Commun. Math. Phys. 201, 619 – 655 (1999)

Communications in
Mathematical Physics

The Initial Boundary Value Problem for Einstein's Vacuum Field Equation

Helmut Friedrich, Gabriel Nagy

Albert-Einstein-Institut, Max-Planck-Institut für Gravitationsphysik, Schlaatzweg 1, 14473 Potsdam, Germany

Received: 11 June 1998 / Accepted: 15 September 1998

Abstract: We study the initial boundary value problem for Einstein's vacuum field equation. We prescribe initial data on an orientable, compact, 3-dimensional manifold S with boundary $\Sigma \neq \emptyset$ and boundary conditions on the manifold $T = \mathbb{R}^+_0 \times \Sigma$. We assume the boundaries Σ and $\{0\} \times \Sigma$ of S and T to be identified in the natural way. Furthermore, we prescribe certain gauge source functions which determine the evolution of the fields. Provided that all data are smooth and certain consistency conditions are met on Σ, we show that there exists a smooth solution to Einstein's equation $\mathrm{Ric}[g] = 0$ on a manifold which has (after an identification) a neighbourhood of S in $T \cup S$ as a boundary. The solution is such that S is space-like, the initial data are induced by the solution on S, and, in the region of T where the solution is defined, T is time-like and the boundary conditions are satisfied.

1. Introduction

In this article we study the initial boundary value problem for Einstein's vacuum field equation. Let S be a smooth, orientable 3-dimensional manifold with boundary $\Sigma \neq \emptyset$. The boundary of the manifold $M = \mathbb{R}^+_0 \times S$ consists then of $S \simeq \{0\} \times S$ and $T = \mathbb{R}^+_0 \times \Sigma$ which are identified along the edge $\Sigma \simeq \{0\} \times \Sigma$ of M. We are interested in answering the following question: Which data do we have to prescribe on S and T such that there exists a (unique) smooth solution g of Einstein's equation

$$\mathrm{Ric}[g] = 0, \tag{1.1}$$

on M for which S is space-like, T is time-like and which is such that g induces the given data on S and T ?

The answer to this question will be of potential interest in any problem concerned with solutions to (1.1) which contain a distinguished time-like hypersurface. It will provide possibilities to construct examples or counterexamples to various conjectures and will give us tools to construct space-times with certain specified properties. We mention just a few such problems.

The motion of ideal fluid bodies with exterior vacuum field is of considerable interest in general relativity but its analytical properties are not well understood. The free time-like boundary, along which the transition of the Einstein–Euler equations into the Einstein vacuum field equations occurs, poses analytical difficulties. Though this situation is different from the one considered in the present article, the study of the initial boundary value problem sheds some light on the problem of the floating fluid balls. Our interest in this problem was one of the reasons to analyse the field equations in this article in a representation which is close to the one considered in [6].

In [2] the modeling of isolated systems in terms of asymptotically flat fields has been criticized. It has been suggested to separate instead the (massive) system of interest by a judiciously chosen time-like cut from the rest of the universe and to study the space-time so obtained as an object of its own. Whether such an approach leads to useful notions characterizing the behaviour of the system as a whole (energy momentum, angular momentum, etc.) and, in particular, whether it allows us to introduce meaningful concepts of incoming/outgoing radiation, etc. requires the understanding of the initial boundary value problem which has de facto been introduced in [2] without ever mentioning it.

In many numerical calculations in general relativity artificial time-like boundaries are introduced to restrict the calculations to finite grids (cf. [7] for possibilities to avoid such boundaries in certain relevant cases). A thorough understanding of the analytical features of the initial boundary value problem for Einstein's equation should be a prerequisite for successful numerical calculations near the boundary.

