Alternating forms and transitive locally grid geometries

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1 Introduction

Let V be a vector space over K. For each nonnegative integer k let $\mathcal{P}_k(V)$ be the set of k-subspaces of V. For positive d ($\leq \dim V$), let $\mathcal{L}_d(V)$ be the set of pairs (U, W) with $U \in \mathcal{P}_{d-1}(V)$, $W \in \mathcal{P}_{d+1}(V)$, and $U \leq W$. We also set $\mathcal{B}_d^-(V) = \mathcal{P}_{d-1}(V)$, $\mathcal{B}_d^+(V) = \mathcal{P}_{d+1}(V)$, and $\mathcal{B}_d(V) = \mathcal{B}_d^-(V) \cup \mathcal{B}_d^+(V)$.

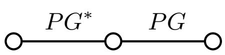
In the case $V = K^4$ and $d = 2$, we can view $\mathcal{P}_2(V)$ and $\mathcal{L}_2(V)$ as the points and lines of the Klein quadric. Then $(\mathcal{B}_2^-(V), \mathcal{P}_2(V), \mathcal{B}_2^+(V))$ is the associated rank 3 geometry of type $D_3\,(= A_3)$ and $(\mathcal{P}_2(V), \mathcal{L}_2(V), \mathcal{B}_2(V))$ the associated rank 3 polar space of type C_3.

We look at several geometries related to these for the Klein quadric. Let $\mathcal{A}_d(V) = (\mathcal{P}_d(V), \mathcal{L}_d(V))$. (Here and throughout, d is some positive integer but $\dim_K V$ may be infinite unless stated otherwise.) $\mathcal{A}_d(V)$ is a partial linear space with point set $\mathcal{P}_d(V)$ and line set $\mathcal{L}_d(V)$ whose members (U, W) we often identify with the $1 + |K|$ distinct d-spaces (that is, incident points) in between the $(d-1)$-space U and the $(d+1)$-space W. $\mathcal{A}_d(V)$ is called a *Grassmann space* and, sometimes, a *d-Grassmann space*.

We associate two rank 3 geometries with $\mathcal{A}_d(V)$. The first is the *Grassmannian geometry* (or *d-Grassmannian geometry*)

$$\mathcal{G}_d(V) = (\mathcal{B}_d^-(V), \mathcal{P}_d(V), \mathcal{B}_d^+(V))$$

with diagram



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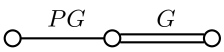
If d is 1 or $\dim_K V - 1$, then $\mathcal{B}_d^-(V)$ or $\mathcal{B}_d^+(V)$ is empty (respectively), and we have the rank 2 geometry of points and lines for a projective space over K or K^{op}.

The second geometry is

$$\mathcal{CG}_d(V) = (\mathcal{P}_d(V), \mathcal{L}_d(V), \mathcal{B}_d(V))$$

with diagram



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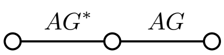
Again if d is 1 or $\dim_K V - 1$, one of the $\mathcal{B}_d^\epsilon(V)$ is empty; so the line set and block set of $\mathcal{CG}_d(V)$ are identical, being equal to the line set of $\mathcal{G}_d(V)$. For this reason, we will only consider the geometry $\mathcal{CG}_d(V)$ for d not equal to 1 or $\dim_K V - 1$. With this restriction in mind, we may (when convenient) identity $\mathcal{CG}_d(V)$ with the collinearity graph Γ of $\mathcal{A}_d(V)$ and $\mathcal{G}_d(V)$, since the three types of objects correspond to three identifiable classes of cliques in Γ. (If d is 1 or $\dim_K V - 1$, the collinearity graph Γ is complete.) The vertices of Γ are the points of $\mathcal{P}_d(V)$ with P_1 and P_2 adjacent precisely when collinear in $\mathcal{A}_d(V)$, that is, when $P_1 \cap P_2$ is a $(d-1)$-space or, equivalently, $\langle P_1, P_2 \rangle$ is a $(d+1)$-space. The maximal cliques in this graph are precisely the members of $\mathcal{B}_d(V)$ (each identified with the set of those points incident to it). Two distinct maximal cliques intersect trivially, in a vertex, or in a line of $\mathcal{L}_d(V)$. The diagram indicates that the residue of a point in the geometry $\mathcal{CG}_d(V)$ is a grid graph. More precisely, the collinearity graph Γ is locally a $|K|$-clique extension of a $u \times v$ grid, where $1 + |K|u$ is the number of d-spaces containing a fixed $(d-1)$-space (from $\mathcal{B}_d^-(V)$, giving maximal "minus" cliques) and $1 + |K|v$ is the number of d-spaces contained in a fixed $(d+1)$-space (from $\mathcal{B}_d^+(V)$, giving maximal "plus" cliques). A plus clique and minus clique are either disjoint or they intersect in the $|K| + 1$ points of a line.

A *geometric hyperplane* of $\mathcal{A}_d(V) = (\mathcal{P}_d(V), \mathcal{L}_d(V))$ is a proper subset $\mathcal{H}$ of $\mathcal{P}_d(V)$ such that, for every line $\ell \in \mathcal{L}_d(V)$, either $\ell \subseteq \mathcal{H}$ or $|\ell \cap \mathcal{H}| = 1$. If d is 1, then a geometric hyperplane is a hyperplane in the usually sense, and dually when $d = \dim_K V - 1$. It is convenient to allow the possibility $d = \dim_K V$, where $\mathcal{A}_d(V)$ has a single point and $\mathcal{H}$ is empty. A geometric hyperplane is always a subspace of the partial linear space $\mathcal{A}_d(V)$.

With $\mathcal{H}$ we associate the rank 3 geometry $\mathcal{G}_d(V)_\mathcal{H} = (\mathcal{B}_\mathcal{H}^-, \mathcal{P}_\mathcal{H}, \mathcal{B}_\mathcal{H}^+)$, which has been called an *affine Grassmannian*. The geometry $\mathcal{G}_d(V)_\mathcal{H}$ consists of point set $\mathcal{P}_\mathcal{H} = \mathcal{P}_d(V) \backslash \mathcal{H}$, the hyperplane complement consisting of all d-spaces not in $\mathcal{H}$, together with the set $\mathcal{B}_\mathcal{H}^-$ of all $(d-1)$-spaces contained in a point of $\mathcal{P}_\mathcal{H}$ and the set $\mathcal{B}_\mathcal{H}^+$ of all $(d+1)$-spaces containing a point of $\mathcal{P}_\mathcal{H}$. For d equal to 1 or $\dim_K V - 1$ this gives the usual affine geometries for V and its dual.

When d is not 1 or $\dim_K V - 1$, the geometry $\mathcal{G}_d(V)_\mathcal{H}$ has diagram



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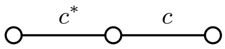
Its collinearity graph $\mathcal{CG}_d(V)_\mathcal{H}$ is locally a $(|K| - 1)$-clique extension of a $u \times v$ grid graph. The lines of $\mathcal{L}_\mathcal{H}$ contain only $|K|$ points of $\mathcal{P}_\mathcal{H}$, since each meets $\mathcal{H}$ in a point. Of particular note is the case $K = \mathbb{F}_2$. There the affine Grassmannian $\mathcal{G}_d(V)_\mathcal{H}$ is a (c^*, c)-geometry, belonging to the diagram



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and the collinearity graph $\mathcal{CG}_d(V)_{\mathcal{H}}$ is a locally grid graph.

There are earlier papers on affine Grassmannians over arbitrary fields and division rings [2, 7, 12, 20]. (Note that the "affine Grassmannian" terminology has also been used elsewhere with a related but different definition.) The most fundamental and deep result in the area is due to Shult:

(1.1) Theorem. (Shult [20, Theorem 2].) *Let V be a finite dimensional vector space over a field, and let $\mathcal{H}$ be a geometric hyperplane in $\mathcal{A}_d(V)$. Then for some nonzero alternating d-linear form f on V, the set $\mathcal{H} = \mathcal{H}_f$ consists of all the f-degenerate d-subspaces of V.*

Thus the study of geometric hyperplanes of $\mathcal{A}_d(V)$ is equivalent to the study of alternating bilinear forms on V.

We say that the subgroup G of $\mathrm{Aut}(\mathcal{G}_d(V)_{\mathcal{H}})$ is *transitive* on $\mathcal{G}_d(V)_{\mathcal{H}}$ if G is transitive on $\mathcal{P}_{\mathcal{H}}$. The subgroup G of $\mathrm{Aut}(\mathcal{G}_d(V)_{\mathcal{H}})$ is *flag-transitive* on $\mathcal{G}_d(V)_{\mathcal{H}}$ if G is transitive on incident triples (U, P, W) (*chambers* or *maximal flags*) with $U \in \mathcal{B}^-_{\mathcal{H}}$, $P \in \mathcal{P}_{\mathcal{H}}$, and $W \in \mathcal{B}^+_{\mathcal{H}}$. Similarly the subgroup G of $\mathrm{Aut}(\mathcal{CG}_d(V)_{\mathcal{H}})$ is *flag-transitive* on $\mathcal{CG}_d(V)_{\mathcal{H}}$ if G is transitive on incident triples (P, ℓ, B) with point $P \in \mathcal{P}_{\mathcal{H}}$, line $\ell \in \mathcal{L}_{\mathcal{H}}$, and maximal clique $B \in \mathcal{B}^-_{\mathcal{H}} \cup \mathcal{B}^+_{\mathcal{H}}$. If the chamber (P, ℓ, B) has $B \in \mathcal{B}^-_{\mathcal{H}}$, then it is a $-$-*chamber*, whereas when $B \in \mathcal{B}^+_{\mathcal{H}}$ it is a $+$-*chamber*. The geometry is *transitive* (respectively, *flag-transitive*) if its full automorphism group is transitive (respectively, flag-transitive) on it.

The focus of this paper is an interesting class of transitive affine Grassmannians.

(1.2) Theorem. *Let $s\colon V \times V \longrightarrow K$ be a symplectic form on V. Let $\mathcal{H} = \mathcal{H}_{s,d}$ be the set of all d-subspaces P of V for which $s|_P$ is degenerate. Assume additionally that $d \geq \mathrm{rank}(V, s) \neq 0$, so that $\mathcal{H}$ does not consist of all d-subspaces of V.*

Then $\mathcal{H}$ is a geometric hyperplane of $\mathcal{A}_d(V) = (\mathcal{P}_d(V), \mathcal{L}_d(V))$, and the associated affine Grassmannian $\mathcal{G}_d(V)_{\mathcal{H}}$ is connected and transitive. Indeed $\mathcal{G}_d(V)_{\mathcal{H}}$ is flag-transitive if and only if either

(a) s is nondegenerate or

(b) $d = \mathrm{rank}(V, s)$.

