Notes on composition algebras

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1. Composition algebras

An *algebra* over the field K is a K-vector space A combined with a bilinear product $\pi\colon\ A \times A \longrightarrow A$. The algebra admits *composition* if there is defined on A a nondegenerate quadratic form $q\colon A \longrightarrow K$ with the additional property that $q(a)q(b) = q(ab)$, for all $a, b \in A$.

One goal is Hurwitz'

(1.1) THEOREM. *A finite dimensional composition algebra has dimension* 1*,* 2*,* 4*, or* 8 *over the field* K*.*

In each of these dimensions examples always exist, and we find out a great deal about the examples as well. Indeed, if we were only interested in a proof of the theorem, then the usual doubling methods (see [6] or Section 6.4 below) are quicker. We also wish to study carefully the related geometries (and groups, although we do not really get to them much).

Hurwitz' theorem does not require the hypothesis of finite dimensionality. The doubling proof makes no distinctions; see Corollary 6.2 below. The proof of Section 4 might adapt in some form to include infinite dimension (countable suffices since any finitely generated algebra has at most countable dimension).

Our main motivation/reference is the chapter by Buekenhout and Cohen [2], which is in turn greatly influenced by Van der Blij and Springer [1]. Also of help are the books of Chevalley [3] and Jacobson [6]. Two good references on quaternion and octonion algebras are the article of Curtis [4] and the chapters by Koecher and Remmert in [5].

2. Some geometry

Let $q\colon A \longrightarrow K$ be a quadratic form on the finite dimensional K-space A. That is,

$$q(\alpha x) = \alpha^2 q(x)\,,$$

for all $\alpha \in K$ and $x \in A$, and the associated form $(\cdot|\cdot)\colon A \times A \longrightarrow K$ given by

$$(a|b) = q(a+b) - q(a) - q(b)$$

is bilinear. For $B \subseteq A$, we let $B^\perp = \{x \in A \,|\, (x|b) = 0,\ b \in B\}$, a K-subspace of A. The form q is *nondegenerate* if $A^\perp = 0$.

For any subset B of A, we let $[B]$ be the K-subspace of A spanned by B.

(2.1) PROPOSITION. *If A has dimension 2 and q is nondegenerate on A, then A is either anisotropic (that is, $q(a) = 0$ if and only if $a = 0$) or hyperbolic. If q is anisotropic, then there is a quadratic extension F of K with $F \otimes_K A$ hyperbolic.*

PROOF. Choose a basis $x, y \in A$ with $(x|y) = 1$, and set $q(x) = a$ and $q(y) = b$. Then $q(\alpha x + \beta y) = a\alpha^2 + \alpha\beta + b\beta^2$. Thus A is anisotropic if $f(t) = at^2 + t + b$ is irreducible in $K[t]$. If $f(t)$ is reducible, then we could have originally chosen an x with $a = q(x) = 0$. Then, after we replace y with $-bx+y$, our new x and y form a hyperbolic pair: $q(x) = q(y) = 0$ and $(x|y) = 1$. (By definition, a hyperbolic 2-space is one spanned by a hyperbolic pair.)

If $\{x, y\}$ is a hyperbolic pair, as in the proof, then

$$q(x + \alpha y) = q(x) + \alpha^2 q(y) + \alpha(x|y) = \alpha\,.$$

Therefore in a hyperbolic 2-space every element of the field K is realized as a q-value.

A subset S of A is *singular* (or sometimes even *totally singular*) if the restriction of q to S is identically 0. Notice that if U is a singular subspace, then nondegenerate q induces a nondegenerate quadratic form on the quotient space $U^\perp/U$.

(2.2) PROPOSITION. *If q is a nondegenerate quadratic form on the K-space A of finite dimension, then the following are equivalent:*

(1) *there is a singular subspace of dimension at least* $\dim_K(A)/2$*;*
(2) *every maximal singular subspace has dimension* $\dim_K(A)/2$*;*
(3) *there are maximal singular subspaces M and N with $A = M \oplus N$;*
(4) *A is a perpendicular direct sum of hyperbolic 2-spaces.*

PROOF. (1) is an easy consequence of all the others, and (4) easily implies (3). The remainder we prove by induction on $\dim(A)$, with Proposition 2.1 providing the initial step (the result being trivial in dimension 1).

We first show that each of (1) and (3) implies (4) Let x be a nonzero singular vector in the maximal singular subspace M (of dimension at least $\dim(A)/2$ for (1)). Then, for any $y \in A \backslash x^\perp$ (chosen from N for (3)), the 2-space $[x, y]$ is hyperbolic by Proposition 2.1. By induction $[x, y]^\perp$ is a perpendicular sum of hyperbolic 2-spaces, giving (4).

We conclude by proving that (4) implies (2). Let x be nonzero in the maximal singular space M. The quotient space $x^\perp/[x]$ is a perpendicular direct sum of hyperbolic 2-spaces, since (1) implies (4). Therefore by induction the maximal singular subspace $M/[x]$ has dimension half that of $x^\perp/[x]$.

In the situation of the proposition, we say that q and A are *split* (or *hyperbolic*). The *index* of q (and A) is then $k = \dim_K(A)/2$.

(2.3) PROPOSITION. *Let the quadratic form q be nondegenerate and split of finite index k on the K-space A.*

(1) *Every singular subspace of codimension* 1 *in a maximal singular subspace is contained in exactly two maximal singular subspaces.*

(2) *The graph on the set of maximal singular subspaces $\mathcal{M}$, with two such adjacent when their intersection has codimension* 1 *in each, is connected bipartite of diameter k. In this graph, the distance between two maximal singular subspaces M and N equals the codimension of $M \cap N$ in each.*

PROOF. If U has codimension 1 in a maximal singular subspace, then $U^\perp/U$ is a hyperbolic 2-space; so (1) follows from Proposition 2.1.

Let $\mathcal{I}$ be the graph described on the set $\mathcal{M}$, and write $M \sim N$ when M is adjacent to N. Let $d(M,N)$ be the distance between M, N in $\mathcal{I}$.

We first claim that, for all $S \in \mathcal{M}$ and $T_1 \sim T_2$ in $\mathcal{I}$, we have

$$|\dim(S \cap T_1) - \dim(S \cap T_2)| = 1\,.$$

Let $U = T_1 \cap T_2$ of codimension 1 in each, and set $R = S \cap U$. If necessary passing to $R^\perp/R$, we may assume $R = 0$ in proving the claim. Then $U^\perp$ has dimension $k+1$ and so intersects S nontrivially. Therefore $T = [U, U^\perp \cap S]$ is totally singular of dimension k. By (1), T is equal to exactly one of T_1 or T_2. Thus

$$\{\dim(S \cap T_1), \dim(S \cap T_2)\} = \{0,1\}\,,$$

giving the claim.

Again by part (1), $d(M,N) \le k - \dim(M \cap N)$. In particular the graph is connected. To prove $d(M,N) = k - \dim(M \cap N)$, we induct on $d(M,N)$. The result is true by definition for $d(M,N) = 0,1$. Suppose $d(M,N) = d$, and choose a $T \in \mathcal{M}$ with $T \sim N$ and $d(T,M) = d-1$. Then by induction $d-1 = k - \dim(M \cap T)$. By the above and the claim $d \le k - \dim(M \cap N) = (d-1) \pm 1 \le d$, as desired.

It remains to prove $\mathcal{I}$ bipartite. Otherwise, there is a minimal cycle $\mathcal{C}$ of odd length, say $2m+1$. But for $S \in \mathcal{C}$, the two vertices T_1 and T_2 at distance m from S in $\mathcal{C}$ are adjacent with $\dim(S \cap T_1) - \dim(S \cap T_2) = 0$, contradicting the earlier claim.

The graph of Proposition 2.3(2) is the *incidence graph* $\mathcal{I}(\mathcal{M})$ of maximal singular subspaces. The two parts of its bipartition (uniquely determined by connectivity) are $\mathcal{M}_\rho$ and $\mathcal{M}_\lambda$.

The following is an easy and familiar calculation:

(2.4) PROPOSITION. *For any nonsingular $a \in A$, the symmetry $s_a \colon x \mapsto x - \frac{(x|a)}{q(a)}a$ has order* 2 *and is an isometry of the quadratic form q on A (and so also of $(\cdot|\cdot)$). That is, for all $x \in A$, we have $q(s_a(x)) = q(x)$.*

3. Some basics

Let A be a composition algebra over K with associated nondegenerate quadratic form q. As A is a K-algebra, we immediately have

(3.1) LEMMA. *The maps $L_a\colon x \mapsto ax$ and $R_a\colon x \mapsto xa$ are K-linear transformations of A.*

The composition law, when written

$$q(L_a x) = q(R_a x) = q(a)q(x)\,,$$

reveals the maps L_a and R_a to be *similarities* for q with respect to the scaling constant $q(a)$. They are then also similiarities for the associated bilinear form $(\cdot|\cdot)$; and we find, for all $a, x, y \in A$,

$$(xa|ya) = (ax|ay) = q(a)(x|y)\,.$$

(3.2) LEMMA. *Every composition algebra is isotopic to one with an identity element* 1.

PROOF. See Jacobson [6, 418–419]. Choose $f \in A$ with $q(f) \neq 0$, and set $e = q(f)^{-1}f^2$ so that $q(e) = 1$. Therefore L_e and R_e are orthogonal hence invertible. Then $x \cdot y = ((R_e)^{-1}x)((L_e)^{-1}y)$ is a K-algebra product on A that admits composition with respect to q and has identity element e^2.

From now on, we assume additionally that A has an identity 1. In this case $q(1) = q(1)^2 = 1$ since q is nondegenerate.

(3.3) LEMMA. *If F is an extension field for K, then the algebra $F \otimes_K A$ also admits composition with respect to the induced quadratic form.*

PROOF. See Chevalley [3, II.2.8, p. 127–128]. The multiplication and forms on A admit unique extension to $F \otimes_K A$ by bilinearity.