There are available in the literature various discussions of the Einstein equation in the neighbourhood of time-like boundaries (see e.g. [1, 10, 15]), but it appears that the existence of solutions to the initial boundary value problem for Einstein's equation has not been discussed so far in any generality. A general study of the initial boundary value problem for Einstein's equation with negative cosmological constant has been given in [4], but there the boundary data are prescribed on the conformal boundary at space-like and null infinity. Due to the fact that this boundary is defined intrinsically by the nature of the geometry, there occur certain simplifications which allow us to characterize the data on the boundary in a covariant way. In contrast to this, in the situation studied in this paper the boundary is not singled out by a geometric consideration but "is put in by hand". This leads to various complications in the detailed analysis of the present problem. Nevertheless, some general ideas and some specific techniques developed in [4] apply to the present problem.

It may be of interest to compare the methods and the results obtained in the present article with the completely different techniques for analysing the field near time-like boundaries used in [10]. This may give a deeper understanding of the problem and should shed light on the relative efficiencies of the different methods.

The basic step in our study is to reduce the geometrical initial boundary value problem for Einstein's equation to an initial boundary value problem for a hyperbolic system to which the general results on "maximally dissipative" initial boundary value problems (cf. Sect. 3) apply. A central difficulty here arises from the need to control the conservation of the constraints if the fields are evolved by a suitable hyperbolic system of reduced equations. In our treatment of the problem this means essentially that we have to show that those equations in the system (2.5) which contain only derivatives in space-like directions are preserved in the course of the evolution. This difficulty largely motivates our choice of the basic equations in Sect. 2, our choice of the gauge conditions in Sect. 4, and our choice of the reduced equations in Sect. 5. It is shown in Theorem 6.1 that our reduced problem, to which certain general results available in the mathematical literature apply, yields in fact solutions to the Einstein equation.

It is a most remarkable feature of Einstein's equation that the nature of the boundary condition does not play any role in this conclusion. This is most important for us, since the way we prescribe the boundary conditions on T does not allow us to check by direct calculations on T whether any constraints on T, either the intrinsic constraints induced on T or the constraints mentioned above, are satisfied on T.

In Sect. 7 we discuss the initial and boundary data which can be prescribed freely. While the initial data are well known from the study of the Cauchy problem for Einstein's equation and while it is also clear that the initial and the boundary data will have to satisfy certain consistency conditions on the edge Σ, the boundary conditions require a more careful study. In the local problem the boundary conditions are suggested by the nature of our reduced equations and by the theory of maximally dissipative boundary value problems. The question of how to prescribe boundary conditions in regions which cannot be handled solely in terms of one choice of gauge, sheds sharp light on some peculiar features of our problem.

It turns out that we need to specify, in an implicit form, a time-like unit vector field e_0 tangent to the boundary T. All other boundary conditions refer to this vector field in one way or another.

The boundary hypersurface is essentially singled out (imagining our prospective solution for the moment as a part of a larger space-time) by prescribing the mean extrinsic curvature χ of the boundary. However, the specification of χ is tied to that of e_0 and while locally the boundary could be specified by one real function, the situation is more complicated if long time evolutions are studied (cf. Sect. 8).

After the specification of the boundary in terms of χ, the basic freedom on the boundary consists in prescribing on T two arbitrary real functions and their coupling to the conformal Weyl curvature. We provide some explanation of the nature of this coupling (cf. Sect. 7) but we avoid speaking of incoming/outgoing gravitational radiation. Any such interpretation would depend on the time-like unit vector field e_0 on T, the choice of which is rather arbitrary as long as no further assumptions are introduced.

In Theorem 8.1 we state our general existence result, which is local in time but global along the edge Σ. We do not show the uniqueness of the solution in the general case. This is due to some open question concerning our gauge conditions (cf. Sect. 4) which we intend to make a topic of a separate investigation. However, in the particular case where the mean extrinsic curvature is constant on the boundary, local uniqueness of the solution is demonstrated.

There are certainly many possibilities to discuss the initial boundary value problem and there will be as many ways of stating boundary conditions. However, all of these should be just modifications of the boundary conditions given in our theorem.