We then have

(1.3) Corollary. *Let $1 < d < \dim_K V - 1$. The locally grid geometry $\mathcal{CG}_d(V)_{\mathcal{H}}$ is connected and transitive. Indeed $\mathcal{CG}_d(V)_{\mathcal{H}}$ is flag-transitive if and only if $2d = \dim_K V$ and*

(a) s is nondegenerate or

(b) $d = \mathrm{rank}(V, s)$.

These are proved (in slightly greater detail) in Section 3. We refer to all the geometric hyperplanes of Theorem 1.2 as being of *Pfaffian* type for reasons that will become clear in Section 4 below. Similarly the related affine Grassmannians and locally grid geometries will be called Pfaffian.

The affine Grassmannians of Theorem 1.2(b) and Corollary 1.3(b) with d equal to $\mathrm{rank}(V, s)$ are examples of *attenuated spaces* [21, 22]. If R is a fixed subspace of codimension d in V, then the set of all d-subspaces of V that meet R nontrivially is a geometric hyperplane $\mathcal{H}_R$ of $\mathcal{A}_d(V)$ called an *attenuated hyperplane*. The associated affine Grassmannian and locally grid geometry will also be called attenuated. The examples of Theorem 1.2(b) come from the attenuated hyperplane $\mathcal{H}_R$ for $R = \mathrm{Rad}(V, s)$ of codimension d (so attenuated affine Grassmannians are Pfaffian if and only if d is even). If d is 1 or $\dim_K V - 1$, then all geometric hyperplanes are actual hyperplanes of V or V^* (respectively) and so are attenuated.

The author first noted the constructions and results of Theorem 1.2 and Corollary 1.3 early in the 1980's in the context of locally grid graphs, which corresponds to the case $K = \mathbb{F}_2$. Also in 1981 Yoshimi Egawa [9] pointed out that attenuated spaces over $\mathbb{F}_2$ (presented in a slightly different form) give rise to locally grid graphs.

Meixner and Pasini's census of flag-transitive locally grid graphs [18] lists attenuated spaces but not the Pfaffian examples. Shult's work and surveys [21, 22] on (c^*, c)-geometries (and locally grid graphs) mention affine Grassmannians over $\mathbb{F}_2$ as a source, but only the attenuated examples are discussed specifically.

Shult suggests that attenuated spaces might provide the only examples of flag-transitive affine Grassmannians. As we have seen above, that is not the case. But Shult's feeling that the condition is highly restrictive is certainly correct. In fact, the finite attenuated and nondegenerate Pfaffian examples are characterized by flag-transitivity.

(1.4) THEOREM. *Let $\mathcal{H}$ be a geometric hyperplane of $\mathcal{A}_d(V)$ with $V = \mathbb{F}_q^n$. Then $\mathcal{G}_d(V)_{\mathcal{H}}$ is flag-transitive if and only if we have one of*

(a) $\mathcal{H} = \mathcal{H}_R$, an attenuated hyperplane with respect to some R of codimension d in V;

(b) d is even and $\mathcal{H} = \mathcal{H}_{s,d}$ for some nondegenerate symplectic s form on V.

The theorem is related to Witt's theorem [24, 7.4], since it classifies alternating forms admitting semisimilarity groups that are transitive on all subconfigurations of a certain isometry type.

(1.5) COROLLARY. *Let $1 < d < \dim_K V - 1$. Let $\mathcal{H}$ be a geometric hyperplane of $\mathcal{A}_d(V)$ with $V = \mathbb{F}_q^n$. Then $\mathcal{C}\mathcal{G}_d(V)_{\mathcal{H}}$ is flag-transitive if and only if $2d = n$ and we have one of*

(a) $\mathcal{H} = \mathcal{H}_R$, an attenuated hyperplane with respect to some R of codimension d in V;

(b) d is even and $\mathcal{H} = \mathcal{H}_{s,d}$ for some nondegenerate symplectic s form on V.

Of particular interest is the locally grid graph case $q = 2$.

In Section 2 we give various properties of alternating bilinear forms and the related geometric hyperplanes. Section 3 contains a proof of Theorem 1.2 and

Corollary 1.3. In Section 4 we construct alternating d-linear forms that give rise to the attenuated and Pfaffian examples. Section 5 is devoted to the proof of Theorem 1.4 and Corollary 1.5.

2 Alternating forms and related hyperplanes

An *alternating d-linear form f* on the vector space V over the field K is a map $f\colon V^d \longrightarrow K$ that is linear in each variable and is 0 whenever two arguments are equal. We also view an alternating d-linear form as an element of $(\bigwedge^d V)^*$. A *symplectic form* on the K-space V is an alternating 2-linear form. The *radical* of f on V is the subspace

$$\mathrm{Rad}(V,f) = \{\, v \in V \mid f(v_1,\dots,v_{d-1},v) = 0, \text{ for all } v_1,\dots,v_{d-1} \in V \,\}\,.$$

The *rank* of V, $\mathrm{rank}(V,f)$, is the codimension of its radical. When $d = 2$ the rank is always even; for this and other well-known geometry, see [24].

Alternating d-linear forms with $d > 2$ are less familiar; see [1, 20]. After Aschbacher [1], for each subspace P of V we set

$$P^\theta = \{\, v \in V \mid f(p_1,\dots,p_{d-1},v) = 0, \text{ for all } p_1,\dots,p_{d-1} \in P \,\}\,.$$

Especially V^θ is $\mathrm{Rad}(V,f)$. More generally $P \cap P^\theta = \mathrm{Rad}(P, f|_P)$. The subspace P (possibly V itself) is *f-degenerate* (or just *degenerate* if the context is clear) when its radical $\mathrm{Rad}(P, f|_P)$ is nonzero. If $\mathrm{Rad}(P, f|_P) = 0$, then P is *f-nondegenerate* or *nondegenerate*.

The *zero form* is identically 0 on V^d.

(2.1) Lemma. *Let f be an alternating d-linear form on the K-space V, and let P be a subspace of dimension at most d. Then either f restricted to P is the zero form and $P \le P^\theta$ or $\dim_K P = d$, P is nondegenerate, and $V = P \oplus P^\theta$.*

Proof. The arguments of [1, (1.4-5)] for $d = 3$ go over to the general case.

Let $v_1,\dots,v_d$ have span W with basis $\{\, w_j \mid j \in J \,\}$. Then $f(v_1,\dots,v_d)$ is a K-linear combination of the $f(w_{j_1},\dots,w_{j_d})$ with $j_i \le j_{i+1}$. In particular $f(v_1,\dots,v_d) = 0$ if $\dim_K W < d$. Indeed, for any subspace W of dimension k, the codimension in V of W^θ is at most $\binom{k}{d-1}$.

If $\dim_K P < d$ or $P \cap P^\theta \neq 0$, then (with an appropriate choice of the w_j) each $f(w_{j_1},\dots,w_{j_d})$ is 0; so the restriction of f to P is zero and $P \le P^\theta$. Otherwise $P \cap P^\theta = 0$, and nondegenerate P has dimension $d = \binom{d}{d-1}$; so we have $V = P \oplus P^\theta$.

Given an alternating d-linear form f on V and a map $A \in \Gamma\mathrm{L}(V)$, semilinear with respect to the automorphism α of the field F, we can define a new alternating d-linear form f^A on V by

$$f^A(v_1,\dots,v_i,\dots,v_n) = f(v_1A^{-1},\dots,v_iA^{-1},\dots,v_nA^{-1})^\alpha\,.$$

The semilinear map A is then a *semisimilarity* of (V,f) if there is a nonzero scalar c with $cf^A = f$. It is an *isometry* if additionally $c = 1$ and $\alpha = 1$.

(2.2) Proposition. *Let f be a nonzero alternating d-linear form on V, and let $\mathcal{H} = \mathcal{H}_f$ be the associated geometric hyperplane of $\mathcal{A}_d(V)$. Then the following groups are all equal:*

(*a*) *the stablizer $\Gamma\mathrm{L}(V)_{\mathcal{H}}$ of $\mathcal{H}$ in $\Gamma\mathrm{L}(V)$;*
(*b*) *the group of all semisimilarites of (V, f);*
(*c*) *the group of all $A \in \Gamma\mathrm{L}(V)$ for which*

$$f(v_1, \dots, v_i, \dots, v_n) = 0 \iff f(v_1A, \dots, v_iA, \dots, v_nA) = 0\,,$$

for all $v_1, \dots, v_n \in V$.

Proof. For any $A \in \Gamma\mathrm{L}(V)$,

$$f(v_1, \dots, v_i, \dots, v_n) = 0 \iff f^A(v_1A, \dots, v_iA, \dots, v_nA) = 0\,.$$

Especially semisimilarites of (V, f) belong to the group of (c). In turn, semilinear maps A as in (c) take f-degenerate d-spaces to f-degenerate d-spaces and so are contained in the stabilizer $\Gamma\mathrm{L}(V)_{\mathcal{H}}$.

It remains to prove that the stabilizer consists of semisimilarities. Let $A \in \Gamma\mathrm{L}(V)_{\mathcal{H}}$. The implication of the previous paragraph shows that $\mathcal{H}_f = \mathcal{H}_{f^A}$. By [20, Cor. 2.1.1] we have $f = cf^A$, for some nonzero constant c. Therefore A is a semisimilarity, as desired.

The induced semisimilarity group $\mathrm{P}\Gamma\mathrm{L}(V)_{\mathcal{H}}$ is clearly a subgroup of the automorphism groups $\mathrm{Aut}(\mathcal{G}_d(V)_{\mathcal{H}})$ and $\mathrm{Aut}(\mathcal{C}\mathcal{G}_d(V)_{\mathcal{H}})$, and usually we have equality.

(2.3) Theorem. *Let V be a K-space, and let $\mathcal{H}$ be geometric hyperplane of $\mathcal{A}_d(V)$. Assume also that $2 < d < \dim_K V - 2$ or that $|K| > 2$. Then*

$$\mathrm{Aut}(\mathcal{G}_d(V)_{\mathcal{H}}) = \mathrm{Aut}(\mathcal{G}_d(V))_{\mathcal{H}} = \mathrm{P}\Gamma\mathrm{L}(V)_{\mathcal{H}}\,.$$

This is due to Blok [2, 3] and answers a question of Shult [22]. For $K = \mathbb{F}_2$ there are genuinely exceptional cases when d is any of 1, 2, $\dim_K V - 2$, or $\dim_K V - 1$; see [3]. (The equality $\mathrm{Aut}(\mathcal{G}_d(V)) = \mathrm{P}\Gamma\mathrm{L}(V)$ is Chow's Theorem [6].)

Theorem 2.3 allows us largely to restrict our attention to automorphisms of affine Grassmannians that are induced by semilinearities of the parent vector space, that is, semisimilarities of (V, f).