We give the full argument for quadratic extensions F of K, the case of primary interest to us. In that case the extension of q to $F \otimes_K A$ is given by

$$q(x + \alpha y) = q(x) + \alpha^2 q(y) + \alpha(x|y)\,,$$

for all $x, y \in A$ and $\alpha \in F$. To show that this induced form admits composition, we must prove the extended law

$$q(x + \alpha y)q(w + \alpha z) = q((x + \alpha y)(w + \alpha z))\,,$$

for all $x, y, w, z \in A$ and $\alpha \in F$. By the composition law and similarity in A we have

$$q(x+\alpha y)q(w+\alpha z) - q((x+\alpha y)(w+\alpha z)) = \alpha^2((x|y)(w|z) - (xz|yw) - (xw|yz))\,.$$

Setting $\alpha = 1$, we find within A the identity

$$(x|y)(w|z) = (xz|yw) + (xw|yz) \qquad (*)$$

since A does admit composition. This in turn implies that

$$q(x+\alpha y)q(w+\alpha z) - q((x+\alpha y)(w+\alpha z)) = 0$$

identically, for all choices of α. In particular, choosing α so that the quadratic extension F equals $K + K\alpha$, we prove that the composition law extends to all $F \otimes_K A$, as desired.

In general, we must prove

$$q(\sum_i \alpha_i x_i)q(\sum_i \alpha_i z_i) = q((\sum_i \alpha_i x_i)(\sum_i \alpha_i z_i))\,,$$

for $\alpha_i \in F$ (coming from some K basis for F) and arbitrary $x_i, z_i \in A$. But this follows, again using similarity and the identity $(*)$.

As an immediate corollary of Proposition 2.1 and Lemma 3.3, we have

(3.4) Corollary. *If A has dimension at least 2 over K then, by tensoring with an appropriate quadratric extension F, we get a composition algebra $F \otimes_K A$ containing nonzero singular elements.*

We define the operation of *conjugation* on A by $x \mapsto \bar{x} = -x + (x|1)1$.

(3.5) Lemma.
(1) $\bar{\bar{x}} = x$
(2) $q(x) = q(\bar{x})$ *and* $(x|y) = (\bar{x}|\bar{y})$

Proof. $\bar{x} = -s_1(x)$, so this follows from Proposition 2.4.

(3.6) Proposition.
(1) $\bar{x}(xy) = q(x)y = (yx)\bar{x}$. *In particular* $\bar{x}x = q(x)1 = x\bar{x}$.
(2) $\bar{x}(yz) + \bar{y}(xz) = (x|y)z$ *and* $(zy)\bar{x} + (zx)\bar{y} = (x|y)z$.
(3) $(x|\bar{v}y) = (vx|y)$ *and* $(x|y\bar{v}) = (xv|y)$.

Proof. In each case, we only prove the first identity.
We first prove (3):

$$\begin{aligned}(x|\bar{v}y) &= (x|((1|v)-v)y)\\ &= (x|y)(1|v) - (x|vy)\\ &= (x|y)(q(1+v) - q(1) - q(v)) - (x|vy)\\ &= ((1+v)x|(1+v)y) - (x|y) - (vx|vy) - (x|vy) \quad \text{by similarity}\\ &= (vx|y)\,.\end{aligned}$$

Next, for (1):

$$\begin{aligned}(\bar{x}(xy)|z) &= (xy|xz) && \text{by (3)}\\ &= q(x)(y|z) && \text{by similarity}\\ &= (q(x)y|z)\,,\end{aligned}$$

for all z. Therefore by nondegeneracy $\bar{x}(xy) = q(x)y$, giving (1).

We linearize (1) to get (2):

$$\begin{aligned} (\bar{x}+\bar{y})((x+y)z) &= q(x+y)z && \text{by (1)} \\ \bar{x}(xz)+\bar{y}(yz)+\bar{x}(yz)+\bar{y}(xz) &= q(x)z+q(y)z+(x|y)z \\ \bar{x}(yz)+\bar{y}(xz) &= (x|y)z\,. \end{aligned}$$

(3.7) Corollary.

(1) $x^2-(x|1)x+q(x)=0$.

(2) $\overline{xy}=\bar{y}\bar{x}$.

Proof. By definition $\bar{x}x = (-x+(x|1)1)x = -x^2+(x|1)x$, so (1) follows directly from Proposition 3.6(1).

For (2), we follow [2, Prop. 14.2.4] and use Proposition 3.6(3) many times:

$$\begin{array}{rclcl} (\overline{xy}|z) &=& (1|(xy)z) &=& (\bar{z}|xy) \\ &=& (\bar{z}\bar{y}|x) &=& (\bar{y}|zx) \\ &=& (\bar{y}\bar{x}|z) && , \end{array}$$

for all z. Therefore, by nondegeneracy, $\overline{xy}=\bar{y}\bar{x}$.

4. Some proofs

We assume throughout that A is a finite dimensional algebra with identity 1 over K admitting composition with respect to nondegenerate q and that there exist nonzero singular elements in A. Let $\mathcal{S}$ be the set of nonzero singular vectors in A.

(4.1) Lemma. *q is split. Indeed if $x,y\in\mathcal{S}$ with $(x|y)\neq 0$, then $A = xA\oplus yA = Ax\oplus Ay$ with each xA and Ax maximal singular.*

Proof. First note that $q(xA)=0=q(Ax)$, for all $x\in\mathcal{S}$.

Let $x\in\mathcal{S}$ and $a\notin x^\perp$. Then there is a second singular vector y with $[x,y]=[x,a]$ and $(x|y)=(\bar{x}|\bar{y})\neq 0$ by Proposition 2.1. Now, by Proposition 3.6(2), for every $z\in A$, we have

$$\begin{aligned} (\bar{x}|\bar{y})z &= x(\bar{y}z)+y(\bar{x}z)\in xA+yA \quad \text{and} \\ (\bar{x}|\bar{y})z &= (z\bar{y})x+(z\bar{x})y\in Ax+Ay\,. \end{aligned}$$

Thus $A=xA+yA=Ax+Ay$ as claimed. As q is nondegenerate, $xA\cap yA = 0$ and both are maximal singular. By Proposition 2.2, q is split. A similar argument proves the claims for Ax and Ay. (Here and elsewhere, lefthanded and righthanded versions of a result can be proven by similar arguments or seen to be equivalent using Corollary 3.7(2).)

Denote by $\mathcal{M}$ the set of all maximal singular subspaces of A. Let k be the dimension of each member of $\mathcal{M}$ (the index of q), so that A has K-dimension $2k$.

(4.2) LEMMA.

(1) *If $x \in \mathcal{S}$, then the image of L_x is xA and its kernel is $\bar{x}A$.*

(2) *If $x \in \mathcal{S}$, then the image of R_x is Ax and its kernel is $A\bar{x}$.*

(3) *If x is nonsingular, then L_x and R_x are invertible.*

PROOF. For (1) certainly the image of L_x is xA, and

$$\dim(xA) + \dim(\bar{x}A) = k + k = \dim(A)\,.$$

So here it remains to prove that $\bar{x}A$ is contained in the kernel of L_x, which comes from Proposition 3.6(1). A similar argument gives (2).

For (3), $q(x)^{-1}L_{\bar{x}}$ equals L_x^{-1} by Proposition 3.6(1), and $q(x)^{-1}R_{\bar{x}}$ is R_x^{-1}.

(4.3) LEMMA. *If q has index at least 2 and $x, y \in \mathcal{S}$ then $xA \neq Ay$.*

PROOF. Suppose $xA = Ay$. Then $\bar{x}(Ay) = 0$ by Lemma 4.2. By Proposition 3.6(2), for all $a \in A$,

$$\bar{x}(ay) + \bar{a}(xy) = (x|a)y\,.$$

Thus $\bar{a}(xy) = (x|a)y$. In particular $A(xy) \leq [y]$ has dimension at most 1. By Lemma 4.2, for nonzero w, R_w has rank k or $2k$. This forces $xy = 0$. Then $(x|a) = 0$, for all a, contradicting nondegeneracy of q.

(4.4) LEMMA. *Assume q has index at least 2. If $x, y \in \mathcal{S}$ with $xy = 0$, then $xA \cap Ay$ has codimension 1 in each and is equal to $x(y^\perp) = (x^\perp)y$.*

PROOF. The codimension of $x(y^\perp)$ in xA is at most one, so by Lemma 4.3 it is enough to prove $x(y^\perp) \subseteq Ay$ (and similarly $(x^\perp)y \subseteq xA$).

Let $v \in y^\perp$. Then by Proposition 3.6(2)

$$(xv)\bar{y} + (xy)\bar{v} = (v|y)x\,.$$

Thus $(xv)\bar{y} = 0$. Therefore $xv \in \ker(R_{\bar{y}}) = Ay$, by Lemma 4.2.

(4.5) LEMMA. *Assume q has index at least 2.*

(1) *Let x be singular and U a maximal singular subspace with $xA \cap U$ of codimension 1 in each. Then there is a singular y with $xy = 0$, $U = Ay$, and $xA \cap U = xA \cap Ay = x(y^\perp) = (x^\perp)y$.*

(2) *Let x be singular and U a maximal singular subspace with $Ax \cap U$ of codimension 1 in each. Then there is a singular y with $yx = 0$, $U = yA$, and $Ax \cap U = yA \cap Ax = y(x^\perp) = (y^\perp)x$.*

PROOF. We only prove (1). Let $U_0 = U \cap xA$, of codimension 1 in each. Let W be the preimage of U_0 under L_x, so that W has codimension 1 in A. By Lemma 4.2, $\ker(L_x) = \bar{x}A$ is contained in W. As W has codimension 1 in A, there is a y, uniquely determined up to scalar multiple, with $W = y^\perp$, hence $U_0 = L_x W = xW = x(y^\perp)$. Furthermore, $[y] = W^\perp \subseteq (\bar{x}A)^\perp = \bar{x}A$, hence $y \in \mathcal{S}$. Also $0 = xy \in x(\bar{x}A)$, by Proposition 3.6 or Lemma 4.2.

By the previous paragraph and Lemma 4.4, we have

$$xA \cap Ay = x(y^\perp) = U_0 = xA \cap U\,.$$

Therefore $Ay = U$ by Proposition 2.3(1).