2. The Field Equations

We shall use a frame formalism in which the metric g will be represented in terms of a frame field $\{e_k\}_{k=0,1,2,3}$ which satisfies the orthonormality condition $g_{ik} \equiv g(e_i, e_k) = diag(1, -1, -1, -1)$ and for which e_0 is future directed. All fields (with the possible exception of the fields e_k themselves) will be in the following expressed in terms of this frame.

The basic unknowns in our representation of the field equations are given by the fields

$$e^{\mu}_{k}, \quad \Gamma_k{}^i{}_j, \quad C^i{}_{jkl}.$$

The functions $e^{\mu}{}_k = e_k(x^{\mu})$ are the coefficients of the frame in a suitably chosen coordinate system $\{x^{\mu}\}_{\mu=0,1,2,3}$. In these coordinates the coefficients of the contravariant form of the metric are then given $g^{\mu\nu} = g^{jk}\, e^{\mu}{}_j\, e^{\nu}{}_k$. The $\Gamma_k{}^i{}_j$ are the connection coefficients in our frame such that $\nabla_k\, e_j \equiv \nabla_{e_k}\, e_j = \Gamma_k{}^i{}_j\, e_i$, where ∇ denotes the Levi–Civita connection of g. The fact that the connection is metric is expressed by the condition $\Gamma_i{}^l{}_k\, g_{lj} = -\Gamma_i{}^l{}_j\, g_{lk}$. Finally, $C^i{}_{jkl}$ is a tensor field which is required to possess the algebraic properties of a conformal Weyl tensor and which will in fact represent that tensor.

The curvature of the connection ∇ is given by

$$\begin{aligned} r^i{}_{jkl} = e_k(\Gamma_l{}^i{}_j) - e_l(\Gamma_k{}^i{}_j) + \Gamma_k{}^i{}_m \Gamma_l{}^m{}_j - \Gamma_l{}^i{}_m \Gamma_k{}^m{}_j \\ - \Gamma_m{}^i{}_j(\Gamma_k{}^m{}_l - \Gamma_l{}^m{}_k). \end{aligned} \tag{2.1}$$

For later discussions it will be convenient to introduce tensor fields $T_i{}^k{}_j$, $\Delta^i{}_{jkl}$, H_{jkl} by setting

$$T_i{}^k{}_j\, e_k = -[e_i, e_j] + (\Gamma_i{}^k{}_j - \Gamma_j{}^k{}_i)\, e_k, \tag{2.2}$$

$$\Delta^i{}_{jkl} = r^i{}_{jkl} - C^i{}_{jkl}, \tag{2.3}$$

$$H_{jkl} = \nabla_i\, C^i{}_{jkl}. \tag{2.4}$$

The Einstein equation can then be expressed by the equations

$$T_i{}^k{}_j = 0, \quad \Delta^i{}_{jkl} = 0, \quad H_{jkl} = 0. \tag{2.5}$$

The first of these equations implies that the connection ∇ is torsion free and therefore, since it is metric, that it is the Levi–Civita connection of the metric g. The equation allows us to determine the connection coefficients in terms of the frame coefficients and their first derivatives. The second equation requires that the curvature of ∇ coincides with the Weyl curvature and thus implies Einstein's equation (1.1). The third equation is the once contracted vacuum Bianchi identity. We refer to it as to the Bianchi equation.

One of the reasons why we chose this representation of the Einstein equation is that it simplifies the analysis of our problem. The equations contain direct geometric information. They are easily adapted to our situation and then entail immediate projection formalisms. Moreover, certain features of the Bianchi equation which are important for the discussion of initial boundary value problems are well understood [4]. Finally, the evolution equations for gravitating ideal fluid bodies derived in [6], which we want to use for analysing the problem of the "floating fluid ball", extend the equations above.

In the frame formalism there exists a natural decomposition of the Bianchi equation. We set $n = e_0$ and study the decomposition of H_{jkl} with respect to n and its orthogonal complement, which carries the induced metric $h_{ij} = g_{ij} - n_i\, n_j$. We denote by ϵ_{lijk} the totally antisymmetric tensor with $\epsilon_{0123} = 1$ and set $\epsilon_{ijk} = n^l\, \epsilon_{lijk}$. Furthermore we set $l_{ij} = h_{ij} - n_i\, n_j$.