Assume that V has finite dimension $n > 2$ over the field K. Then V possesses many *dualities*. These are bijections that take i-subspaces to j-subspaces, for $i + j = n$, and that respect inclusion. If δ is a duality, then δ^2 belongs to the group $\mathrm{P}\Gamma\mathrm{L}(V)$; so $\mathrm{P}\Gamma\mathrm{L}(V)$ has index 2 in $\mathrm{P}\Gamma\mathrm{L}^*(V)$, its extension by all dualities. Correspondingly the induced stabilizer $\mathrm{P}\Gamma\mathrm{L}^*(V)_{\mathcal{H}}$ can be twice as big as $\mathrm{P}\Gamma\mathrm{L}(V)_{\mathcal{H}}$.

(2.4) Lemma. *Let $\dim_K V = n > d$. Further let δ be a duality of V and $\mathcal{H}$ a geometric hyperplane of $\mathcal{A}_d(V)$.*

(1) *$\mathcal{H}^\delta$ is a geometric hyperplane of $\mathcal{A}_{n-d}(V)$.*

(2) *If $\mathcal{H} = \mathcal{H}^\delta$, then $n = 2d$ and δ induces an automorphism of $\mathcal{CG}_d(V)_\mathcal{H}$ that switches $\mathcal{B}^-_\mathcal{H}$ and $\mathcal{B}^+_\mathcal{H}$. In this case $\mathrm{P\Gamma L}^*(V)_\mathcal{H} \setminus \mathrm{P\Gamma L}(V)_\mathcal{H}$ is the set of all such stabilizing dualities.*

Proof. (1) Let A, B, C be subspaces of V with $A \leq B \leq C$. Then $A^\delta \geq B^\delta \geq C^\delta$. Thus $(A, B, C) \in \mathcal{A}_d(V)$ if and only if $(C^\delta, B^\delta, A^\delta) \in \mathcal{A}_{n-d}(V)$.

(2) If $\mathcal{H} = \mathcal{H}^\delta$, then $\mathcal{A}_d(V) = \mathcal{A}_{n-d}(V)$ and $d = n - d$. Thus δ switches $\mathcal{B}^-$ and $\mathcal{B}^+$ and fixes $\mathcal{H}$, so $(\mathcal{B}^-_\mathcal{H})^\delta = \mathcal{B}^+_\mathcal{H}$. The stabilizing dualities are those of the coset $\delta\,\mathrm{P\Gamma L}(V)_\mathcal{H}$.

The following important observation is essentially due to Shult [20] and has several helpful consequences.

(2.5) Proposition. (1) *If $\mathcal{H}$ and $\mathcal{J}$ are geometric hyperplanes of $\mathcal{A}_d(V)$ with $\mathcal{H} \subseteq \mathcal{J}$, then $\mathcal{H} = \mathcal{J}$.*

(2) *For any geometric hyperplane $\mathcal{H}$ of $\mathcal{A}_d(V)$, the associated affine Grassmannian $\mathcal{G}_d(V)_\mathcal{H}$ and locally grid geometry $\mathcal{CG}_d(V)_\mathcal{H}$ are connected.*

Proof. It is a well-known and elementary fact that a geometric hyperplane is maximal if and only its complement is connected. For finite dimensional V a full proof of (2) can be found in [3, Lemma 3.3], and the infinite dimensional case follows easily.

(2.6) Theorem. *Let $\mathcal{H}$ be a geometric hyperplane of $\mathcal{A}_d(V)$. Then $\mathrm{P\Gamma L}(V)_\mathcal{H}$ is flag-transitive on $\mathcal{G}_d(V)_\mathcal{H}$ but reducible on V if and only if we have $\mathcal{H} = \mathcal{H}_R$, an attenuated hyperplane with respect to some R of codimension d in V.*

Proof. Let R be a nontrivial invariant subspace.

Transitivity on incident pairs from $(\mathcal{B}^-_\mathcal{H}, \mathcal{P}_\mathcal{H})$ implies, for $P \in \mathcal{P}_\mathcal{H}$, that either $P \leq R$ or $P \cap R = 0$. If $P \leq R$, then $\langle \mathcal{P}_\mathcal{H} \rangle \leq R$. Since R is proper in V there are $(d+1)$-spaces W with $W \cap R = P$. But then, for any hyperplane U of P, the line (U, W) meets $\mathcal{P}_\mathcal{H}$ in the unique point P, an impossibility. Thus $P \cap R = 0$.

Transitivity on $\mathcal{B}^+_\mathcal{H}$ and the existence of members of $\mathcal{B}^+_\mathcal{H}$ meeting R nontrivially imply that everything in $\mathcal{B}^+_\mathcal{H}$ meets R nontrivially. This forces the codimension of R to be d.

Now $\mathcal{H}$ contains $\mathcal{H}_R$, so we have equality by Proposition 2.5.

(2.7) Lemma. *Let $1 < d < \dim_K V - 1$, and let $\mathcal{H}$ be a geometric hyperplane of $\mathcal{A}_d(V)$. Then $\mathrm{Aut}(\mathcal{G}_d(V)_\mathcal{H})$ is a subgroup of of index at most 2 in $\mathrm{Aut}(\mathcal{CG}_d(V)_\mathcal{H})$. If the index is equal to 2, then $2d = n$ and the automorphisms of $\mathrm{Aut}(\mathcal{CG}_d(V)_\mathcal{H}) \setminus \mathrm{Aut}(\mathcal{G}_d(V)_\mathcal{H})$ switch $\mathcal{B}^-_\mathcal{H}$ and $\mathcal{B}^+_\mathcal{H}$.*

Proof. The lines of $\mathcal{G}_d(V)_\mathcal{H}$ and $\mathcal{CG}_d(V)_\mathcal{H}$ are exactly those cliques of $\mathcal{CG}_d(V)_\mathcal{H}$ of size at least 2 that occur as the intersection of two distinct maximal cliques. In particular $\mathrm{Aut}(\mathcal{G}_d(V)_\mathcal{H}) \leq \mathrm{Aut}(\mathcal{CG}_d(V)_\mathcal{H})$.

The locally grid geometry $\mathcal{CG}_d(V)_\mathcal{H}$ is connected by Proposition 2.5. On the maximal clique set $\mathcal{B}_\mathcal{H}$ of $\mathcal{CG}_d(V)_\mathcal{H}$ define a relation by $B_1 \sim B_2$ when $B_1 \cap B_2$ contains a unique point of $\mathcal{P}_\mathcal{H}$. $\mathcal{B}_\mathcal{H}$ then has exactly two connected components under $\sim$, namely $\mathcal{B}^-_\mathcal{H}$ and $\mathcal{B}^+_\mathcal{H}$. Any automorphism g of $\mathcal{CG}_d(V)_\mathcal{H}$ must therefore either switch these two components or fix both. In the second case, g induces an automorphism of $\mathcal{G}_d(V)_\mathcal{H}$.

In the first case, there is a maximal clique $B^- \in \mathcal{B}^-$ with $(B^-)^g = B^+ \in \mathcal{B}^+_\mathcal{H}$. If $K = \mathbb{F}_q$, we have $q^d = |B^-|$ and $|B^+| = q^{n-d}$, where $n = \dim_K V$. If K is not finite, then the lines of $\mathcal{L}$ give B^- the structure of an affine space K^d and B^+ the structure of K^{n-d}. In either event, $\dim_K V = n$ is finite and $d = n - d$, as desired.

3 Proof of Theorem 1.2

This section is mainly concerned with proving of Theorem 1.2 and Corollary 1.3. Theorem 3.5 presents similar results for attenuated hyperplanes and the associated attenuated spaces. Also Proposition 3.6 gives the complete classification of hyperplanes in the cases where d is one of 1, 2, $\dim V - 2$, or $\dim V - 1$.

The result which motivated this paper is

(3.1) PROPOSITION. *Let V be a vector space over the field K and s a symplectic form on V. Let d be even with $0 < d \le \mathrm{rank}(V, s)$. Then*

$$\mathcal{H}_{s,d} = \{P \le V \,|\, \dim_K P = d,\ \mathrm{Rad}(P, s|_P) \ne 0\,\}$$

is a geometric hyperplane of the d-Grassmann space $\mathcal{A}_d(V)$.

PROOF. As even $d \le \mathrm{rank}(V, s)$, $\mathcal{H}_{s,d}$ is not all of $\mathcal{P}_d(V)$. Let $U \le P \le W$ be a chamber of the Grassmannian $\mathcal{G}_d(V)$. If the line (U, W) is not entirely within $\mathcal{H}$, then we may choose P to be nondegenerate (under $s|_P$). In that case $\mathrm{Rad}(W, s|_W) = \langle r\rangle$ has dimension 1, and any d-space complement to $\langle r\rangle$ in W is nondegenerate. In particular, of the $|K| + 1$ distinct d-spaces $\langle U, w\rangle \le W$, only $\langle U, r\rangle$ is degenerate.

(3.2) LEMMA. *Let $\mathcal{H} = \mathcal{H}_{s,d}$ be as in Proposition 3.1. Then in its action on $\mathcal{G}_d(V)_\mathcal{H}$ the group $PGL(V)_\mathcal{H}$ has chamber orbits*

$$\{(U, P, W) \,|\, U \in \mathcal{B}^-_\mathcal{H},\ P \in \mathcal{P}_\mathcal{H},\ W \in \mathcal{B}^+_\mathcal{H},\ W \cap \mathrm{Rad}(V, s) = 0\,\}$$

and

$$\{(U, P, W) \,|\, U \in \mathcal{B}^-_\mathcal{H},\ P \in \mathcal{P}_\mathcal{H},\ W \in \mathcal{B}^+_\mathcal{H},\ W \cap \mathrm{Rad}(V, s) \ne 0\,\}\,.$$

In particular $\mathrm{P\Gamma L}(V)_\mathcal{H}$ *is transitive on the points of $\mathcal{G}_d(V)_\mathcal{H}$ (and $\mathcal{CG}_d(V)_\mathcal{H}$). Indeed* $\mathrm{P\Gamma L}(V)_\mathcal{H}$ *acts flag-transitively on $\mathcal{G}_d(V)_\mathcal{H}$ if and only if we have one of*

(*a*) *s is nondegenerate or*

(*b*) *$d = \mathrm{rank}(V, s)$, the codimension of* $\mathrm{Rad}(V, s)$.

PROOF. Clearly $\mathrm{Sp}(V, s) \leq \Gamma\mathrm{L}(V)_{\mathcal{H}}$.

Let $P \in \mathcal{P}_{\mathcal{H}}$ and $X = P^{\perp}$, so that $V = P \oplus X$.