(4.6) Corollary. *Assume q has index at least 2. For every maximal singular subspace U, there is a singular x with U equal to xA or Ax. The two parts of the incidence graph $\mathcal{I}(\mathcal{M})$ on the set $\mathcal{M}$ of maximal singular subspaces are $\mathcal{M}_\rho = \{Ax \mid x \in \mathcal{S}\}$ and $\mathcal{M}_\lambda = \{xA \mid x \in \mathcal{S}\}$.*

Proof. Consider the two sets of maximal singular subspaces $\{Ax \mid x \in \mathcal{S}\}$ and $\{xA \mid x \in \mathcal{S}\}$. They are disjoint by Lemma 4.3. By Lemma 4.5 every edge on yA in the incidence graph $\mathcal{I}(\mathcal{M})$ goes to $\{Ax \mid x \in \mathcal{S}\}$, and every edge of $\mathcal{I}(\mathcal{M})$ on Ay goes to $\{xA \mid x \in \mathcal{S}\}$. By Proposition 2.3(2) $\mathcal{I}(\mathcal{M})$ is bipartite and connected, so these sets are the two parts of the bipartition.

(4.7) Lemma. *Assume q has index at least 3. Let $x, y \in \mathcal{S}$ be with $[x] \neq [y]$.*

(1) *If $(x|y) = 0$ then $xA \cap yA$ has codimension 2 in each and $Ax \cap Ay$ has codimension 2 in each.*

(2) *$xA \neq yA$ and $Ax \neq Ay$.*

Proof. Let U_0 be singular of dimension $k-1\,(\geq 2)$ and containing $[x, y]$. By Lemma 4.5 and Corollary 4.6, there are $w, z \in \mathcal{S}$ with $U_0 = wA \cap Az$. As $[x, y] \subseteq Az$, we have $x\bar{z} = y\bar{z} = 0$ by Lemma 4.2. Therefore $xA \cap A\bar{z}$ and $yA \cap A\bar{z}$ both have dimension $k-1$ by Lemma 4.4. This implies that $xA \cap yA$ has dimension at least $k-2$. The dimension of $xA \cap yA$ can not be $k-1$ by Lemmas 4.3 and 4.5, so (1) will follow from (2).

If $xA = yA$, then $(x|y) = 0$; so in proving (2) we can make use of the previous paragraph. By Lemma 4.4 again $xA \cap A\bar{z} = yA \cap A\bar{z}$ equals the $k-1$ space $(x^\perp)\bar{z} = (y^\perp)\bar{z}$. Its preimage under $R_{\bar{z}}$ is then $x^\perp = y^\perp$. This forces $[x] = [y]$, which is not the case.

Starting again with $\bar{w}x = \bar{w}y = 0$, we find the rest of the lemma.

(4.8) Theorem. *A has dimension 2, 4, or 8.*

Proof. We must prove that k is 1, 2, or 4. In doing this, clearly we can assume that $k \geq 3$. Consider the part $\mathcal{M}_\lambda = \{xA \mid x \in \mathcal{S}\}$ of the graph $\mathcal{I}(\mathcal{M})$ and distances within it.

By Proposition 2.3 and Corollary 4.6, the distance from xA to yA in $\mathcal{I}(\mathcal{M})$ is even and equal to the codimension of $xA \cap yA$ in each. Every even number in the range 0 to k must be realized, since $\mathcal{I}(\mathcal{M})$ is connected of diameter k. But by Lemmas 4.1 and 4.7, the only distances realized within $\mathcal{M}_\lambda = \{xA \mid x \in \mathcal{S}\}$ are 0 (when $[x] = [y]$), 2 (when $(x|y) = 0$ but $[x] \neq [y]$), and k (when $(x|y) \neq 0$). This forces k to be even and $2 \geq k-2\,(\geq 1)$. That is, $k = 4$.

5. Some properties

We discuss some properties of composition algebras with identity for arbitrary K in dimensions 1, 2, 4, and 8.

In dimension 1 there is really nothing to say. The algebra is unique. The form q is just squaring and clearly admits composition. In the other dimensions, there are often many examples; but if we restrict our attention to split algebras, then they are essentially unique (a further consequence of the doubling approach [6] and Section 6.4).

Let A now be, as in the previous section, a finite dimensional split composition algebra with identity and nondegenerate form q. Let $k = \dim_K(A)/2$ be the index of q. We collect together many properties of A coming from our work of the previous two sections. In the case of index $k = 1$, the algebra A is a hyperbolic 2-space, and most of the statements in the next theorem are either vacuous or trivial.

Compare Theorem 14.3.1 of Buekenhout and Cohen [2].

(5.1) THEOREM. *Let A be split of index k equal to 2 or 4. Throughout $x, y \in \mathcal{S}$, the set of nonzero singular vectors.*

(1) *$\{ zA \,|\, z \in \mathcal{S}\}$ and $\{ Az \,|\, z \in \mathcal{S}\}$ are the two classes of maximal singular subspaces of A.*

(2) *$a \in xA$ if and only if $\bar{x}a = 0$ if and only if $\bar{a}x = 0$. $a \in Ax$ if and only if $a\bar{x} = 0$ if and only if $x\bar{a} = 0$.*

(3) *Always $xA \neq Ay$. For $k = 2$, $xA = yA$ if and only if $y \in xA$ and $Ax = Ay$ if and only if $y \in Ax$. For $k = 4$, $xA = yA$ if and only if $Ax = Ay$ if and only if $[x] = [y]$.*

(4) *Assume $(x|y) \neq 0$. Then $A = xA \oplus yA = Ax \oplus Ay$.*

(5) *Assume $k = 2$, $(x|y) = 0$, and $[x] \neq [y]$. Then either $xA = yA$ and $Ax \cap Ay = 0$ or $Ax = Ay$ and $xA \cap yA = 0$.*

(6) *Assume $k = 4$, $(x|y) = 0$, and $[x] \neq [y]$. Then $xA \cap yA$ and $Ax \cap Ay$ both have dimension 2, with $xA \cap yA = x(\bar{y}A) = y(\bar{x}A)$ and $Ax \cap Ay = (A\bar{y})x = (A\bar{x})y$.*

(7) *Assume $xy = 0$. Then $xA \cap Ay = x(y^\perp) = (x^\perp)y$ of codimension 1 in both xA and Ay.*

(8) *Assume $xy \neq 0$. Then $xA \cap Ay = [xy]$ of dimension 1.*

PROOF. (1) follows from Corollary 4.6.

(2) comes from Corollary 3.7 and Lemma 4.2.

(3) By (1) or Lemma 4.3, $xA \neq Ay$. For $k = 4$, the rest is Lemma 4.7(2). For $k = 2$, $xA \cap yA$ is nonzero if and only if $xA = yA$ by (1). That is, $y \in xA$ if and only if $xA = yA$.

(4) is contained in Lemma 4.1.

(5) As in (3), again $xA \cap yA$ is either 0 or $xA = yA$ and similarly for $Ax \cap Ay$. Here $[x, y]$ is itself maximal singular. Therefore either $[x, y] = xA = yA$, in which case $Ax \cap Ay$ must be zero by (1), or $[x, y] = Ax = Ay$ and $xA \cap yA = 0$.

(6) The dimensions are correct by Lemma 4.7(1). For the rest of (6), we prove $xA \cap yA \supseteq x(\bar{y}A) = y(\bar{x}A)$ of dimension 2. Since $(x|y) = 0$, Proposition 3.6(2) gives $x(\bar{y}A) = y(\bar{x}A)$, clearly contained in $xA \cap yA$. The space $x(\bar{y}A)$ is the image of $\bar{y}A$ under L_x, and so has dimension equal to $\dim(\bar{y}A) - \dim(\ker(L_x) \cap \bar{y}A) = \dim(\bar{y}A) - \dim(\bar{x}A \cap \bar{y}A) = 4 - 2 = 2$, as desired.

(7) is Lemma 4.4.

(8) By (1) the dimension of $xA \cap Ay$ is odd and at most k, and $xA \cap Ay$ visibly contains xy. This gives the $k = 2$ case. For $k = 4$, the only other possibility is for $xA \cap Ay$ to have dimension 3. Then by Lemma 4.5(1) there is a second y' with $Ay = Ay'$ and $xy' = 0$, which contradicts (3).

6. Some examples

As mentioned previously, a composition algebra with identity and having dimension 1 over K must be K with associated form $q(x) = x^2$ (and the characteristic of K cannot be 2). In particular, it is unique up to isomorphism.

We now investigate split composition K-algebras A with identity. Thus A has dimension $2k$ and index k, for $k = 1, 2, 4$.

6.1. Index 1

Suppose $\dim_K(A) = 2$, hence $k = 1$. Then A is hyperbolic. Choose $z \in A$ with $q(z) = 0$ and $(z|1) = 1$. Then $\bar{z}$ is also singular and $z + \bar{z} = 1$. We have $A = \{\alpha z + \delta \bar{z} \,|\, \alpha, \delta \in K\}$. By Corollary 3.7(1), $z^2 = (z|1)z - q(z) = z$ and, similarly, $\bar{z}^2 = \bar{z}$. Also $z\bar{z} = \bar{z}z = q(z) = 0$ by Proposition 3.6(1). Therefore z and $\bar{z}$ are a spanning pair of orthogonal idempotents in A, and multiplication in A is completely determined. Furthermore

$$q(\alpha z + \delta \bar{z}) = \alpha^2 q(z) + \delta^2 q(\bar{z}) + \alpha\delta(z|\bar{z}) = \alpha\delta\,.$$

Therefore A is uniquely determined up to isomorphism.

6.2. Index 2

Split composition K-algebras with identity and of index 2 and 4 are also unique up to isomorphism (a consequence of uniqueness for index 1 and the doubling Proposition 6.1 below.)

A composition algebra of dimension 4 is usually called a *quaternion* algebra.

There is a canonical example of a split composition K-algebra of index 2 and dimension 4, namely the algebra of all 2×2 matrices over K with the usual multiplication and with $q(x) = \det(x)$:

$$\det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \alpha\delta - \beta\gamma\,.$$

Notice that the subalgebra of diagonal matrices is isomorphic to the index 1 example given above with

$$z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \bar{z} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

6.3. Index 4

A composition algebra of dimension 8 is usually called an *octonion* or *Cayley* algebra.