The electric and magnetic part of the conformal Weyl tensor are defined with respect to n by $E_{ik} = h_i{}^m\, h_k{}^n\, C_{mjnl}\, n^j\, n^l$ and $B_{ik} = h_i{}^m\, h_k{}^n\, C^*_{mjnl}\, n^j\, n^l$ with the right dual of the conformal Weyl tensor given by $C^*_{ijkl} = \frac{1}{2}\, C_{ijmn}\, \epsilon^{mn}{}_{kl}$. We have $E_{ij} = E_{ji}$,

$E_{ij}\,n^j = 0$, $E_i{}^i = 0$. The same relations hold for B_{ij}. With these conventions the conformal Weyl tensor can be written

$$C_{ijkl} = 2\left(l_{j[k}\,E_{l]i} - l_{i[k}\,E_{l]j} - n_{[k}\,B_{l]m}\,\epsilon^m{}_{ij} - n_{[i}\,B_{j]m}\,\epsilon^m{}_{kl}\right).$$

Using the symmetries of the Weyl tensor and the identities

$$\epsilon_{ijp}\,\epsilon^{klp} = -2\,h_{[i}{}^k\,h_{j]}{}^l, \quad \epsilon_{ipq}\,\epsilon^{jpq} = -2\,h_i{}^j,$$

we get the decomposition

$$H_{jkl} = 2\,n_j\,P_{[k}\,n_{l]} + h_{j[k}\,P_{l]} + Q_i\,(n_j\,\epsilon^i{}_{kl} - \epsilon^i{}_{j[k}\,n_{l]}) - 2\,P_{j[k}\,n_{l]} - Q_{ji}\epsilon^i{}_{kl}, \tag{2.6}$$

where we set

$$P_k = n^j\,h_k{}^l\,n^m\,\nabla_i\,C^i{}_{jlm}, \quad Q_k = -\frac{1}{2}\,n^j\,\epsilon_k{}^{lm}\,\nabla_i\,C^i{}_{jlm}, \tag{2.7}$$

$$P_{jk} = h_{(j}{}^m\,h_{k)}{}^n\,n^l\,\nabla_i\,C^i{}_{mln}, \quad Q_{jk} = \frac{1}{2}\,h_{(j}{}^l\epsilon_{k)}{}^{mn}\nabla_i C^i{}_{lmn}. \tag{2.8}$$

In terms of these fields the Bianchi equation is equivalent to

$$P_k = 0, \quad Q_k = 0, \quad P_{jk} = 0, \quad Q_{jk} = 0. \tag{2.9}$$

To obtain more explicit expressions we set $K_{ij} = h_i{}^k\,\nabla_k\,n_j$, $K = h^{ij}\,K_{ij}$, $a^i = n^j\,\nabla_j\,n^i$, $\mathcal{D}_k\,E_{ij} = h_k{}^l\,h_i{}^m\,h_j{}^n\,\nabla_l\,E_{mn}$, etc. such that

$$K_{ij} = -h_i{}^k\,\Gamma_k{}^0{}_j, \quad K = -h^{pq}\,\Gamma_p{}^0{}_q, \quad a^i = \Gamma_0{}^i{}_0.$$