If P_1 and P_2 are two points, then $\langle P_1, P_2 \rangle$ can be embedded in a finite dimensional subspace Y of V with $\mathrm{Rad}(Y, s|_Y) = Y \cap \mathrm{Rad}(V, s)$. A symplectic basis for each of P_1 and P_2 can be extended to symplectic bases for Y. Thus there is a $g \in \mathrm{Sp}(Y, s|_Y)$ taking P_1 to P_2. As Y has a perpendicular complement in V, all members of $\mathrm{Sp}(Y, s|_Y)$ extend to elements of $\mathrm{Sp}(V, s)$.

We now need only prove that the stabilizer of P is transitive on the chambers $U \leq P \leq W$.

Any $(d-1)$-space $U \leq P$ equals $u^{\perp} \cap P$ for a unique 1-space $\langle u \rangle \leq P$. Any $(d+1)$-space $W \geq P$ equals $\langle P, x \rangle$ for the unique 1-space $\langle x \rangle = X \cap W$. Thus the orbits of the stabilizer of P in $\Gamma\mathrm{L}(V)_{\mathcal{H}}$ are the same as the orbits on 1-spaces within P (for $\mathcal{B}^-_{\mathcal{H}}$) and within $X = P^{\perp}$ (for $\mathcal{B}^+_{\mathcal{H}}$). As before $\mathrm{Sp}(P, s|_P) \times \mathrm{Sp}(X, s|_X) \leq \Gamma\mathrm{L}(V)_{\mathcal{H}}$. The factors act independently on P and X. $\mathrm{Sp}(P, s|_P)$ is then transitive on those $U \in \mathcal{B}^-_{\mathcal{H}}$ incident to P. $\mathrm{Sp}(X, s|_X)$ is transitive on those 1-spaces of X contained in $\mathrm{Rad}(X, s|_S) = \mathrm{Rad}(V, s)$ and on those 1-spaces of X outside of $\mathrm{Rad}(X, s|_X)$. This gives the two orbits described. $\Gamma\mathrm{L}(V)_{\mathcal{H}}$ acts flag-transitively precisely when one of these two orbits is empty. The second is empty if and only if $\mathrm{Rad}(V, s) = 0$, that is, when s is nondegenerate as in (a). The first is empty when every $(d+1)$-space that contains the nondegenerate d-space P meets $\mathrm{Rad}(V, s)$. This is the case if and only if d is equal to the codimension of $\mathrm{Rad}(V, s)$ as in (b).

No element of $\mathrm{P}\Gamma\mathrm{L}(V)_{\mathcal{H}}$ can switch $\mathcal{B}^-_{\mathcal{H}}$ and $\mathcal{B}^+_{\mathcal{H}}$; so, for $1 < d < \dim_K V - 1$, the group $\mathrm{P}\Gamma\mathrm{L}(V)_{\mathcal{H}}$ can never act flag-transitively on $\mathcal{CG}_d(V)_{\mathcal{H}}$. But the group $\mathrm{P}\Gamma\mathrm{L}^*(V)_{\mathcal{H}}$ sometimes can.

(3.3) LEMMA. *Let $\mathcal{H}$ be a geometric hyperplane of $\mathcal{A}_d(V)$ with $1 < d < \dim_K V - 1$. The group $\mathrm{P}\Gamma\mathrm{L}(V)_{\mathcal{H}}$ acts flag-transitively on $\mathcal{G}_d(V)_{\mathcal{H}}$ if and only if it is transitive on the set of $-$-chambers and the set of $+$-chambers of $\mathcal{CG}_d(V)_{\mathcal{H}}$.*

PROOF. First suppose that $\mathrm{P}\Gamma\mathrm{L}(V)_{\mathcal{H}}$ is flag-transitive on $\mathcal{G}_d(V)_{\mathcal{H}}$. Let (P_1, ℓ_1, B_1) and (P_2, ℓ_2, B_2) be two chambers of $\mathcal{CG}_d(V)_{\mathcal{H}}$, with $B_1, B_2 \in \mathcal{B}^+$, say (a similar argument handling the case $B_1, B_2 \in \mathcal{B}^-$). Let B_1^- and B_2^- from $\mathcal{B}^-$ be chosen so that each line ℓ_i is contained in the line of $\mathcal{CG}_d(V)$ determined by the incident pair (B_i^-, B_i). By flag-transitivity, there is a $g \in \mathrm{P}\Gamma\mathrm{L}(V)_{\mathcal{H}}$ with $(B_1^-, P_1, B_1)^g = (B_2^-, P_2, B_2)$. This forces $\ell_1^g = \ell_2$. Therefore g is an element of $\mathrm{P}\Gamma\mathrm{L}(V)_{\mathcal{H}}$ with $(P_1, \ell_1, B_1)^g = (P_2, \ell_2, B_2)$, as desired.

Now suppose that $\mathrm{P}\Gamma\mathrm{L}(V)_{\mathcal{H}}$ is transitive on the set of $-$-chambers and the set of $+$-chambers of $\mathcal{CG}_d(V)_{\mathcal{H}}$. Let (U_1, P_1, W_1) and (U_2, P_2, W_2) be two chambers of $\mathcal{G}_d(V)_{\mathcal{H}}$. By assumption, there is a $g \in \mathrm{P}\Gamma\mathrm{L}(V)_{\mathcal{H}}$ with $(P_1, W_1)^g = (P_2, W_2)$, so we may take $P = P_1 = P_2$ and $W = W_1 = W_2$. Each $U_i \in \mathcal{B}^-_{\mathcal{H}}$, with (U_i, P, W) a chamber of $\mathcal{G}_d(V)_{\mathcal{H}}$, meets W in a line $\ell_i = U_i \cap W$, which contains the point P; and $\ell_i = \ell_j$ if and only if $i = j$. Again by flag-transitivity, there is an $f \in \mathrm{P}\Gamma\mathrm{L}(V)_{\mathcal{H}}$ with $(P, \ell_1, W)^f = (P, \ell_2, W)$. But then $(U_1, P, W)^f = (U_2, P, W)$, as desired.

(3.4) COROLLARY. *Let $\mathcal{H} = \mathcal{H}_{s,d}$ be as in Proposition 3.1 with $1 < d < \dim_K V - 1$. The group $\mathrm{P\Gamma L}^*(V)_{\mathcal{H}}$ acts flag-transitively on $\mathcal{CG}_d(V)_{\mathcal{H}}$ if and only if $\mathrm{P\Gamma L}(V)_{\mathcal{H}}$ is flag-transitive on $\mathcal{G}_d(V)_{\mathcal{H}}$ and $\dim_K V = 2d$.*

PROOF. If $\mathrm{P\Gamma L}^*(V)_{\mathcal{H}}$ is flag-transitive on $\mathcal{CG}_d(V)_{\mathcal{H}}$, then $\mathrm{P\Gamma L}(V)_{\mathcal{H}}$, of index 2, has at most two orbits on the chambers of $\mathcal{CG}_d(V)_{\mathcal{H}}$. These must be the $+$-chambers and the $-$-chambers; so, by the lemma, $\mathrm{P\Gamma L}(V)_{\mathcal{H}}$ is flag-transitive on $\mathcal{G}_d(V)_{\mathcal{H}}$. As $\mathrm{P\Gamma L}(V)_{\mathcal{H}}$ has two orbits, it is proper in $\mathrm{P\Gamma L}^*(V)_{\mathcal{H}}$; therefore we have $2d = n$ by Lemma 2.4.

To prove the converse, we consider the two cases of Lemma 3.2 in turn. If, as in Lemma 3.2(a), s is nondegenerate, then the polarity $X \mapsto X^{\perp}$ induces an automorphism of $\mathcal{CG}_d(V)_{\mathcal{H}}$. Indeed it maps the chamber (U, P, W) to the chamber $(W^{\perp}, P^{\perp}, U^{\perp})$.

In Lemma 3.2(b), let $R = \mathrm{Rad}(V, s)$. Then there is a duality δ of V (here definitely not associated with s) that takes R to itself. But then δ induces an automorphism of $\mathcal{CG}_d(V)_{\mathcal{H}}$ that interchanges $\mathcal{B}^-_{\mathcal{H}}$ and $\mathcal{B}^+_{\mathcal{H}}$, as desired.

The dual of an attenuated affine Grassmannian is again attenuated. As seen in the proof of the previous corollary, when the parent symplectic form is nondegenerate on V, a dual Pfaffian affine Grassmannian is Pfaffian. On the other hand, Lemma 3.2 proves that the dual of a Pfaffian affine Grassmannian is not Pfaffian when the form is degenerate of rank larger than d.

The results on Pfaffian geometric hyperplanes are, in particular, valid for attenuated hyperplanes for which the subspace R has even codimension in the space V. This restriction on codimension is spurious.

(3.5) THEOREM. *Let R be a nonzero subspace of finite codimension d $(0 < d < \dim V)$ in the space V over the field K. Set*

$$\mathcal{H} = \mathcal{H}_R = \{P \leq V \mid \dim_K P = d,\ P \cap R \neq 0\}.$$

(1) *The set $\mathcal{H}_R$ is a geometric hyperplane of the d-Grassmann space $\mathcal{A}_d(V)$.*

(2) *The affine Grassmannian $\mathcal{G}_d(V)_{\mathcal{H}}$ is connected and admits $\mathrm{PGL}(V)_R = \mathrm{PGL}(V)_{\mathcal{H}}$ acting flag-transitively.*

(3) *The locally grid geometry $\mathcal{CG}_d(V)_{\mathcal{H}}$ is connected and transitive, and it admits $\mathrm{P\Gamma L}^*(V)_R = \mathrm{P\Gamma L}^*(V)_{\mathcal{H}}$ acting flag-transitively if and only if $\dim_K V = 2d$.*

PROOF. (1) Let P_1 and P_2 be two d-spaces that are disjoint from R with $\dim(P_1 \cap P_2) = d - 1$ and $\dim\langle P_1, P_2\rangle = d + 1$. Then $\langle P_1, P_2\rangle \cap R = T$ has dimension 1, and $Q = \langle P_1 \cap P_2, T\rangle$ is the unique d-subspace of $\langle P_1, P_2\rangle$ that meets R nontrivially. Therefore $\mathcal{H}$ is a geometric hyperplane.

(2) Connectivity is true in general by Proposition 2.5.