Again by the doubling Proposition 6.1, a split Cayley algebra (index 4 and dimension 8) over K is unique up to isomorphism. Buekenhout and Cohen [2] give Zorn's construction:

$$\begin{bmatrix} x_1 & x_{234} \\ x_{567} & x_8 \end{bmatrix} \begin{bmatrix} y_1 & y_{234} \\ y_{567} & y_8 \end{bmatrix} =$$

$$\begin{bmatrix} x_1 y_1 - x_{234} \cdot y_{567} & x_1 y_{234} + x_{234} y_8 + x_{567} \times y_{567} \\ x_{567} y_1 + x_8 y_{567} + x_{234} \times y_{234} & x_8 y_8 - x_{567} \cdot y_{234} \end{bmatrix},$$

where $x_1, x_8, y_1, y_8 \in K$ and $x_{234}, x_{567}, y_{234}, y_{567} \in K^3$. Here, in addition to scalar multiplication (from both sides), the two products $\cdot$ and $\times$ are, respectively, the usual dot product

$$(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and cross product (vector product)

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 b_3 - a_3 b_2, a_1 b_3 - a_3 b_1, a_2 b_1 - a_1 b_2).$$

The associated quadratic form is

$$x_1 x_8 + x_{234} \cdot x_{567}.$$

I prefer the version got by replacing the 567 entry with its negative:

$$\begin{bmatrix} x_1 & x_{234} \\ x_{567} & x_8 \end{bmatrix} \begin{bmatrix} y_1 & y_{234} \\ y_{567} & y_8 \end{bmatrix} =$$

$$\begin{bmatrix} x_1 y_1 + x_{234} \cdot y_{567} & x_1 y_{234} + x_{234} y_8 + x_{567} \times y_{567} \\ x_{567} y_1 + x_8 y_{567} - x_{234} \times y_{234} & x_8 y_8 + x_{567} \cdot y_{234} \end{bmatrix} =$$

$$\begin{bmatrix} x_1 y_1 + x_{234} \cdot y_{567} & x_1 y_{234} + x_{234} y_8 \\ x_{567} y_1 + x_8 y_{567} & x_8 y_8 + x_{567} \cdot y_{234} \end{bmatrix} + \begin{bmatrix} 0 & x_{567} \times y_{567} \\ -x_{234} \times y_{234} & 0 \end{bmatrix},$$

with associated quadratic form

$$x_1 x_8 - x_{234} \cdot x_{567}.$$

This demonstrates a connection with the usual matrix multiplication and determinant. One easily sees three 4-dimensional subalgebras represented as matrix algebras, as in the previous subsection:

$$\left\{ \begin{bmatrix} x_1 & x_{234} \\ x_{567} & x_8 \end{bmatrix} \middle| x_{234}, x_{567} \in E \right\}$$

where the 1-space E is one of $[(1,0,0)]$, $[(0,1,0)]$, or $[(0,0,1)]$. Indeed, for any 1-space $E=[e]$ with $e\cdot e=\mu\neq 0$, this is a 4-dimensional matrix subalgebra. (Set $x_1=\alpha$, $x_8=\delta$, $x_{234}=\beta e$ and $x_{567}=(\gamma/\mu)e$.)

(For $e\cdot e=0$, we have

$$\left\{\begin{bmatrix}\alpha & 0\\ 0 & \delta\end{bmatrix}\right\}\perp\left\{\begin{bmatrix}0 & \beta e\\ \gamma e & 0\end{bmatrix}\right\},$$

for $\alpha,\beta,\gamma,\delta\in K$, where the first factor is nondegenerate of dimension 2, as before, and the second factor is a 0-subalgebra of dimension 2.)

6.4. Doubling

See Jacobson [6, 423–425]. The fundamental result is

(6.1) PROPOSITION. *Let B be a nondegenerate subalgebra of a composition algebra (not necessarily split) with B containing the identity (and so invariant under the conjugation map $b\mapsto\bar b$). Choose $t\in B^\perp$ with $q(t)=-\gamma\neq 0$. Then $A=B+Bt=B\oplus Bt$ is a nondegenerate subalgebra of dimension twice that of B with multiplication given by*

$$(u+vt)(x+yt)=(ux+\gamma\bar y v)+(yu+v\bar x)t\,,$$

for $u,v,x,y\in B$.

PROOF. For $a,b\in B$, we have by Proposition 3.6(3), $(a|bt)=(\bar b a|t)=0$. Therefore $A=B+Bt$ is the perpendicular direct sum of nondegenerate B and Bt. As $\gamma\neq 0$, Bt has dimension equal to that of B (by Lemma 4.2) and is itself nondegenerate by similarity. Therefore A is nondegenerate.

It remains to prove that A satisfies the stated multiplication rule. For $a,b\in B$ and $r,s\in Bt$, we have the fundamental identities:
(*i*) $r=-\bar r$;
(*ii*) $br=r\bar b$;
(*iii*) $(ab)r=b(ar)$;
(*iv*) $(ar)s=(rs)a$.
The first is clear, since $(r|1)=0$. For the second, we start with Proposition 3.6(2):

$$(ab)r+(a\bar r)\bar b=(\bar r|b)a=0$$

hence $(ab)r=-(a\bar r)\bar b=(ar)\bar b$ by (*i*). Specializing to $a=1$ gives (*ii*). We then in turn have $(ab)r=(ar)\bar b=b(ar)$ by (*ii*), and this is (*iii*).

For (*iv*), we again use Proposition 3.6(2):

$$(rs)a+(r\bar a)\bar s=(\bar a|s)r=0$$

Hence $(rs)a=-(r\bar a)\bar s=(r\bar a)s=(ar)s$ by (*i*) and (*ii*).

Therefore

$$\begin{aligned}(u+vt)(x+yt) &= ux+u(yt)+(vt)x+(vt)(yt)\,; \\ &= ux+(yu)t+\bar{x}(vt)+(t(yt))v \qquad \text{by } (iii),(ii),(iv)\,; \\ &= ux+(yu)t+(v\bar{x})t-(\bar{t}(t\bar{y}))v \qquad \text{by } (iii),(i),(ii)\,; \\ &= ux+(yu)t+(v\bar{x})t-(q(t)\bar{y})v \qquad \text{by (3.6)(1)}\,; \\ &= (ux+\gamma\bar{y}v)+(yu+v\bar{x})t\,;\end{aligned}$$

as desired.

In particular we get the infinite dimensional analogue of Hurwitz' theorem.

(6.2) Corollary. *There are no composition algebras of infinite dimension.*

Proof. If there were such an algebra, then within it we could construct nondegenerate composition subalgebras of dimension 16, contradicting Hurwitz' Theorem 1.1.

The theorem has an important converse, the "doubling construction." If B is an arbitrary K-algebra with identity and admitting composition with respect to the form q_B, and if γ is a arbitrary nonzero element of K, then this formula turns $A = B \oplus Bt$ into a K-algebra with identity that may admit composition with respect to the quadratic form $q_A(x+yt) = q_B(x) - \gamma q_B(y)$. Conjugation is given by $\overline{x+yt} = \bar{x} - yt$.

For a split composition algebra, every possible value $-\gamma$ is attained by q on each nondegenerate split subspace. Thus we usually fix $-\gamma = 1$.

The standard uniqueness, existence, nonexistence proof for split composition algebras uses this construction:

1. start from $K = A_1$ itself, commutative and associative with the conjugation map trivial (in characteristic 2, must start with A_2);
2. the double of A_1 is A_2, a uniquely determined composition algebra of dimension 2, which is commutative and associative but has nontrivial conjugation;
3. A_2 has unique double A_4, a composition algebra of dimension 4, associative but no longer commutative;
4. A_4 doubles to a unique algebra A_8 of dimension 8, which still admits composition but is now neither commutative nor associative;
5. finally, from A_8 the double A_{16} of dimension 16 no longer admits composition.

We have all that is needed for a formal proof, but see Jacobson [6] for the details.

The doubling construction can be expressed nicely in 2×2 matrix form:

$$\begin{bmatrix} u & v \\ \gamma\bar{v} & \bar{u} \end{bmatrix}\begin{bmatrix} x & y \\ \gamma\bar{y} & \bar{x} \end{bmatrix} = \begin{bmatrix} ux+\gamma\bar{y}v & yu+v\bar{x} \\ \gamma x\bar{v}+\gamma\bar{u}\bar{y} & \gamma\bar{v}y+\bar{x}\bar{u} \end{bmatrix}.$$

The selection $\gamma = -1$ then gives the usual matrix construction of the complex numbers from the reals and the quaternions from the complexes.

When the algebra B is commutative and conjugation is trivial, the resulting A is commutative. As long as B is commutative, the product on A is just the regular matrix product and so is associative. For associative B admitting composition, the algebra A admits composition with respect to the "determinant" form $x\bar{x} - \gamma\bar{y}y$.

7. Some triality

We give a version of triality for D_4 geometries based upon the treatment of Buekenhout and Cohen [2].

Let A be be a vector space K^8 equipped with a nondegenerate split (hyperbolic) quadratic form q. The associated D_4 geometry $\mathcal{A}$ is a 4-partite graph $\mathcal{E}_0\dot{\cup}\mathcal{E}_1\dot{\cup}\mathcal{E}_2\dot{\cup}\mathcal{E}_3$. The singular (projective) points $\mathcal{E}_0$ are the singular 1-spaces of A with respect to q. The singular (projective) lines $\mathcal{E}_1$ are the associated singular 2-spaces. $\mathcal{E}_2$ and $\mathcal{E}_3$ are the two classes of maximal singular 4-spaces given by Proposition 2.3. Incidence between a point and a line or 4-space is given by containment. Similarly a line is incident to those 4-spaces that contain it. Finally $U_2 \in \mathcal{E}_2$ and $U_3 \in \mathcal{E}_3$ are incident if and only if they intersect in a 3-space (so the subgraph induced on $\mathcal{E}_2\dot{\cup}\mathcal{E}_3$ is exactly the incidence graph $\mathcal{I}(\mathcal{M})$ discussed earlier).

We denote adjacency in $\mathcal{E}$ by $\sim$.

