1}|_{G_1}$ has 3 composition factors (counting multiplicities) of degree $(3^n - 1)(3^{n-1} - 1)/8$. If $U^{+1}|_{G_1}$ has another nonlinear composition factor,

$\dim U^{+1}$ would be at least $4(3^n-1)(3^{n-1}-1)/8+1$ since the smallest degree of nonlinear irreducible 2-Brauer character of G_1 is $(3^n-1)(3^{n-1}-1)/8$ (see Table 1 of [4]). This contradicts the fact that $U'^{+1} \in \{X_1, Y_1, Z_1\}$, whence $U^{+1}|_{G_1}$ has exactly 3 nonlinear composition factors, all of degree $(3^n-1)(3^{n-1}-1)/8$.

Recall that $U^{+1}|_{G_1}$ is a submodule of $\mathbb{F}P_1^{+1} \oplus 2\mathbb{F}P_1^{-1}$. It follows that $U^{+1}_{G_1}$ has at most 6 composition factors of dimension 1 (see Table 1). Combining this with the conclusion of the previous paragraph, we obtain $\dim U^{+1} \leq 3(3^n-1)(3^{n-1}-1)/8+6$. This forces $\dim U^{+1} = \dim X_1 + 1$ and therefore $U'^{+1} \cong X_1$ again by $U'^{+1} \in \{X_1, Y_1, Z_1\}$.

The arguments for $U'^{-1} \cong Y_1$ are similar. Since U'^{κ} appears at least twice as a composition factor of $\mathbb{F}P^{\kappa}$ (see Lemma 3.2), both U'^{+1} and U'^{-1} are constituents of $\overline{\psi}$. Therefore $\overline{\psi} = \beta(X_1) + \beta(Y_1)$ by comparing degrees. $\square$

Proposition 7.4. *Theorem 1.2 holds when $G = O_{2n+1}(3), n \geq 3$ and $\ell = 2$.*

Proof. **Case 4**: n even. We know from Lemma 7.3 that $U'^{+1} \cong X_1$. The self duality of U'^{+1} from Lemma 3.2 then implies that $U'^{+1\perp}/U'^{+1}$ has composition factors: $\mathbb{F}$ (twice), Z_1, and Y_1. Using Lemma 4.1 and inspecting the structure of $\mathbb{F}P^0$ (see Figure 6 of [11]), we see that Y_1 must be a submodule of $U'^{+1\perp}/U'^{+1}$ but Z_1 is not. Therefore, the structure of $U'^{+1\perp}/U'^{+1}$ is $Y_1 \oplus (\mathbb{F} - Z_1 - \mathbb{F})$ and hence that of $\mathbb{F}P^1$ is determined.

Now we determine the structure of $\mathbb{F}P^{-1}$. Since $[P^{-1}] \notin S(\mathbb{F}P^{-1})$, $\mathbb{F}P^{-1} = T(\mathbb{F}P^{-1}) \oplus S(\mathbb{F}P^{-1})$. Also, $U'^{-1} \cong Y_1$ is the socle of $S(\mathbb{F}P^{-1})$. Since $S(\mathbb{F}P^{-1})$ is self-dual, its head is also (isomorphic to) Y_1. Hence $S(\mathbb{F}P^{-1})/\operatorname{Ker}(Q_{-1,0}|_{S(\mathbb{F}P^{-1})}) \cong \operatorname{Im}(Q_{-1,0}|_{S(\mathbb{F}P^{-1})})$ has Y_1 as head. From the submodule structure of $\mathbb{F}P^0$, we find that $\operatorname{Im}(Q_{-1,0}|_{S(\mathbb{F}P^{-1})})$ is uniserial with socle series $\mathbb{F} - Z_1 - \mathbb{F} - Y_1$. So $S(\mathbb{F}P^{-1})$ has a quotient $\mathbb{F} - Z_1 - \mathbb{F} - Y_1$. Again by its self-duality, it has a submodule $Y_1 - \mathbb{F} - Z_1 - \mathbb{F}$, which implies that $U'^{-1\perp}/U'^{-1}$ has a submodule $\mathbb{F} - Z_1 - \mathbb{F}$. Notice that $U'^{-1\perp}/U'^{-1}$ is self-dual and has composition factors: $\mathbb{F}$ (twice), X_1, and Z_1. Its structure must be $\mathbb{F} \oplus X_1 \oplus (\mathbb{F} - Z_1 - \mathbb{F})$, as described in Table 3.

Case 5: n odd. First we find the structure of $\mathbb{F}P^{+1}$. Composition factors of $U'^{+1\perp}/U'^{+1}$ are $\mathbb{F}$, Y_1, and Z_1. Therefore, the structure of $U'^{+1\perp}/U'^{+1}$ is simply $\mathbb{F} \oplus Y_1 \oplus Z_1$ by its self-duality.

Now we turn to $\mathbb{F}P^{-1}$. By Proposition 3.1, the socle of $\operatorname{Im}(Q_{0,-1})$ is either Y_1 ($\cong U'^{-1}$) or $\mathbb{F} \oplus Y_1$ ($\cong U^{-1}$). Notice that $\operatorname{Im}(Q_{0,-1}) \cong \mathbb{F}P^0/\operatorname{Ker} Q_{0,-1}$ and $\mathbb{F}P^0$ has only one quotient having such socle, which is $Y_1 - (\mathbb{F} \oplus Z_1)$ (see the structure of $\mathbb{F}P^0$ in Figure 6 of [11]). We deduce that $\mathbb{F}P^{-1}$ has a submodule of structure $Y_1 - (\mathbb{F} \oplus Z_1)$. We temporarily set $\mathbb{F}_1 := T(\mathbb{F}P^{-1})$ and $\mathbb{F}_2 := \mathbb{F}P^{-1}/S(\mathbb{F}P^{-1})$. Then the submodule of $\mathbb{F}P^{-1}$ of structure $Y_1 - (\mathbb{F} \oplus Z_1)$ must be $Y_1 - (\mathbb{F}_2 \oplus Z_1)$. It follows that $U'^{-1\perp}/U'^{-1}$ has a submodule $\mathbb{F}_2 \oplus Z_1$. Recall that $U'^{-1\perp}/U'^{-1}$ is self-dual and has composition factors: $\mathbb{F}_1$, $\mathbb{F}_2$, Z_1, and X_1, we conclude that its structure is $\mathbb{F}_1 \oplus \mathbb{F}_2 \oplus Z_1 \oplus X_1$. $\square$

Propositions 7.2 and 7.4 complete the proof of Theorem 1.2 for the orthogonal groups in odd dimension.

Acknowledgement

The authors are grateful to Ulrich Meierfrankenfeld for his helpful suggestions leading to the proof of Lemma 6.5. They are also grateful to an anonymous referee for valuable comments and suggestions. Part of this work was done while the second author was a postdoctoral fellow in the Department of Mathematics at Michigan State University. It is a pleasure to thank the department for the supportive and hospitable work environment.

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