The definition of $\mathcal{H}$ uniquely determines R, so $\mathrm{GL}(V)_{\mathcal{H}} = \mathrm{GL}(V)_R$. Let $\bar{v}_1, \dots \bar{v}_d$ be a basis of $\bar{V} = V/R$, and assume that the two d-spaces P_1 and P_2 of $\mathcal{G}_d(V)_{\mathcal{H}}$ have bases $x_1, \dots, x_d$ and $y_1, \dots y_d$, respectively, with $\bar{x}_i = \bar{v}_i = \bar{y}_i$, for all i. Then the map $x_i \mapsto y_i$, for all i, extends to a member of $\mathrm{GL}(V)_R$ that is trivial on R and $\bar{V}$ and takes P_1 to P_2. As $\mathrm{GL}(V)_{R,P} \simeq \mathrm{GL}(\bar{V}) \times \mathrm{GL}(R)$ is transitive on the pairs of $(d-1)$- and $(d+1)$-spaces incident to the fixed d-space P of $\mathcal{G}_d(V)_{\mathcal{H}}$, we have the desired flag-transitivity.

(3) This follows directly from (2) as in Corollary 3.4.

REMARK. Parts (1) and (2) of the theorem remain valid over arbitrary division rings without change to either statement or proof. Part (3) must be changed to say that $\mathrm{P\Gamma L}^*(V)_{\mathcal{H}}$ is flag-transitive on $\mathcal{CG}_d(V)_{\mathcal{H}}$ if and only if $\dim_K V = 2d$ and K has an anti-isomorphism. The backwards implication follows as before, but the direct part requires additional proof.

As we have seen in Theorem 2.3, the cases in which d is one of 1, 2, $\dim_K V-2$, or $\dim_K V - 1$ are exceptional. But these are the cases that admit full classification of the geometric hyperplanes.

(3.6) PROPOSITION. *Let V be a vector space over K and d a positive integer. Let $\mathcal{H}$ be a geometric hyperplane of $\mathcal{A}_d(V)$.*

(1) *If $d = 1$ or $d = \dim_K V - 1$, then $\mathcal{H}$ is an attenuated hyperplane and $\mathrm{Aut}(\mathcal{G}_d(V)_{\mathcal{H}})$ is flag-transitive on $\mathcal{G}_d(V)_{\mathcal{H}}$.*

(2) *Let $d = 2$. Then $\mathcal{H} = \mathcal{H}_{s,2}$ is a Pfaffian hyperplane for some symplectic form s. The group $\mathrm{Aut}(\mathcal{G}_d(V)_{\mathcal{H}})$ is flag-transitive on $\mathcal{G}_d(V)_{\mathcal{H}}$ if and only if either $\mathrm{rank}(V,s) = 2$ or $\mathrm{rank}(V,s) = \dim V$. Furthermore $\mathrm{Aut}(\mathcal{CG}_d(V)_{\mathcal{H}})$ is flag-transitive on $\mathcal{CG}_d(V)_{\mathcal{H}}$ if and only if either $\mathrm{rank}(V,s) = 2$ or $\mathrm{rank}(V,s) = \dim V = 4$.*

(3) *Let $d = \dim_K V - 2$. Then $\mathcal{H}$ is the dual of a Pfaffian hyperplane $\mathcal{H}_{s,2}$ for some symplectic form s. Again $\mathrm{Aut}(\mathcal{G}_d(V)_{\mathcal{H}})$ is flag-transitive on $\mathcal{G}_d(V)_{\mathcal{H}}$ if and only if either $\mathrm{rank}(V,s) = 2$ or $\mathrm{rank}(V,s) = \dim V$. Furthermore $\mathrm{Aut}(\mathcal{CG}_d(V)_{\mathcal{H}})$ is flag-transitive on $\mathcal{CG}_d(V)_{\mathcal{H}}$ if and only if either $\mathrm{rank}(V,s) = 2$ or $\mathrm{rank}(V,s) = \dim V = 4$.*

PROOF. Part (1) is immediate, and (3) follows directly from (2) by duality.

When $d = 2$, as in (2), Shult's Theorem 1.1 implies that $\mathcal{H} = \mathcal{H}_{s,2}$ for some symplectic form s. The examples given are indeed flag-transitive by Lemma 3.2 and Corollary 3.3. It remains to prove that these are the only flag-transitive examples.

Suppose that $\mathrm{rank}(V,s) > 2$ and that $\mathcal{G}_d(V)_{\mathcal{H}}$ is flag-transitive. Assume that s is degenerate. By Lemma 3.2, $\mathrm{P\Gamma L}(V)_{\mathcal{H}}$ has two orbits on chambers—that of the radical chambers (B^-, P, B^+) with $B^+ \cap \mathrm{Rad}(V,s) \neq 0$ and that of the nonradical chambers. We must show that $\mathrm{Aut}(\mathcal{G}_d(V)_{\mathcal{H}})$ does not fuse these two orbits. But this is clear, since every nonradical chamber is in a nondegenerate Pfaffian subgeometry $\mathcal{G}_2(W)_{\mathcal{H}'}$, for an s-nondegenerate 4-space W containing B^+ and $\mathcal{H}' = \mathcal{H} \cap \mathcal{P}_2(W)$, while no radical chamber is in a subgeometry with this isomorphism type. The contradiction proves s to be nondegenerate.

The previous paragraph and Lemma 2.7 show that the only flag-transitive $\mathcal{CG}_d(V)_{\mathcal{H}}$ are the stated examples.

Proof of Theorem 1.2 and Corollary 1.3:

Proposition 2.5(2) shows that the geometries $\mathcal{G}_d(V)_\mathcal{H}$ of Theorem 1.2 and $\mathcal{CG}_d(V)_\mathcal{H}$ of Corollary 1.3 are both connected. All other parts of Theorem 1.2 and Corollary 1.3 then follow from the various results of this section.

4 Alternating forms and Pfaffians

Shult's Theorem 1.1 encourages us to look for alternating forms on V that give rise to attenuated and Pfaffian geometric hyperplanes.

First consider the case of the attenuated hyperplane $\mathcal{H}_R$. Let P be a d-space complement to R in V, and let $e_1, \dots, e_d$ be a K-basis of P. For each $v_i \in V$, we have $v_i = \sum_{j=1}^d a_{i,j} e_j + r_i$, for unique $r_i \in R$ and $a_{i,j} \in K$. Then $f(v_1, \dots, v_d) = \det([a_{i,j}]_{i,j})$ is an alternating d-linear form. The form f is 0 precisely when the span $Q = \langle v_1, \dots, v_d \rangle$ projects onto a proper subspace of P, that is, when Q intersects R nontrivially and so belongs to $\mathcal{H}_R$. The projection λ_j of each v_i onto its j-th coordinate $a_{i,j}$ is a linear functional on V. We can think of the construction of f in terms of the canonical embedding of $\bigwedge^d V^*$ in $(\bigwedge^d V)^*$ given by

$$(v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_d) \cdot (\lambda_1 \wedge \cdots \wedge \lambda_j \wedge \cdots \wedge \lambda_d) = \det([v_i \lambda_j]_{i,j}) .$$

(4.1) Proposition. *Let $\mathcal{H}_R$ be an attenuated hyperplane of $\mathcal{A}_d(V)$. Let $\lambda_1, \dots, \lambda_d$ be a basis of $\ker_{V^*} R$. Then $\lambda_R = \wedge_{j=1}^d \lambda_j$ is an alternating d-linear form on V for which the λ_R-degenerate d-spaces of V are exactly those of $\mathcal{H}_R$. Conversely, for each pure vector λ of $\wedge^d V^*$ (thought of as an element of $(\wedge^d V)^*$), the pure vectors of the hyperplane $\ker \lambda$ give an attenuated hyperplane of $\mathcal{A}_d(V)$.*

We next want to find an alternating form that produces the Pfaffian examples. This is particularly easy when $K = \mathbb{F}_2$.

(4.2) Proposition. *Let s be a symplectic form on the $\mathbb{F}_2$-space V. The map $\delta(v_1, \dots, v_d) = \det([s(v_i, v_j)]_{i,j})$ is an alternating d-linear form on V with $\delta(v_1, \dots, v_d) \neq 0$ if and only if $\langle v_1, \dots, v_d \rangle$ is an s-nondegenerate d-space of V.*

Proof. Since, in Theorem 4.3 below, we will prove the appropriate generalization to arbitrary fields, we only sketch the proof here.

Consider the map $D\colon V^d \times V^d \longrightarrow K$ given by

$$D(v_1, \dots, v_d; w_1 \dots, w_d) = \det([s(v_i, w_j)]_{i,j})$$

and the related

$$\delta(v_1, \dots, v_d) = D(v_1, \dots, v_d; v_1 \dots, v_d) .$$

Over the field K (initially arbitrary), we use the basic properties of determinants and the Gram matrix $[s(v_i, v_j)]_{i,j}$ to see that the maps D and δ have the properties:

(i) $D(v_1, \ldots, v_d; w_1 \ldots, w_d) = 0$ if $v_i = v_j$ or $w_i = w_j$ for some pair of indices $i \neq j$;

(ii) $D(v_1, \ldots, v_d; w_1 \ldots, w_d)$ is linear in each variable.

(iii) $\delta(v_1, \ldots, v_d) \neq 0$ if and only if $\langle v_1, \ldots, v_d \rangle$ is an s-nondegenerate d-space of V;

By (i) and (ii) D induces a symmetric bilinear form (also denoted D) on $\wedge^d V \times \wedge^d V$ via

$$D(v_1 \wedge \cdots \wedge v_d, w_1 \wedge \cdots \wedge w_d) = D(v_1, \ldots, v_d; w_1 \ldots, w_d)\,,$$

and δ is the quadratic form coming from the diagonal of D. By (iii) the pure vectors of $\wedge^d V$ that are nonsingular for δ are exactly those coming from d-spaces that are nondegenerate for s.

In the special case $K = \mathbb{F}_2$, the quadratic form δ is in fact a linear form, giving the result.

Degeneracy of a form is detected by whether or not its *discriminant*, the determinant of a Gram matrix, is zero. In the proof of the previous proposition, we have seen that the discriminant provides a symmetric bilinear and diagonal quadratic form on the exterior power that indicate degeneracy. To find a linear form, as desired, we wish to “take the square root” of this quadratic form. In characteristic 2 the quadratic form is additive and thus over $\mathbb{F}_2$ is its own square root, but this is an anomaly. We must instead look for a general way of finding a square root for the discriminant. Luckily, for symplectic forms there is such a general function, namely the Pfaffian. Our presentation of Pfaffians is similar to that of Chevalley [5, p. 57] which uses exponentials in exterior algebras.

As before, let V be a K-vector space and s a symplectic form on V. For every tuple $(v_1, \ldots, v_d)$ from V, we define an element of the exterior algebra $\bigwedge V$ by setting

$$\mathrm{Pf}(v_1, \ldots, v_d) = \prod_{\{i,j\} \in \mathcal{U}} (1 + s(v_i, v_j) v_i v_j)\,,$$

the product being over the set $\mathcal{U}$ of all unordered pairs $\{i, j\}$ from $\{1, \ldots, d\}$. (Note that $s(v_i, v_j)v_i v_j = s(v_j, v_i) v_j v_i$.) Every term of Pf has even degree, so the calculation of Pf can be done within the commutative subalgebra $\bigwedge_{\mathrm{even}} V$ of $\bigwedge V$.