(7.1) LEMMA. *Let $\mathcal{E} = \mathcal{E}_0\dot{\cup}\mathcal{E}_2\dot{\cup}\mathcal{E}_3$ be the tripartite subgraph induced on the set of singular points and 4-spaces. For each singular line $\ell \in \mathcal{E}_1$, let $\mathcal{E}_\ell$ be the subgraph of $\mathcal{E}$ of those singular points and 4-spaces incident to ℓ.*

(1) *$\mathcal{E}_\ell$ is a complete tripartite subgraph of $\mathcal{E}$, with each part of cardinality $|K| + 1$ (a projective line).*

(2) *If T is a complete tripartite subgraph of $\mathcal{E}$ meeting at least two parts of $\mathcal{E}$ in at least two vertices, then there is a unique singular line $\ell \in \mathcal{E}_1$ with $T \subseteq \mathcal{E}_\ell$.*

PROOF. The points incident to ℓ are certainly incident to any singular subspace containing it. Now let $M \in \mathcal{E}_\ell \cap \mathcal{E}_2$ and $N \in \mathcal{E}_\ell \cap \mathcal{E}_3$. $M \cap N$ has odd codimension in each (see Proposition 2.3(2)) and dimension at least 2. Thus $\dim(M \cap N) = 3$, and M and N are incident in $\mathcal{E}$. This proves $\mathcal{E}_\ell$ to be complete tripartite. To prove (1) it remains to show that each $\mathcal{E}_\ell \cap \mathcal{E}_i$ (for $i = 0, 2, 3$) has the structure of a projective line over K. This is clear for $i = 0$. Consider a singular 3-space H containing ℓ. This represents an arbitrary singular 1-space in the quotient orthogonal geometry $\ell^\perp/\ell$, split of dimension 4. By Proposition 2.3(1), this is contained in exactly two maximal 4-spaces, one in $\mathcal{E}_2$ and the other in $\mathcal{E}_3$. Thus each $\mathcal{E}_\ell \cap \mathcal{E}_i$, for $i = 2, 3$, induces a partition of the singular projective points of the geometry $\ell^\perp/\ell$ into singular projective lines. As M/ℓ runs through the projective lines of $(\mathcal{E}_\ell \cap \mathcal{E}_2)/\ell$, the projective line $N/\ell \in (\mathcal{E}_\ell \cap \mathcal{E}_3)/\ell$ meets each in exactly one projective point. This completes (1). (We have the two rulings of the associated quadric, transverse to each other.)

For (2), first suppose distinct $[x], [y] \in T \cap \mathcal{E}_0$ and distinct $M, M' \in T \cap \mathcal{E}_2$. Then $[x], [y]$ are contained in M; so they span a singular line ℓ, which in turn is incident to any 4-space incident to both $[x]$ and $[y]$, including all those of T. By the previous paragraph $M \cap M' = \ell$; so any point of T, being incident to both M and M', must also belong to ℓ. Thus $T \subseteq \mathcal{E}_\ell$, as desired. The case in which T is known to meet both $\mathcal{E}_2$ and $\mathcal{E}_3$ in sets of size at least 2 is similar. (In fact, we only need the result when T meets each part of $\mathcal{E}$ in at least 2 points.) This completes (2).

We now assume additionally that A is an algebra with identity that admits composition with respect to q. By Corollary 4.6 or Theorem 5.1(1) we may set

$$\mathcal{E}_2 = \{\, zA \mid [z] \in \mathcal{E}_0 \,\} \text{ and } \mathcal{E}_3 = \{\, Az \mid [z] \in \mathcal{E}_0 \,\}\,.$$

(7.2) LEMMA. *For $[x], [y] \in \mathcal{E}_0$, the following are equivalent:*
(1) $xy = 0$;
(2) $[y] \sim \bar{x}A$;
(3) $[x] \sim A\bar{y}$;
(4) $[\bar{y}] \sim Ax$;
(5) $[\bar{x}] \sim yA$;
(6) $xA \sim Ay$;
(7) $\bar{y}A \sim A\bar{x}$.

PROOF. By Lemma 4.2, $\bar{x}A$ is the kernel of L_x, so $y \in \bar{x}A$ if and only if $xy = 0$. Similarly $[x] \in A\bar{y} = \ker(R_y)$ if and only if $xy = 0$. Also $[\bar{y}] \in Ax$ if and only if $\bar{y}\bar{x} = 0$ if and only if $xy = 0$ by Corollary 3.7(2), and similarly for $[\bar{x}] \in yA$.

By Theorem 5.1 $xA \cap Ay$ has codimension 1 in each if and only if $xy = 0$, and similarly $\bar{y}A \cap A\bar{x}$ has codimension 1 in each if and only if $\bar{y}\bar{x} = 0$.

Define on $\mathcal{E}$ the map τ, for all $[x] \in \mathcal{E}_0$:

$$[x] \xrightarrow{\tau} \bar{x}A \xrightarrow{\tau} A\bar{x} \xrightarrow{\tau} [x]\,,$$

so that τ has order 3 and permutes the three parts of $\mathcal{E} = \mathcal{E}_0 \dot\cup \mathcal{E}_2 \dot\cup \mathcal{E}_3$ cyclically.

(7.3) THEOREM. *The map τ is an automorphism of $\mathcal{E}$ and extends uniquely to an order 3 automorphism of the associated D_4 geometry $\mathcal{A}$, which we also denote τ, a triality automorphism of $\mathcal{A}$.*

PROOF. We have τ acting on pairs:

$$([y], \bar{x}A) \xrightarrow{\tau} (\bar{y}A, A\bar{x}) \xrightarrow{\tau} (A\bar{y}, [x]) \xrightarrow{\tau} ([y], \bar{x}A)\,.$$

By the proposition, any one of these is an edge of $\mathcal{E}$ if and only if $xy = 0$, in which case they are all edges. Therefore τ is an automorphism of the graph $\mathcal{E}$.

By Lemma 7.1(2), the subgraphs $\mathcal{E}_\ell$ of $\mathcal{E}$ are the maximal complete tripartite subgraphs having each part of size greater than 1. Any automorphism of $\mathcal{E}$ must act on the set of such subgraphs and so on $\mathcal{E}_1$, that is, it extends to an automorphism of the full D_4 geometry $\mathcal{A}$. This extension must be unique, since again by Lemma 7.1(2) any automorphism of $\mathcal{A}$ that is trivial on $\mathcal{E}$ is trivial on $\mathcal{A}$.

Let κ be the permutation of the D_4 geometry $\mathcal{A}$ determined by the conjugation map in A:

$$\kappa([x]) = [\bar{x}] \in \mathcal{E}_0;\ \kappa([x, y]) = [\bar{x}, \bar{y}] \in \mathcal{E}_2;\ \kappa(xA) = A\bar{x};\ \kappa(Ax) = \bar{x}A\,.$$

(7.4) Proposition. *κ is an automorphism of the D_4 geometry $\mathcal{A}$ of order 2 that inverts the triality automorphism τ.*

Proof. As before, we only need check this on the edges of $\mathcal{E}$. We have on pairs

$$([y], \bar{x}A) \stackrel{\kappa}{\longleftrightarrow} ([\bar{y}], Ax) \text{ and } (\bar{y}A, A\bar{x}) \stackrel{\kappa}{\longleftrightarrow} (Ay, xA)\,.$$

Again by Lemma 7.2, any of these pairs is an edge if and only if $xy = 0$, in which case all are edges. Futhermore

$$[x] \xrightarrow{\kappa} [\bar{x}] \xrightarrow{\tau} xA \xrightarrow{\kappa} A\bar{x}\,,$$

and so forth, leading to

$$[x] \xrightarrow{\kappa\tau\kappa} A\bar{x} \xrightarrow{\kappa\tau\kappa} \bar{x}A \xrightarrow{\kappa\tau\kappa} [x]\,.$$

Therefore $\kappa\tau\kappa = \tau^{-1}$, as claimed.

Of course, it should be no surprise that κ is an automorphism of $\mathcal{A}$. From Proposition 2.4, we see that κ is induced by the negative of the orthogonal symmetry s_1 on A.

An element of the D_4 geometry $\mathcal{A}$ is *absolute* for τ if it is incident with its image under τ. (For a singular line $\ell \in \mathcal{E}_1$, this means $\ell^\tau = \ell$.)

(7.5) Lemma. *For $[x] \in \mathcal{E}_0$, the following are equivalent:*
(1) *$[x]$ is absolute for τ;*
(2) *xA is absolute for τ;*
(3) *Ax is absolute for τ;*
(4) *$x^2 = 0$;*
(5) *$(x|1) = 0$;*
(6) *$\bar{x} = -x$.*

Proof. By Lemma 7.2 we have any one of (1) $[x] \sim \tau([x]) = \bar{x}A$, (2) $xA \sim \tau(xA) = Ax$, and (3) $Ax \sim \tau(Ax) = [\bar{x}]$, if and only if $xx = x^2 = 0$, in which case we have all three.

By Corollary 3.7(1), $x^2 - (x|1)x + q(x) = 0$. For $[x] \in \mathcal{E}_0$, we thus have $x^2 = 0$ if and only if $(x|1) = 0$, which, by the definition of conjugation, holds if and only if $\bar{x} = -x$.

(7.6) LEMMA. *For $[x],[y] \in \mathcal{E}_0$, set $\ell = [x,y]$. The following are equivalent:*
(1) *$\ell \in \mathcal{E}_1$ is absolute for τ;*
(2) *the algebra product in A is identically 0 on ℓ;*
(3) *$x^2 = y^2 = xy = 0$;*
(4) *$x^2 = xy = yx = 0$.*

PROOF. As multiplication is bilinear, the algebra product on ℓ being trivial is equivalent to $x^2 = y^2 = yx = xy = 0$. Suppose this is the case, as in (2). Then x and y are absolute, $x \in \ker(R_y) \cap \ker(L_y) = \bar{y}A \cap A\bar{y}$, and similarly $y \in \bar{x}A \cap A\bar{x}$. Therefore $T = \{[x],[y],\bar{x}A,\bar{y}A,A\bar{x},A\bar{y}\}$ is a complete tripartite subgraph of $\mathcal{E}$, which is visibly left invariant by τ. By Lemma 7.1(2), this subgraph T is contained in $\mathcal{E}_{\ell'}$, for some line $\ell' \in \mathcal{E}_1$. As $x,y \in T$, we must have $\ell' = [x,y] = \ell$. Furthermore, T is in the intersection of $\mathcal{E}_\ell$ and $\mathcal{E}_\ell^\tau$. Again using Lemma 7.1(2), we find that $\mathcal{E}_\ell = \mathcal{E}_\ell^\tau$, hence $\ell = \ell^\tau$ is absolute. That is, (2) implies (1).