Observing that

$$h_i{}^m\,h_j{}^n\,n^k\,\nabla_k\,E_{mn} = \mathcal{L}_n\,E_{ij} - E_{lj}\,K_i{}^l - E_{il}\,K_j{}^l,$$

where $\mathcal{L}_n$ denotes the Lie derivative in the direction of n, we get

$$P_i = \mathcal{D}^j\,E_{ji} + 2\,K^{jl}\,\epsilon^k{}_{l(i}\,B_{j)k}, \tag{2.10}$$

$$Q_i = \mathcal{D}^j\,B_{ji} + \epsilon_i{}^{kl}\,(2\,K^j{}_k - K_k{}^j)\,E_{lj}, \tag{2.11}$$

$$\begin{aligned} P_{jl} = \mathcal{L}_n\,E_{jl} + \mathcal{D}_i\,B_{k(j}\,\epsilon_{l)}{}^{ik} - 3\,K_{(j}{}^i\,E_{l)i} - 2\,K^i{}_{(j}\,E_{l)i} \\ -2\,a_i\,\epsilon^{ik}{}_{(j}\,B_{l)k} + h_{jl}\,K^{ik}\,E_{ik} + 2\,K\,E_{jl}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} Q_{jl} = \mathcal{L}_n\,B_{jl} - \mathcal{D}_i\,E_{k(j}\,\epsilon_{l)}{}^{ik} + 2\,a_i\,\epsilon^{ik}{}_{(j}\,E_{l)k} \\ -K^i{}_{(j}\,B_{l)i} - 2\,K_{(j}{}^i\,B_{l)i} + K\,B_{jl} - K_{ik}\,B_{pq}\,\epsilon^{pi}{}_{(j}\,\epsilon^{kq}{}_{l)}. \end{aligned} \tag{2.13}$$

3. Maximally Dissipative Boundary Value Problems

We need to remove the gauge freedom in Eqs. (2.5) and to extract from the resulting equations a "reduced system" which will allow us to discuss initial boundary value problems. To motivate our choice of gauge conditions and reduced system, we shall outline briefly the argument which leads to maximally dissipative boundary conditions.

We consider on $M = \{x \in \mathbb{R}^4 | x^0 \geq 0, x^3 \geq 0\}$ a real linear symmetric hyperbolic system

$$A^\mu \partial_\mu u = B\, u + f(x) \tag{3.1}$$

for an $\mathbb{R}^N$-valued unknown u on M, i.e. the matrices $A^\mu = A^\mu(x)$, $\mu = 0, 1, 2, 3$, are smooth functions on M which take values in the set of symmetric $N \times N$-matrices, there exists a 1-form ξ_μ such that $A^\mu\, \xi_\mu$ is positive definite, $B = B(x)$ is a smooth matrix-valued function and $f(x)$ a smooth $\mathbb{R}^N$-valued function on M. For convenience we assume that the positivity condition is satisfied with $\xi_\mu = \delta^0{}_\mu$.

Set

$$S = \{x \in M | x^0 = 0\}, \quad T = \{x \in M | x^3 = 0\},$$

and define for $\tau \geq 0$ the sets

$$M_\tau = \{x \in M | 0 \leq x^0 \leq \tau\}, \quad S_\tau = \{x \in M | x^0 = \tau\},$$

$$T_\tau = \{x \in M | 0 \leq x^0 \leq \tau, x^3 = 0\}.$$

We prescribe data as follows: We choose $g \in C^\infty(S, \mathbb{R}^N)$ and require as initial condition

$$u(x) = g(x), \quad x \in S.$$

We choose a smooth map Q of T into the set of linear subspaces of $\mathbb{R}^N$ and require as boundary condition

$$u(x) \in Q(x), \quad x \in T.$$

The type of maps Q admitted here is suggested by the structure of the equations. Suppose that u is a solution of (3.1) of spatially compact support in M. Then (3.1) implies

$$\partial_\mu({}^t u\, A^\mu\, u) = {}^t u\, K\, u + 2\, {}^t u\, f \quad \text{with} \quad K = B + {}^t B + \partial_\mu A^\mu.$$

Integration over M_τ gives

$$\int_{S_\tau} {}^t u\, A^0\, u\, dS = \int_S {}^t u\, A^0\, u\, dS + \int_{M_\tau} \{{}^t u\, K\, u + 2\, {}^t u\, f\}\, dV + \int_{T_\tau} {}^t u\, A^3\, u\, dS.$$