We now define a map $\mathrm{pf}\colon V^d \longrightarrow K$ by looking at projection onto the d-graded component of Pf:

$$\mathrm{pf}(v_1, \ldots, v_d) v_1 \cdots v_d = \mathrm{Pf}(v_1, \ldots, v_d)|_d\,.$$

Especially $\mathrm{pf}(v_1, v_1) = s(v_1, v_2)$. Here it is understood that $\mathrm{pf}(v_1, \ldots, v_d)$ is 0 whenever $v_1 \cdots v_d$ is the 0 element of $\bigwedge V$. (In particular, since $\mathrm{Pf} \in \bigwedge_{\mathrm{even}} V$, pf is identically 0 for odd d.) We call $\mathrm{pf}(v_1, \ldots, v_d)$ the *Pfaffian* of the ordered set of vectors $v_1, \ldots, v_d \in V$.

The remainder of this section is devoted to a proof of

(4.3) THEOREM. *The map* pf *is an alternating d-linear form on V with* $\mathrm{pf}(v_1, \dots, v_d) \neq 0$ *if and only if* $\langle v_1, \dots, v_d \rangle$ *is an s-nondegenerate d-space of V.*

The map pf is indeed alternating.

(4.4) PROPOSITION. (1) *If $v_i = v_j$ for $i \neq j$, then* $\mathrm{pf}(v_1, \dots, v_d) = 0$.
(2) *If $\sigma \in \mathrm{Sym}(d)$, then* $\mathrm{pf}(v_1, \dots, v_d) = \mathrm{sgn}(\sigma)\ \mathrm{pf}(v_{\sigma(1)}, \dots, v_{\sigma(d)})$.

PROOF. (1) is clear, since in this case $v_1 \cdots v_d$ is 0 in $\bigwedge V$.

For (2) we start with $\mathrm{Pf}(v_{\sigma(1)}, \dots, v_{\sigma(d)}) = \mathrm{Pf}(v_1, \dots, v_d)$ from the definition and remarks following it. Therefore, the d-graded piece is

$$\begin{aligned} \mathrm{pf}(v_{\sigma(1)}, \dots, v_{\sigma(d)}) v_{\sigma(1)} \cdots v_{\sigma(d)} &= \mathrm{pf}(v_1, \dots, v_d) v_1 \cdots v_d \\ &= \mathrm{pf}(v_1, \dots, v_d) \mathrm{sgn}(\sigma)\, v_{\sigma(1)} \cdots v_{\sigma(d)}\,, \end{aligned}$$

as desired.

We first prove Cayley's theorem, showing that the Pfaffian provides the desired square root of the discriminant. This is the fundamental property of the Pfaffian and is crucial in our proof of Theorem 4.3.

(4.5) THEOREM. $\mathrm{pf}(v_1, \dots, v_d)^2 = \det[s(v_i, v_j)]_{i,j}$.

PROOF. If $\{v_1, \dots, v_d\}$ is linearly dependent, then the lefthand side is 0 by definition. But in that case, the Gram matrix $[s(v_i, v_j)]_{i,j}$ will be that of a form with rank less than d, so its determinant will also be 0. Therefore we may assume that $v_1, \dots, v_d$ form a linearly independent set in V. Both sides are also 0 if d is odd, so we may assume $d = 2m$.

We begin with a standard reduction ([11, Lemma 7.2.1],[23, Prop.2.2]) in the calculation of the discriminant

$$\det[s(v_i, v_j)]_{i,j} = \sum_{\sigma \in \mathrm{Sym}(d)} \mathrm{sgn}(\sigma) \prod_i s(v_i, v_{\sigma(i)})\,.$$

Write $\mathrm{Sym}(d) = \mathcal{F} \cup \mathcal{O} \cup \mathcal{E}$, where $\mathcal{F}$ consists of those permutations with a fixed point, $\mathcal{O}$ consists of those permutations with no fixed points but with orbits of odd length, and $\mathcal{E}$ consists of those permutations with all orbits of even length.

For $\sigma \in \mathcal{F}$, $\prod_i s(v_i, v_{\sigma(i)}) = 0$ as $s(v_j, v_j) = 0$ for a fixed point j. For $\sigma \in \mathcal{O}$, among the odd cycles of σ, let c be the cycle containing the point of $\{1, \dots d\}$ of smallest value. Set $\sigma' = c^{-2}\sigma$. Then
(*i*) $\sigma \neq \sigma' \in \mathcal{O}$ and $(\sigma')' = \sigma$;
(*ii*) $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\sigma')$;
(*iii*) $\prod_i s(v_i, v_{\sigma(i)}) = -\prod_i s(v_i, v_{\sigma'(i)})$.
We conclude that

$$\det[s(v_i, v_j)]_{i,j} = \sum_{\sigma \in \mathcal{E}} \mathrm{sgn}(\sigma) \prod_i s(v_i, v_{\sigma(i)})\,. \qquad (1)$$

We will prove that squaring the Pfaffian gives this calculation of the discriminant.

A 1-factor of the complete graph on $\{1,2,\ldots,d=2m\}$ is an m edge subgraph of valency 1, one edge on each vertex. We let $\mathcal{I}$ be the set of all such 1-factors. Starting from the definition of the Pfaffian, we find in $\bigwedge_{\text{even}}$

$$\begin{aligned} \mathrm{pf}(v_1,\ldots,v_d)v_1\cdots v_d &= \mathrm{Pf}(v_1,\ldots,v_d)|_d \\ &= \Big(\prod_{\{i,j\}\in\mathcal{U}}(1+s(v_i,v_j)v_iv_j)\Big)\Big|_d \\ &= \sum_{I\in\mathcal{I}}\prod_{\{i,j\}\in I}s(v_i,v_j)v_iv_j\,. \end{aligned} \tag{2}$$

Let R have dimension d. We replace V by $V\oplus R$ and extend the form s so that R is in its radical. Assume that $r_1,\ldots,r_d$ are linearly independent elements of R, and set $w_i=v_i+r_i$, for $i=1,\ldots,d$. Then $\{v_1,\ldots,v_d,w_1,\ldots,w_d\}$ is linearly independent and

$$s(v_i,v_j)=s(v_i,w_j)=s(w_i,v_j)=s(w_i,w_j)\,,$$

for all i,j. By definition and (2) we have

$$\begin{aligned} &\mathrm{pf}(v_1,\ldots,v_d)^2v_1\cdots v_dw_1\cdots w_d= \\ &\quad= \mathrm{pf}(v_1,\ldots,v_d)v_1\cdots v_d\,\cdot\,\mathrm{pf}(w_1,\ldots,w_d)w_1\cdots w_d \\ &\quad= \Big(\sum_{I\in\mathcal{I}}\prod_{\{i,j\}\in I}s(v_i,v_j)v_iv_j\Big)\Big(\sum_{J\in\mathcal{I}}\prod_{\{k,l\}\in J}s(w_k,w_l)w_kw_l\Big)\,. \end{aligned}\tag{3}$$

We now need a bijection between the permutations of $\mathcal{E}$ and ordered pairs of 1-factors:

$$\mathcal{E}\longleftrightarrow\mathcal{I}_G\times\mathcal{I}_W\quad\text{given by}\quad\sigma\longleftrightarrow(I,J)\,,$$

where, in order to distinguish 1-factors, we call the I from $\mathcal{I}_G$ $(=\mathcal{I})$ green and the J from $\mathcal{I}_W$ $(=\mathcal{I})$ white.

First, let I (green) and J (white) be two 1-factors. The graph $I\cup J$ has degree 2 and so is a union of cycles. Each cycle has edges of alternating colors (green, white, green, white,...) and so has even length. To find a unique $\sigma\in\mathcal{E}$, we only need to orient each cycle c of σ (for instance, to distinguish $(1,2,3,4)$ from $(4,3,2,1)$). This we do by finding the smallest value i_c in c and then orienting the cycle in the direction of the green edge on i_c.

Conversely, if we begin with $\sigma\in\mathcal{E}$, then each cycle c breaks into two partial 1-factors. To find unique corresponding $I\in\mathcal{I}_G$ and $J\in\mathcal{I}_W$ we must color each partial 1-factor. We do this by coloring each edge $i_c\sim\sigma(i_c)$ green. This gives the bijection.

From (1) and (3), we will be done when we have proven

$$\begin{aligned} &\mathrm{sgn}(\sigma)\Big(\prod_i s(v_i,v_{\sigma(i)})\Big)\,v_1\cdots v_d\,w_1\cdots w_d= \\ &\qquad\prod_{\{i,j\}\in I}s(v_i,v_j)v_iv_j\prod_{\{k,l\}\in J}s(w_k,w_l)w_kw_l\,, \end{aligned}\tag{4}$$

for any fixed $\sigma \longleftrightarrow (I,J)$. To do this, we rewrite the righthand side.

Let $z_i = v_i$ and $z'_{\sigma(i)} = v_{\sigma(i)}$, if the edge $i \sim \sigma(i)$ is green. Let $z_i = w_i$ and $z'_{\sigma(i)} = w_{\sigma(i)}$, if the edge $i \sim \sigma(i)$ is white. Then

$$\prod_{\{i,j\}\in I} s(v_i,v_j)v_iv_j \prod_{\{k,l\}\in J} s(w_k,w_l)w_kw_l = \prod_i s(z_i,z'_{\sigma(i)}) \prod_i z_i\, z'_{\sigma(i)}\,.$$

Since $s(z_i,z'_{\sigma(i)}) = s(v_i,v_{\sigma(i)})$, the proof of (4) is reduced to verification of

$$\operatorname{sgn}(\sigma)\, v_1\cdots v_d\, w_1\cdots w_d = \prod_i z_i\, z'_{\sigma(i)}\,. \tag{5}$$

We rewrite $\prod_i z_i\, z'_{\sigma(i)}$ using a suggestive two line notation:

$$\prod_i z_i\, z'_{\sigma(i)} = \prod_i \begin{matrix} z_i \\ z'_{\sigma(i)} \end{matrix} = \begin{matrix} z_1 & \cdots & z_i & \cdots & z_d \\ z'_{\sigma(1)} & & z'_{\sigma(i)} & & z'_{\sigma(d)} \end{matrix}\,.$$

For instance, the product $abhkxy$ would be written $\begin{matrix} a & h & x \\ b & k & y \end{matrix}$. In the exterior algebra $\bigwedge V$, the identity

$$z_j\, z'_p\, z_{j+1}\, z'_q = -z_j\, z'_q\, z_{j+1}\, z'_p$$

thus becomes

$$\begin{matrix} z_j & z_{j+1} \\ z'_p & z'_q \end{matrix} = - \begin{matrix} z_j & z_{j+1} \\ z'_q & z'_p \end{matrix}\,.$$