Conversely, ℓ is absolute for τ if and only if $\mathcal{E}_\ell^\tau = \mathcal{E}_\ell$. In this case T is a complete tripartite subgraph of $\mathcal{E}_\ell$. In particular, $[x]$ and $[y]$ are themselves absolute with $x \in \tau([y]) = \bar{y}A$ and $y \in \tau([x]) = \bar{x}A$. That is, $x^2 = y^2 = yx = xy = 0$; so (1) implies (2).

It remains to prove that the relations $x^2 = y^2 = yx = xy = 0$ are equivalent to their subsets (3) and (4). Starting with (3), we have $\bar{x} = -x$ and $\bar{y} = -y$ since $x^2 = y^2 = 0$ (by Lemma 7.5). Then $0 = xy = \overline{xy} = \bar{y}\bar{x} = (-y)(-x) = yx$.

Now assume $x^2 = xy = yx = 0$, as in (4). Then

$$\begin{aligned}
(x|y)1 &= y\bar{x} + x\bar{y} && \text{by } (3.6)(2)\\
&= -yx + x\bar{y} && \text{as } x^2 = 0\\
&= x\bar{y}\\
&= x(-y + (y|1)1)\\
&= -xy + (y|1)x\\
&= (y|1)x && .
\end{aligned}$$

As $[1] \neq [x]$, we have $(x|y) = (y|1) = 0$. So $y^2 = 0$ by Lemma 7.5.

8. Some hexagons

With the material of the previous section we can easily describe the G_2 and ${}^3D_4$ hexagons, again following Buekenhout and Cohen [2]. These hexagons and their duals provide the only known examples of thick, finite generalized hexagons.

For our purposes, a thick generalized hexagon is a partial linear space $\mathcal{H} = (\mathcal{P},\mathcal{L})$ whose incidence graph $\mathcal{I}(\mathcal{H})$ is connected of diameter 6 with girth 12 and having every vertex of degree at least 3. Note that the dual of a generalized hexagon is also a generalized hexagon. It can be proven that $\mathcal{I}(\mathcal{H})$ is biregular, but we shall show this directly in the cases of interest to us. By convention, in $\mathcal{I}(\mathcal{H})$ the point valency is $1+t$ and the line valency is $1+s$. That is, there are $1+s$ points per line and $1+t$ lines per point. In this case, we say that $\mathcal{H}$ has *order* (s,t).

Let A be the 8-dimensional composition K-algebra with identity of the previous section. Let the point set $\mathcal{P}$ consist of those 1-spaces (projective points) of A with trivial algebra product, and let the line set $\mathcal{L}$ consist of those 2-spaces (projective lines) with trivial algebra product. Thus the points and lines are exactly those absolute for the triality automorphism τ of the associated D_4 geometry. We may identify a line with the set of points contained in it.

(8.1) Theorem. *$\mathcal{H} = (\mathcal{P}, \mathcal{L})$ is a thick generalized hexagon.*

(8.2) Corollary. *If K is finite of order q, then $\mathcal{H} = (\mathcal{P}, \mathcal{L})$ is a thick generalized hexagon of order (q, q).*

We shall learn that, in the collinearity graph $\mathcal{C}(\mathcal{H})$, the points at distance 1 from $[x]$ are those of $xA \cap Ax$, those points at distance 2 are those of $x^\perp \backslash \{xA \cap Ax\}$, and at distance 3 are those of $A \backslash x^\perp$.

(8.3) Lemma. *For $x, y \in A$, the following are equivalent:*
(1) *$[x, y] \in \mathcal{L}$;*
(2) *$x^2 = y^2 = xy = 0$;*
(3) *$x^2 = xy = yx = 0$.*
Thus lines of $\mathcal{L}$ are full projective lines, containing $1 + |K|$ points of $\mathcal{P}$ (hence $s = |K|$). In particular, the incidence graph $\mathcal{I}(\mathcal{H})$ contains no 4-cycles.

Proof. This is immediate from the definitions and Lemma 7.6.

(8.4) Proposition. (1) *The lines of $\mathcal{L}$ on the point $[x] \in \mathcal{P}$ are all within the projective plane $xA \cap Ax$. These are the only lines of $\mathcal{L}$ within xA or Ax. In particular, $\mathcal{I}(\mathcal{H})$ contains no 6-cycles.*

(2) *Every projective line within the plane $xA \cap Ax$ and containing $[x]$ belongs to $\mathcal{L}$. In particular, each point is on exactly $1 + |K|$ lines of $\mathcal{L}$ (so that $t = |K|$).*

Proof. Suppose that $[y]$ is collinear with $[x]$. Then $xy = yx = 0$, so $y \in \ker(L_x) \cap \ker(R_x) = \bar{x}A \cap A\bar{x} = xA \cap Ax$, a 3-space by Theorem 5.1(7). Now suppose that the line $[y, z]$ is in xA or Ax, say xA. We may assume that $[x] \neq [y]$. Then $xz = yz = 0$, and $x, y, z \in \ker L_x \cap \ker L_y = xA \cap yA = [x, y]$ by Theorem 5.1(6). That is, $[y, z] = [x, y]$ contains the point $[x]$, completing (1).

If $y\,(\notin [x])$ belongs to $xA \cap Ax$, then $x^2 = xy = yx = 0$; so by Lemma 8.3 we have $[y] \in \mathcal{P}$ and $[x, y] \in \mathcal{L}$, giving (2).

It is a consequence of Proposition 8.4(1) that, when $[y]$ and $[z]$ are collinear in $\mathcal{H}$ with $[x]$ but not each other, we must have $xy \neq 0$ but $(x|y) = 0$. The converse is also valid.

(8.5) Proposition. *Let $[x], [y] \in \mathcal{P}$ with $(x|y) = 0$ but $z = xy \neq 0$. Then $[z]$ belongs to $\mathcal{P}$ and is the unique point for which $zA \cap Az$ contains $[x, y]$. In particular, $\mathcal{I}(\mathcal{H})$ contains no 8-cycles.*

PROOF. By Theorem 5.1(8),

$$(xA \cap Ax) \cap (yA \cap Ay) \subseteq xA \cap Ay = [xy]\,.$$

Therefore, there is at most one point collinear with both $[x]$ and $[y]$, that being $[z]$. (So $\mathcal{I}(\mathcal{H})$ has no 8-cycles.) On the other hand, if $[z]$ is a point of $\mathcal{P}$, then $x^2 = y^2 = z^2 = xz = zy = 0$; and $[z]$ is indeed collinear with both $[x]$ and $[y]$ by Lemma 8.3.

It remains to demonstrate that $0 = z^2 = (xy)^2$. We have $q(xy) = q(x)q(y) = 0$, so that xy is singular. If we can prove $x, y \in (xy)A \cap A(xy)$, then the result will follow from Theorem 5.1(7),(8). By Theorem 5.1(2), we must show that each of $\bar{x}(xy)$, $\bar{y}(xy)$, $(xy)\bar{x}$, and $(xy)\bar{y}$ is equal to 0. For the first and last, this is a direct consequence of Proposition 3.6(1). For the other two, we have

$$\begin{aligned} &\bar{y}(xy) = -\bar{x}(yy) + (x|y)y = 0 \quad \text{and} \\ &(xy)\bar{x} = -(xx)\bar{y} + (x|y)x = 0 \end{aligned}$$

by Proposition 3.6(2).

(8.6) LEMMA. *$\mathcal{I}(\mathcal{H})$ contains no 10-cycles.*

PROOF. Let the points of a 10-cycle be $[x_0], [x_1], [x_2], [x_3], [x_4]$ so that $[x_i]$ is collinear with $[x_{i-1}]$ and $[x_{i+1}]$ (indices read modulo 5). Then $(x_{i-1}|x_i) = (x_i|x_{i+1}) = 0$ is verified within $x_iA \cap Ax_i$, and $(x_{i-2}|x_i) = (x_i|x_{i+2}) = 0$ within $x_{i-1}A \cap Ax_{i-1}$ and $x_{i+1}A \cap Ax_{i+1}$. Therefore $W = [x_0, x_1, x_2, x_3, x_4]$ is singular. Furthermore it contains the 3-space $[x_{i-1}, x_i, x_{i+1}] = x_iA \cap Ax_i$, for each $i = 0, \ldots, 4$. By Proposition 2.3(1) at least one of x_0A or Ax_0 is equal to at least one of x_1A or Ax_1. This contradicts Theorem 5.1(3).

(8.7) LEMMA. (1) *$\mathcal{I}(\mathcal{H})$ has diameter 6 and contains 12-cycles.*

(2) *If $[x], [y] \in \mathcal{P}$ with $(x|y) \neq 0$, then the number of paths of length 6 in $\mathcal{I}(\mathcal{H})$ connecting $[x]$ and $[y]$ is equal to the number of lines of $\mathcal{H}$ on $[x]$.*

PROOF. Let $[x] \in \mathcal{P}$ and $\ell \in \mathcal{L}$. Then $x^\perp \cap \ell$ contains at least one point $[y] \in \mathcal{P}$, and the distance in $\mathcal{I}(\mathcal{H})$ from $[x]$ to $[y]$ is at most 4 by Propositions 8.4 and 8.5. Therefore $[x]$ and ℓ are at distance at most 5 in $\mathcal{I}(\mathcal{H})$. Since every line contains points and every point is on a line, $\mathcal{I}(\mathcal{H})$ has diameter at most 6.