Therefore

$$\prod_i z_i\, z'_{\sigma(i)} = \begin{matrix} z_1 & \cdots & z_i & \cdots & z_d \\ z'_{\sigma(1)} & & z'_{\sigma(i)} & & z'_{\sigma(d)} \end{matrix} = \operatorname{sgn}(\sigma) \begin{matrix} z_1 & \cdots & z_i & \cdots & z_d \\ z'_1 & & z'_i & & z'_d \end{matrix}\,.$$

Here always $\{z_i, z'_i\} = \{v_i, w_i\}$. Half the edges of $I \cup J$ are white, so for exactly $m = d/2$ values of i we have $z_i = w_i$ and $z'_i = v_i$. Hence

$$\begin{aligned}
\prod_i z_i\, z'_{\sigma(i)} &= \operatorname{sgn}(\sigma)(-1)^m \begin{matrix} v_1 & \cdots & v_i & \cdots & v_d \\ w_1 & & w_i & & w_d \end{matrix} \\
&= \operatorname{sgn}(\sigma)(-1)^m v_1\, w_1 \cdots v_i\, w_i \cdots v_d\, w_d \\
&= \operatorname{sgn}(\sigma)(-1)^m (-1)^{2m(2m-1)/2} v_1 \cdots v_d\, w_1 \cdots w_d \\
&= \operatorname{sgn}(\sigma) v_1 \cdots v_d\, w_1 \cdots w_d\,,
\end{aligned}$$

as desired for (5). This completes the proof of Theorem 4.5.

(4.6) Corollary. *Let* $t_1, \ldots, t_d \in \operatorname{Rad}(V,s)$. *Then*

$$\operatorname{pf}(v_1, \ldots, v_d) = \operatorname{pf}(v_1 + t_1, \ldots, v_d + t_d)\,.$$

PROOF. If $\{v_1, \ldots, v_d\}$ and $\{v_1 + t_1, \ldots, v_d + t_d\}$ are both linearly independent sets, then this is clear from the definition (and we have used this observation in our proof of the theorem). If $\{v_1, \ldots, v_d\}$ and $\{v_1 + t_1, \ldots, v_d + t_d\}$ are both linearly dependent, then both sides are 0 by definition and again the result is clear.

Suppose now that $\{v_1, \ldots, v_d\}$ is linearly dependent but $\{v_1 + t_1, \ldots, v_d + t_d\}$ is linearly independent. Then $\mathrm{pf}(v_1, \ldots, v_d) = 0$ by definition. On the other hand, we must use the exponential to calculate $\mathrm{pf}(v_1 + t_1, \ldots, v_d + t_d)$. By the theorem, its square satisfies

$$\mathrm{pf}(v_1 + t_1, \ldots, v_d + t_d)^2 = \det[s(v_i + t_i, v_j + t_j)]_{i,j} = \det[s(v_i, v_j)]_{i,j} \,.$$

Since the v_i are linearly dependent, this last is again 0, completing the proof of the corollary.

(4.7) PROPOSITION. *For $d > 2$*

$$\mathrm{pf}(v_1, \ldots, v_{d-1}, v_d) = s\left(\sum_{k=1}^{d-1} \alpha_k v_k, v_d\right) ,$$

where $\alpha_k = (-1)^{k+1}\,\mathrm{pf}(v_1, \ldots, \widehat{v}_k, \ldots, v_{d-1})$. Here $(v_1, \ldots, \widehat{v}_k, \ldots, v_{d-1})$ is the $(d-2)$-tuple that results from deleting v_k from $(v_1, \ldots, v_{d-1})$.

PROOF. By the previous corollary, we may assume that $v_1, \ldots, v_d$ is a linearly independent collection of vectors. If d is odd, then the result is trivial; so we assume that d is even. We have

$$\begin{aligned}
\mathrm{pf}(v_1, \ldots, v_d) v_1 \cdots v_d &= \Big(\prod_{\{i,j\} \in \mathcal{U}} (1 + s(v_i, v_j) v_i v_j) \Big)\Big|_d \\
&= \Big(\prod_{d \notin \{i,j\}} (1 + s(v_i, v_j) v_i v_j) \prod_{k=1}^{d-1} (1 + s(v_k, v_d) v_k v_d) \Big)\Big|_d \\
&= \sum_{k=1}^{d-1} \Big(\prod_{k,d \notin \{i,j\}} (1 + s(v_i, v_j) v_i v_j) \Big)\Big|_{d-2} s(v_k, v_d) v_k v_d \\
&= \sum_{k=1}^{d-1} \mathrm{pf}(v_1, \ldots, \widehat{v}_k, \ldots, v_{d-1}) v_1 \cdots \widehat{v}_k \cdots v_{d-1} \cdot s(v_k, v_d) v_k v_d \\
&= \sum_{k=1}^{d-1} (-1)^{d-1-k}\,\mathrm{pf}(v_1, \ldots, \widehat{v}_k, \ldots, v_{d-1}) s(v_k, v_d)\, v_1 \cdots v_k \cdots v_d \\
&= \sum_{k=1}^{d-1} s\big((-1)^{d-1-k}\,\mathrm{pf}(v_1, \ldots, \widehat{v}_k, \ldots, v_{d-1}) v_k, v_d\big)\, v_1 \cdots v_d \,;
\end{aligned}$$

hence $\mathrm{pf}(v_1, \ldots, v_d) = s\big(\sum_{k=1}^{d-1} \alpha_k v_k, v_d\big)$, as claimed.

Proof of Theorem 4.3:

As $\mathrm{pf}(v_1, v_2) = s(v_1, v_2)$, the map pf is linear in its last variable by Proposition 4.7. Therefore pf is linear in all variables by Proposition 4.4.2. It is alternating by Proposition 4.4. Finally, $\mathrm{pf}(v_1, \ldots, v_d)$ is nonzero precisely when $\langle v_1, \ldots, v_d \rangle$ is nondegenerate by Theorem 4.5.

Remarks. There are many treatments of Pfaffians. See Knuth [16] for historical discussion. Often 1-factors (perfect matchings) with appropriate sign conventions are used from the start [11, 13, 16, 23]. In enumerative applications [11, 23] it is appropriate to consider only characteristic 0 (from which the general result can be derived [10, 14]).

Dress and Wentzel [8] use exterior algebra and give a recursive definition of the Pfaffian, which is effectively the identity of Proposition 4.7 initialized by taking $\mathrm{pf}(v_1, v_2) = s(v_1, v_2)$. Their Theorem 2 proves the Pfaffian, under that definition, to be an alternating multilinear form.

Here we sought a direct construction of an alternating form having the properties of Theorem 4.3. Our version is motivated by that of Chevalley [5], which uses the exponential, and is similar to that of [4, 19], which use derivations, and that of [10, pp. 588-589], which uses divided powers. It is free from recursion, restrictions on characteristic, and additional sign conventions.

5 Transitivity of finite affine Grassmannians

This section is concerned with the proofs of Theorem 1.4 and Corollary 1.5.

Theorem 1.2 shows that the Pfaffian affine Grassmannians come from geometric hyperplanes $\mathcal{H}$ for which $\mathrm{P\Gamma L}(V)_{\mathcal{H}}$ is transitive on the d-spaces that are nondegenerate for a particular alternating d-linear form. This is a Witt type property for these forms. It seems possible that all such transitive affine Grassmannians could be classified. Indeed we may be seeing all of them in the Pfaffian and attenuated examples. In this section we prove that this is the case when we restrict attention to finite, flag-transitive affine Grassmannians.

We will need a result from the literature.

(5.1) Theorem. *Let $G \leq \mathrm{\Gamma L}_n(q)$ be transitive on the 1-spaces of $V = \mathbb{F}_q^n$. Then G is also transitive on the 1-spaces of V^*, and we have one of:*

(a) $n \leq 4$;

(b) $G = \mathrm{SL}_2(13)$ *with* $q = 3$ *and* $n = 6$;

(c) $G \leq \mathrm{\Gamma L}_1(q^n)$;

(d) $\mathrm{\Gamma L}_m(q^r) \trianglerighteq G \trianglerighteq \mathrm{SL}_m(q^r)$ *with* $n = mr > 4$ *and* $m \geq 2$;

(e) $\mathrm{\Gamma Sp}_m(q^r) \trianglerighteq G \trianglerighteq \mathrm{Sp}_m(q^r)$ *with* $n = mr > 4$ *and* m *even;*

(f) $\mathrm{\Gamma G}_2(q^r) \trianglerighteq G \trianglerighteq \mathrm{G}_2(q^r)'$ *with* q *even and* $n = mr$ *for* $m = 6$.

In cases (d), (e), and (f), V is the natural $\mathbb{F}_{q^r}$-module of dimension m viewed as a $\mathbb{F}_q$-module.

Proof. See [15, p. 68] or [17, p. 199].

Here $\Gamma\mathrm{Sp}_m(q^r)$ is the full group of semisimilarities of a nondegenerate symplectic form on $\mathbb{F}_{q^r}^m$ and consists of $\mathrm{Sp}_m(q^r)$ extended by the automorphisms and scalars of $\mathbb{F}_{q^r}$. Similarly $\Gamma\mathrm{G}_2(q^r)$ is $\mathrm{G}_2(q^r)$ extended by the automorphisms and scalars of $\mathbb{F}_{q^r}$.

Proof of Corollary 1.5:

One direction is immediate from Corollary 1.3. Assume now that $G = \mathrm{Aut}(\mathcal{CG}_d(V)_\mathcal{H})$ is flag-transitive on $\mathcal{CG}_d(V)_\mathcal{H}$. By Lemma 2.7, $n = 2d$ and the subgroup $H = \mathrm{Aut}(\mathcal{G}_d(V)_\mathcal{H})$ has index 2 in G, the elements of $G \setminus H$ exchanging $\mathcal{B}^-_\mathcal{H}$ and $\mathcal{B}^+_\mathcal{H}$.

If $d \in \{1, 2, n-2, n-1\}$ then Proposition 3.6 gives the corollary, so we may assume that $2 < d < n-2$. Therefore, by Theorem 2.3, we have $H = \mathrm{Aut}(\mathcal{G}_d(V)_\mathcal{H}) = \mathrm{P\Gamma L}(V)_\mathcal{H}$.