By Proposition 8.4(1), the point set $\mathcal{P}$ can not span a singular subspace; so pairs $[x], [y] \in \mathcal{P}$ with $(x|y) \neq 0$ certainly exist. Such a pair of points must be at distance exactly 6, again by Propositions 8.4 and 8.5. Let ℓ be a line on $[x]$. By the previous paragraph, there is a path of length 6 connecting $[x]$ and $[y]$ via ℓ. This path must be unique since two different 5-paths from $[y]$ to ℓ would give rise to a shorter cycle in $\mathcal{I}(\mathcal{H})$. This gives (2). Since there is more than one choice for ℓ on $[x]$, we have 12-cycles, completing (1).

Results 8.3 through 8.7 prove Theorem 8.1 and its corollary.

(8.8) PROPOSITION. *The K-space $1^\perp$ of dimension 7 is the K-span of the point set $\mathcal{P}$ of $\mathcal{H}$.*

Proof. By Lemma 7.5 the set $\mathcal{P}$ consists of those singular 1-spaces within $1^\perp$. Let $[x] \neq [y]$ with $x, y \in M \backslash 1^\perp$, a hyperplane complement in some maximal singular space M of A. Then the identity element 1 is in both the hyperbolic 2-spaces $X = [x, \bar{x}]$ and $Y = [y, \bar{y}]$ with $X \neq Y$. Therefore the 6-spaces $X^\perp$ and $Y^\perp$ are contained in $1^\perp$ and, indeed, in $[\mathcal{P}]$, since both are split. As $X \neq Y$, also $X^\perp \neq Y^\perp$; so $[\mathcal{P}] \supseteq [X^\perp, Y^\perp] = 1^\perp$, as desired.

Next we describe the generalized hexagon of type ${}^3D_4$. Let F be a cubic extension of K with the Galois group of F over K generated by the element σ of order 3.

The F-space ${}_FA = F \otimes A = F^8$ can be given the structure of a split composition F-algebra with identity in such a way that, when σ is extended to a semi-linear transformation (also denoted σ) of the vector space ${}_FA$, the fixed vectors under σ are exactly those of ${}_KA = A = K^8$, the composition K-algebra under discussion previously. We use q to denote the quadratic form on ${}_FA$ as well as its restriction to ${}_KA$. (The construction can be thought of formally as a tensor product as in Lemma 3.3 above or more specifically through the location of some common basis for ${}_FA$ and ${}_KA$. The doubling construction does this easily. Also the Zorn construction makes the containment clear.)

The transformation σ is a semi-isometry of q, in that $q(x^\sigma) = q(x)^\sigma$ (and so σ is also a semi-isometry for the associated bilinear form $(\cdot|\cdot)$). Furthermore, it is an automorphism of ${}_FA$ as K-algebra (but not as F-algebra) and commutes with conjugation since $1 \in A$.

We define a new product on ${}_FA$:

$$x \circ y = \bar{x}^\sigma \bar{y}^{\sigma^2} .$$

This gives ${}_FA$ a new algebra structure, which we denote A_σ.

Notice that A_σ is a K-algebra but not an F-algebra:

(8.9) Lemma. *For $\alpha, \beta \in F$ and $x, y \in A_\sigma$, we have*

$$(\alpha x + \beta y) \circ (\gamma x + \delta y) = \alpha^\sigma \gamma^{\sigma^2} x \circ x + \alpha^\sigma \delta^{\sigma^2} x \circ y + \beta^\sigma \gamma^{\sigma^2} y \circ x + \beta^\sigma \delta^{\sigma^2} y \circ y .$$

A_σ admits the "twisted" composition law:

$$q(\alpha x \circ \beta y) = \alpha^\sigma \beta^{\sigma^2} q(x) q(y) .$$

Again the semi-linear transformation σ is a K-algebra automorphism.

Following the lead of our construction of the generalized hexagon $\mathcal{H}$, we consider a new incidence system $\mathcal{H}_\sigma$ with point set $\mathcal{P}_\sigma$, consisting of those $[x]$ with $x \in A_\sigma$ and $x \circ x = 0$, and line set $\mathcal{L}_\sigma$, consisting of those F-spaces of dimension 2 on which the $\circ$-multiplication is identically 0. By Lemma 8.9 the transformation σ induces an automorphism of $\mathcal{H}_\sigma$.

The basic result is then

(8.10) Theorem. *$\mathcal{H}_\sigma = (\mathcal{P}_\sigma, \mathcal{L}_\sigma)$ is a thick generalized hexagon.*

(8.11) Corollary. *If K is finite of order q, then $\mathcal{H} = (\mathcal{P}, \mathcal{L})$ is a thick generalized hexagon of order (q^3, q).*

As before, in the collinearity graph $\mathcal{C}(\mathcal{H}_\sigma)$, the points of $\mathcal{P}_\sigma$ at distance 1 from $[x]$ are those of $x \circ A_\sigma \cap A_\sigma \circ x$, those points at distance 2 are those of $x^\perp \backslash \{x \circ A_\sigma \cap A_\sigma \circ x\}$, and at distance 3 are those of $A_\sigma \backslash x^\perp$.

On the K-subalgebra ${}_KA$ the $\circ$-multiplication is particularly simple since each vector of ${}_KA$ is fixed by σ. Indeed if $x, y \in {}_KA$, then $x \circ y = \bar{x}\bar{y}$. In particular, if $x^2 = 0$ in ${}_KA$, then $x \circ x = \bar{x}\bar{x} = 0$. That is, each point ${}_K[x] = Kx$ of $\mathcal{H}$ can be identified with a point ${}_F[x] = Fx$ of $\mathcal{H}_\sigma$. Similarly, if $Kx + Ky = {}_K[x, y]$ is a line of $\mathcal{H}$, then $x^2 = y^2 = xy = yx = 0$, hence $Fx + Fy = {}_F[x, y]$ satisfies $x \circ x = y \circ y = x \circ y = y \circ x = 0$ and so is a line of $\mathcal{H}_\sigma$ (by Lemma 8.9).

Therefore we may think of the G_2 hexagon over K as being embedded in the ${}^3D_4$ hexagon over F as a fixed point subgeometry for the automorphism induced by σ. More precisely (but less elegantly), we have

(8.12) Proposition. *The map $\Phi\colon \mathcal{P} \cup \mathcal{L} \longrightarrow \mathcal{P}_\sigma \cup \mathcal{L}_\sigma$ given by $\Phi\colon {}_K[x] \mapsto {}_F[x]$ and $\Phi\colon {}_K[x, y] \mapsto {}_F[x, y]$ is an isomorphism of the incidence graph $\mathcal{I}(\mathcal{H})$ with an induced subgraph of $\mathcal{I}(\mathcal{H}_\sigma)$. We write $\Phi(\mathcal{I}(\mathcal{H})) = {}_K\mathcal{I}$, $\Phi(\mathcal{P}) = {}_K\mathcal{P}$, and $\Phi(\mathcal{L}) = {}_K\mathcal{L}$.*

The proof of Theorem 8.10 is very similar to that of Theorem 8.1. We begin with an abbreviated version of Theorem 5.1. More is true, but we only give those results of specific help in proving Theorem 8.10. Since most of these results are translations of the earlier results into the present language, we maintain parallel numbering.

(8.13) Theorem. *Throughout $x, y \in \mathcal{S}$, the set of nonzero singular vectors.*

(1) $\{\, z \circ A_\sigma \,|\, z \in \mathcal{S}\}$ and $\{\, A_\sigma \circ z \,|\, z \in \mathcal{S}\}$ are the two classes of maximal singular subspaces of A_σ.

(2) $a \in x \circ A_\sigma$ if and only if $a \circ x = 0$, and $a \in A_\sigma \circ x$ if and only if $x \circ a = 0$.

(3) Always $x \circ A_\sigma \neq A_\sigma \circ y$. Also $x \circ A_\sigma = y \circ A_\sigma$ if and only if $A_\sigma \circ x = A_\sigma \circ y$ if and only if $[x] = [y]$.

(6) Assume $(x|y) = 0$, and $[x] \neq [y]$. Then $x \circ A_\sigma \cap y \circ A_\sigma$ and $A_\sigma \circ x \cap A_\sigma \circ y$ both have dimension 2.

(7) Assume $x \circ y = 0$. Then $x \circ A_\sigma \cap A_\sigma \circ y$ has dimension 3.

(8) Assume $x \circ y \neq 0$. Then $x \circ A_\sigma \cap A_\sigma \circ y = [x \circ y]$ of dimension 1.

Proof. We have $z \circ A_\sigma = \bar{z}^\sigma A_\sigma$ and $A_\sigma \circ z = A_\sigma \bar{z}^{\sigma^2}$, so (1) is just a translation of Theorem 5.1(1). (It might be better to write $z \circ A_\sigma = \bar{z}^\sigma {}_FA$ and so forth.)

For (2) we have $a \in x \circ A_\sigma$ if and only if $a \in \bar{x}^\sigma A_\sigma$ if and only if $\overline{\bar{x}^\sigma} a = 0$. This can be rewritten as $0 = x^\sigma a = \bar{a}\bar{x}^\sigma = \bar{a}^\sigma \bar{x}^{\sigma^2} = a \circ x$.

Parts (3), (6), (7), and (8) are also direct consequences of the corresponding parts of Theorem 5.1. For instance $x \circ A_\sigma \cap A_\sigma \circ y = \bar{x}^\sigma A_\sigma \cap A_\sigma \bar{y}^{\sigma^2}$ has dimension

3 or 1, depending upon whether $\bar{x}^\sigma \bar{y}^{\sigma^2} = x \circ y$ is 0 or not. In the second case, the intersection 1-space is exactly $[\bar{x}^\sigma \bar{y}^{\sigma^2}] = [x \circ y]$. This proves (7) and (8).

The next few results, (8.14) through (8.18), are the present counterparts to the earlier (8.3) through (8.7). We do not give certain of the arguments where all that is needed is direct translation of the earlier proofs into the present language (but keep in mind that between Theorem 5.1(2) and Theorem 8.13(2) and elsewhere there are certain left-right distinctions).

(8.14) LEMMA. *For $x, y \in A_\sigma$, the following are equivalent:*
(1) $[x, y] \in \mathcal{L}_\sigma$*;*
(2) $x \circ x = y \circ y = x \circ y = 0$*.*
Thus lines of $\mathcal{L}_\sigma$ are full projective lines, containing $1 + |F|$ points of $\mathcal{P}_\sigma$ (hence $s = |F|$). In particular, the incidence graph $\mathcal{I}(\mathcal{H}_\sigma)$ contains no 4-cycles.