As G has a single chamber orbit on $\mathcal{CG}_d(V)_\mathcal{H}$, the index 2 subgroup H has at most two chamber orbits. Those must be that of the $-$-chambers and that of the $+$-chambers of $\mathcal{CG}_d(V)_\mathcal{H}$. By Lemma 3.3, H is flag-transitive on $\mathcal{G}_d(V)_\mathcal{H}$. Hence by Theorem 1.4, the geometric hyperplane $\mathcal{H}$ is either attenuated or nondegenerate Pfaffian, completing the corollary.

Remark. Except in its use of Theorem 1.4, our proof of Corollary 1.5 does not require K to be finite.

The rest of the section is devoted to our proof of Theorem 1.4. The examples are flag-transitive by Theorem 1.2, so we only need to prove that a flag-transitive finite affine Grassmannian is either attenuated or nondegenerate Pfaffian.

We let $\mathcal{H}$ be a geometric hyperplane of $\mathcal{A}_d(V)$, for $V = \mathbb{F}_q^n$, and assume that $\mathrm{Aut}(\mathcal{G}_d(V)_\mathcal{H})$ is transitive on the chambers of $\mathcal{G}_d(V)_\mathcal{H}$. Throughout we let $G = \Gamma\mathrm{L}(V)_\mathcal{H}$ and $q = p^a$, where p is the characteristic of $K = \mathbb{F}_q$.

We have some initial reductions.

(5.2) Lemma. *In proving Theorem 1.4 we may assume:*

(1) $\mathcal{H}$ *is not attenuated;*

(2) $\mathcal{H}$ *is the set of d-spaces that are degenerate for the alternating d-linear form f on V;*

(3) $2 < d \leq n/2 < n-2$*, especially* $n \geq 6$*;*

(4) $\mathrm{Aut}(\mathcal{G}_d(V)_\mathcal{H}) = \mathrm{P\Gamma L}(V)_\mathcal{H}$*;*

(5) $G = \Gamma\mathrm{L}(V)_\mathcal{H}$ *is irreducible on V and flag-transitive in its action on* $\mathcal{G}_d(V)_\mathcal{H}$.

Proof. We may assume, as in (1), that $\mathcal{H}$ is not attenuated, with the goal of proving that $\mathcal{H}$ is a nondegenerate Pfaffian geometric hyperplane.

By Shult's Theorem 1.1, $\mathcal{H}$ is the collection of d-spaces that are degenerate for an alternating d-linear form f as in (2).

We may take $2 < d < n-2$ by Proposition 3.6. The affine Grassmannian $\mathcal{G}_d(V)_\mathcal{H}$ in $\mathcal{A}_d(V)$ is isomorphic to its dual in $\mathcal{A}_{n-d}(V)$, and the list of conclusions to the theorem is closed under duality; so we may also assume $d \leq n/2$. This gives (3).

By (3) and Theorem 2.3 we have $\mathrm{Aut}(\mathcal{G}_d(V)_{\mathcal{H}}) = \mathrm{P\Gamma L}(V)_{\mathcal{H}}$ as in (4).

By (4) and assumption, the action of $G = \Gamma\mathrm{L}(V)_{\mathcal{H}}$ is flag-transitive. If G is reducible on V, then $\mathcal{H}$ is attenuated by Theorem 2.6, counter to (1).

(5.3) LEMMA. *Let $S = \mathrm{Stab}_G(P)$ be the global stabilizer in G of $P \in \mathcal{G}_d(V)_{\mathcal{H}}$. Then $S \leq \mathrm{Stab}_G(P^\theta)$, and S is transitive on the 1-spaces of P. Indeed, S is transitive on the pairs (X, W) where X is a 1-space of P and W is a $(d+1)$-space containing P.*

PROOF. By Proposition 2.2, $S \leq \mathrm{Stab}_G(P^\theta)$. By flag-transitivity S is transitive on the $(d-1)$-spaces B^- of P, so by duality it is transitive on the 1-spaces of P. Indeed, as S is transitive on the pairs (U, W) with U a $(d-1)$-subspace of P and W and $(d+1)$-space containing P, it is also transitive on pairs (X, W) as described.

(5.4) COROLLARY. *G is transitive on the 1-spaces of V.*

PROOF. The group G is irreducible on V by Lemma 5.2(5), so $\mathrm{Rad}(V, f)$ is trivial. That is, for every $v \in V$ there is a P in $\mathcal{P}_{\mathcal{H}}$ with $v \in V$. By the lemma, the stabilizer of P is transitive on the set of 1-spaces of P. Since $\mathcal{G}_d(V)_{\mathcal{H}}$ is connected by Proposition 2.5, the group G is transitive on the full set of 1-spaces of V.

(5.5) LEMMA. *There is no affine Grassmanian of $\mathcal{A}_3(\mathbb{F}_3^6)$ that admits $\mathrm{SL}_2(13)$ acting flag-transitively.*

PROOF. The number of chambers (B^-, P, B^+) would be $13^2|\mathcal{P}_{\mathcal{H}}|$, but 13^2 does not divide $|\mathrm{SL}_2(13)|$.

(5.6) LEMMA. *The subgroup $N = G^{(\infty)}$ is isomorphic to one of $\mathrm{SL}_n(q)$, $\mathrm{Sp}_n(q)$, or $\mathrm{G}_2(q)'$. Indeed, with an appropriate choice of notation we have, for $n = mr$ and $m \geq 2$, one of:*
(d) $\Gamma\mathrm{L}_m(q^r) \trianglerighteq G \trianglerighteq N = \mathrm{SL}_m(q^r)$;
(e) $\Gamma\mathrm{Sp}_m(q^r) \trianglerighteq G \trianglerighteq N = \mathrm{Sp}_m(q^r)$ *with m even;*
(f) $\Gamma\mathrm{G}_2(q^r) \trianglerighteq G \trianglerighteq N = \mathrm{G}_2(q^r)'$ *with q even and $m = 6$.*

PROOF. We must eliminate cases (*a*)-(*c*) of Theorem 5.1. As $n \geq 6$, case (*a*) does not occur. Case (*b*) is disposed of in Lemma 5.5.

Now suppose as in (*c*) that $G \leq \Gamma\mathrm{L}_1(q^n)$. Recall that $q = p^a$, for prime p; set $b = na$. Thus $|G|$ divides $(q^n - 1)b$.

Let (B^-, P, B^+) be a chamber of $\mathcal{G}_d(V)_{\mathcal{H}}$. Then the stabilizer in G of B^- is transitive on the various $P \in \mathcal{P}_{\mathcal{H}}$ containing it. These form a hyperplane complement in the quotient space V/B^-, so the order of G is divisible by q^{n-d}. Similarly the stabilizer of B^+ is transitive on $\{\, P \in \mathcal{P}_{\mathcal{H}} \mid P \leq B^+ \,\}$, a hyperplane complement in the dual of B^+; so $|G|$ is divisible by q^d. Thus $q^{\lfloor n/2 \rfloor}$ divides $(q^n - 1)b$, whence $(p^a)^{\lfloor n/2 \rfloor}$ divides b. This in turn implies that p^{b-a} divides b^2. As $b = na \geq 6a$, we conclude that p^{5b} divides b^{12}; and so $32^b \leq b^{12}$. This is false for $b \geq 7$. As $6 \leq n$ and $b = na$, we can only have $a = 1$ and $n = b = 6$. Since p^{b-a} must divide b^2, this yields a contradiction.

(5.7) LEMMA. *We have $r = 1$ and $n = m$.*

PROOF. Set $F = \mathrm{End}_K(N) = \mathbb{F}_{q^r}$.

We first claim that $P = FP$ for $P \in \mathcal{P}_{\mathcal{H}}$. Otherwise there is a 1-dimensional K-space $X \leq P$ with $P < FX+P$. In that case, there is a $(d+1)$-dimensional K-space $W_0 \leq FX+P$. By Lemma 5.3, the stabilizer of X and P in $G\,(\leq \Gamma\mathrm{L}_F(V))$ remains transitive on the set of $(d+1)$-dimensional K-spaces W containing P. Therefore $W \leq FX + P$ always, and so $V = FX + P$. Thus

$$\dim_K V = \dim_K (FX + P) = \dim_K FX + \dim_K P - \dim_K FX \cap P\,.$$

Now $\dim_K FX = r \leq n/2$ by Lemma 5.6; and $\dim_K P = d \leq n/2$ by Lemma 5.2; and $FX \cap P \geq X$. Therefore our dimension calculation yields

$$n \leq n/2 + n/2 - 1\,,$$

a contradiction. Thus $P = FP$, as claimed.

Let collinear $P_1, P_2 \in \mathcal{P}_{\mathcal{H}}$. Then by the previous paragraph F acts on the 1-dimensional K-space $P_1/P_1 \cap P_2$. Thus $K = F$, $r = 1$, and $n = m$, as desired.

(5.8) LEMMA. *There is a subgroup $N \simeq \mathrm{Sp}_n(q)$ normal in G and a nondegenerate N-invariant symplectic form s on V with $\mathcal{H} = \mathcal{H}_{s,d}$, as in the Pfaffian case (b) of Theorem 1.4. In particular d is even.*

PROOF. From Lemmas 5.6 and 5.7, we have three possibilities: G normalizes a subgroup N that is one of $\mathrm{SL}_n(q)$, $\mathrm{Sp}_n(q)$, or $\mathrm{G}_2(q)'$ (where in the last case $n = 6$ and q is even) with, respectively, $\mathrm{SL}_n(q) \leq G \leq \Gamma\mathrm{L}_n(q)$, $\mathrm{Sp}_n(q) \leq G \leq \Gamma\mathrm{Sp}_n(q)$, or $\mathrm{G}_2(q)' \leq G \leq \Gamma\mathrm{G}_2(q)$.

By Lemma 5.3, the stabilizer $S \cap N$ of $P \in \mathcal{P}_{\mathcal{H}}$ also fixes P^θ and so leaves invariant the decomposition $V = P \oplus P^\theta$. In particular N is not $\mathrm{SL}_n(q)$, since there the stabilizer of any P is transitive on the set of complements.

If G normalizes $N = \mathrm{G}_2(q)'$, then there is a nondegenerate symplectic form s on V for which G induces semisimilarities [1]. As $2 < d < n - 2$ we must have $d = 3$. By Lemma 5.3, S is irreducible on P; so P is totally isotropic for s. The stabilizer $S \cap N$ of P is then parabolic and stabilizes no complement P^θ [1]. This is a contradiction.

Finally, if G normalizes $N = \mathrm{Sp}_n(q)$, then again there is a nondegenerate symplectic form s on V for which G induces semisimilarities. Again S is irreducible on P and fixes the complement P^θ, and so P is nondegenerate for s and has even dimension d. We conclude that $\mathcal{H}$ contains $\mathcal{H}_{s,d}$. By Proposition 2.5(1) we have $\mathcal{H} = \mathcal{H}_{s,d}$, as desired.

This lemma completes the proof of Theorem 1.4.

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