PROOF. By Lemma 8.9, (1) is equivalent to $0 = x \circ x = y \circ y = x \circ y = y \circ x$. It remains to prove that the first three identities imply the last. If $x \circ x = y \circ y = x \circ y = 0$, then $[x, y] \in y \circ A_\sigma \cap A \circ x$ by Theorem 8.13(2). But then Theorem 8.13(8) forces $y \circ x = 0$.

(8.15) PROPOSITION. *The lines of $\mathcal{L}_\sigma$ on the point $[x] \in \mathcal{P}_\sigma$ are all within the projective plane $x \circ A_\sigma \cap A_\sigma \circ x$. These are the only lines of $\mathcal{L}_\sigma$ within $x \circ A_\sigma$ or $A_\sigma \circ x$. In particular, $\mathcal{I}(\mathcal{H}_\sigma)$ contains no 6-cycles.*

As before, when $[y]$ and $[z]$ from $\mathcal{P}_\sigma$ are collinear in $\mathcal{H}_\sigma$ with $[x]$ but not each other, we have $x \circ y \neq 0$ and $(x|y) = 0$. One important distinction here is that Lemma 8.3(3) and so Proposition 8.4(2) have no counterparts in Lemma 8.14 and Proposition 8.15. Indeed we will see, in Proposition 8.19 below, that there are points in the plane $x \circ A_\sigma \cap A_\sigma \circ x$ not in $\mathcal{P}_\sigma$ and lines in this plane on $[x]$ but not in $\mathcal{L}_\sigma$.

(8.16) PROPOSITION. *Let $[x], [y] \in \mathcal{P}_\sigma$ with $(x|y) = 0$ but $z = x \circ y \neq 0$. Then $[z]$ belongs to $\mathcal{P}_\sigma$ and is the unique point for which $z \circ A_\sigma \cap A_\sigma \circ z$ contains $[x, y]$. In particular, $\mathcal{I}(\mathcal{H}_\sigma)$ contains no 8-cycles.*

(8.17) LEMMA. *$\mathcal{I}(\mathcal{H}_\sigma)$ contains no 10-cycles.*

(8.18) LEMMA. (1) *$\mathcal{I}(\mathcal{H}_\sigma)$ has diameter 6 and contains 12-cycles.*
(2) *If $[x], [y] \in \mathcal{P}_\sigma$ with $(x|y) \neq 0$, then the number of paths of length 6 in $\mathcal{I}(\mathcal{H}_\sigma)$ connecting $[x]$ and $[y]$ is equal to the number of lines of $\mathcal{H}_\sigma$ on $[x]$.*

PROOF. By Propositions 8.12, $\mathcal{I}(\mathcal{H}_\sigma)$ has the subgraph ${}_K\mathcal{I}$, which contains 12-cycles by Lemma 8.7. Furthermore, $[\mathcal{P}_\sigma] \supseteq [{}_K\mathcal{P}]$ of dimension 7 by Proposition 8.8 and so not singular. Therefore the proof of Lemma 8.7 goes over to prove the present lemma, provided we are sure that every point $[x]$ of $\mathcal{P}_\sigma$ is on a line of $\mathcal{L}_\sigma$ (as yet unclear).

Let ℓ be any line of $\mathcal{L}_\sigma$ (lines exist, for instance in ${}_K\mathcal{L}$). Then $[x]^\perp \cap \ell$ contains at least one point $[y] \in \mathcal{P}_\sigma$. By Propositions 8.15 and 8.16, either

$x \circ y = 0$ and $[x]$ and $[y]$ are collinear (perhaps even on ℓ) or $x \circ y \neq 0$, in which case $[x, x \circ y]$ is a line. In both cases there is at least one line on $[x]$, which is all that was need to complete the proof of the lemma.

As before, results 8.14 through 8.18 will prove Theorem 8.10 and its corollary when combined with the final

(8.19) PROPOSITION. *Each point of $\mathcal{P}_\sigma$ is on exactly $1+|K|$ lines of $\mathcal{L}_\sigma$ (that is, $t = |K|$).*

PROOF. By Lemma 8.18, if points $[x]$ and $[y]$ of $\mathcal{P}_\sigma$ have $(x|y) \neq 0$, then they are at distance 6 in $\mathcal{I}(\mathcal{H}_\sigma)$ and the number of lines of $\mathcal{L}_\sigma$ on $[x]$ is equal to the number of lines on $[y]$. To prove the proposition, we show:

(*i*) For every point $[x]$ there is a $[y]$ in the point set ${}_K\mathcal{P}$ of the subhexagon ${}_K\mathcal{I}$ with $(x|y) \neq 0$;

(*ii*) Every point $[y]$ of ${}_K\mathcal{P}$ is on exactly $1 + |K|$ lines of $\mathcal{L}_\sigma$.

For (*i*), note that $[{}_K\mathcal{P}] = 1^\perp$, of dimension 7 by Proposition 8.8. Thus $x^\perp$ can not contain $[{}_K\mathcal{P}]$ or its generating set ${}_K\mathcal{P}$. That is, there is a $[y]$ in ${}_K\mathcal{P}\backslash x^\perp$; and $(x|y) \neq 0$, as needed for (*i*).

For (*ii*), we at least know that there are $1 + |K|$ lines of the subhexagon ${}_K\mathcal{I}$ on $[y]$. Choose $u, v \in {}_KA$ so that $[y, u]$ and $[y, v]$ are two different lines of $\mathcal{L}_\sigma$ (indeed of ${}_K\mathcal{L}$) on $[y]$. Let $\ell = [u, v]$ be the projective line (over F) generated by $[u]$ and $[v]$. (This is not a line of $\mathcal{L}_\sigma$ by Propostion 8.15.)

By Proposition 8.15 every line of $\mathcal{L}_\sigma$ containing $[y]$ intersects ℓ in a unique point $[z] \in \mathcal{P}_\sigma$. Conversely by Lemma 8.14 and Proposition 8.15, for $[z] \in \ell$, the line $[y, z]$ is in $\mathcal{L}_\sigma$ if and only if $z \in \mathcal{P}_\sigma$. It remains to count the points $[z]$ of ℓ with $z \circ z = 0$.

Every point $[z] \in \ell$ either equals $[u]$ or is $[\alpha u + v]$, for some $\alpha \in F$. Those $[\alpha u + v]$ belonging to ${}_K\mathcal{P}$ are then exactly those with $\alpha \in K$. By Lemma 8.9

$$\begin{aligned}(\alpha u + v) \circ (\alpha u + v) &= (\alpha^\sigma)^2\, u \circ u + \alpha^\sigma u \circ v + \alpha^{\sigma^2} v \circ u + v \circ v \\ &= \alpha^\sigma\, u \circ v + \alpha^{\sigma^2} v \circ u\,.\end{aligned}$$

Setting $\alpha = 1$, we find $(u + v) \circ (u + v) = u \circ v + v \circ u$. But $u, v \in {}_KA$ with $[u + v] \in {}_K\mathcal{P}$, so equally well $(u + v) \circ (u + v) = (\overline{u + v})\,(\overline{u + v}) = 0$. Therefore $0 = u \circ v + v \circ u$ and $-u \circ v = v \circ u$. This in turn gives

$$(\alpha u + v) \circ (\alpha u + v) = (\alpha^\sigma - \alpha^{\sigma^2})\, u \circ v\,.$$

Now $u \circ v = \bar{u}\bar{v} \neq 0$ by Proposition 8.4. Therefore, those α for which $(\alpha u + v) \circ (\alpha u + v) = 0$ are exactly those with $0 = \alpha^\sigma - \alpha^{\sigma^2}$ hence $0 = \alpha - \alpha^\sigma$. As K is the fixed field for σ in F, we conclude that $(\alpha u + v) \circ (\alpha u + v) = 0$ if and only if $\alpha \in K$. Thus with $[u]$ included, exactly $1 + |K|$ of the points $[z]$ on ℓ have $z \circ z = 0$. This completes the proposition (and so the theorem and corollary).

9. Some other stuff of interest

1. *Constructions.* It would be instructive to write down the relationship between the doubling construction and the other constructions, particularly Zorn's construction of the split octonions.

More detailed structure results, particularly for the octonions, would be nice.

2. *Automorphisms.* Both the Zorn and doubling constructions make certain automorphisms of the octonions and the associated G_2 hexagon quite apparent. For the Zorn construction, this is handled extensively in [2]. Buekenhout and Cohen give detailed remarks on the BN-properties of the automorphism groups of both the G_2 and ${}^3D_4$ hexagons.

The uniqueness properties of doubling lead to various algebra automorphisms, via the follow corollary to Proposition 6.1.

(9.1) Corollary. *Let A, B, t, and γ be as in Proposition 6.1. For any algebra automorphism g_B of B and any $t' \in B^\perp$ with $q(t') = -\gamma$, there is an isomorphism g of the subalgebras $A = B + Bt$ and $A' = B + Bt'$ with $g|_B = g_B$ and $g(t) = t'$.*

For instance, there is an isomorphism that is trivial on B and takes t to any suitable t'. A consequence is transitivity of the automorphism group of A on the elements of $1^\perp$ with any fixed nonzero q value.

We have shown that the automorphism group of a D_4 geometry has a normal subgroup with quotient the symmetric group of degree three, acting naturally on the D_4 diagram. Clearly the kernel of this homomorphism contains $PO_8^+(K)'$ acting naturally, but we have not proved that this is the full kernel nor have we shown the action of the triality automorphism on the kernel. This is discussed well in Van der Blij and Springer [1], parts of which are presented in [2].

3. *Identities.* Our presentation is missing a proof of the alternative law $(xy)y = x(y^2)$ and the Moufang identity $(xy)(zx) = (x(yz))x$. The alternative law can be derived easily from Proposition 3.6 and Corollary 3.7. The Moufang identity then follows but is difficult; see [2].

The Moufang identity is important for a thorough study of the topics mentioned here. For instance it is intimately connected with automorphisms of D_4 geometries [1]. It is surprising that the Moufang identity does not play a role in our other discussions. Perhaps that means we are missing some easy and enlightening arguments.

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