If the last term on the right-hand side were non-positive, we could use this equation to obtain the energy estimates which are basic for proving existence and uniqueness of solutions to symmetric hyperbolic systems. Thus the structure of the "normal matrix" A^3 plays a prominent role in formulating the boundary conditions. We shall assume the following conditions to be satisfied:

(i) The set T is a characteristic of (3.1) of constant multiplicity, i.e.

$$\dim(\ker A^3(x)) = \text{const.} > 0, \quad x \in T.$$

(ii) The map Q is chosen such as to ensure the desired non-positivity

$${}^t u\, A^3(x)\, u \leq 0, \quad u \in Q(x), \quad x \in T.$$

(iii) The subspace $Q(x)$, $x \in T$, is a maximal with (ii), i.e. dim$(Q(x))$ = number of non-positive eigenvalues of A^3 counting multiplicity.

The last condition implies in particular that ker $A^3(x) \subset Q(x)$.

We discuss the specification of Q in terms of linear equations. Since A^3 is symmetric, we can assume, possibly after a transformation of the dependent unknown, that at $x \in T$,

$$A^3 = \kappa \begin{bmatrix} -I_p & 0 & 0 \\ 0 & 0_k & 0 \\ 0 & 0 & I_q \end{bmatrix}, \quad \kappa > 0,$$

where I_p is a $p \times p$ unit matrix, 0_k is a $k \times k$ zero matrix, etc. and $p + k + q = N$. Writing $u = {}^t(a, b, c) \in \mathbb{R}^p \times \mathbb{R}^k \times \mathbb{R}^q$ we find that at x the linear subspaces admitted as values of Q are necessarily given by equations of the form

$$0 = c - H\,a,$$

where $H = H(x)$ is a $q \times p$ matrix satisfying

$$-\,{}^t a\,a + {}^t a\,{}^t H\,H\,a \leq 0, \quad a \in \mathbb{R}^p, \quad \text{i.e.} \quad {}^t H\,H \leq I_p.$$

We note that there is no freedom to prescribe data for the component b of u associated with the kernel of A^3. More specifically, if $A^3 \equiv 0$ on T, energy estimates are obtained without imposing conditions on T and the solutions are determined uniquely by the initial condition on S.

By subtracting a suitable smooth function from u and redefining the function f, we can convert the homogeneous problem above to an inhomogeneous problem and vice versa. Inhomogeneous maximal dissipative boundary conditions are of the form

$$q = c - H\,a,$$

with $q = q(x)$, $x \in T$, a given $\mathbb{R}^q$-valued function representing the free boundary data on T.

The linear maximally dissipative boundary value problem has been analysed by Rauch [12] under weak smoothness assumptions, results for higher smoothness can be found e.g. in [13] and in the literature given there. In the case of quasi-linear equations the matrices A^μ depend on the unknown u as well. Thus the fact that the normal matrix A^3 depends also on u has to be observed in formulating the boundary conditions. Initial boundary value problems for quasi-linear equations with a general form of A^0 and boundary conditions as indicated above have been discussed e.g. in [8, 14].

To illustrate the discussion above in a simple case we consider Maxwell fields. This will also allow us to point out some specific features of Einstein's equation. We assume that a metric g is given on the set M above and that Maxwell's equations are expressed in an orthonormal frame e_k. The notation introduced in the previous chapter will be employed throughout. Maxwell's equations are given by

$$0 = H_k \equiv \nabla^j\,F_{jk} - 4\,\pi\,J_k, \quad 0 = H^*_k \equiv \nabla^j\,F^*_{jk},$$

with $F^*_{jk} = \frac{1}{2}\epsilon_{jk}{}^{il}F_{il}$. In terms of the electric field $E^i = -h^i{}_j\,n_k\,F^{jk}$ and the magnetic field $B^i = h^i{}_j\,n_k\,F^{*jk}$ we get decompositions

$$F_{ij} = -2\,E_{[i}\,n_{j]} + \epsilon_{ijk}\,B^k, \quad F^*_{ij} = \epsilon_{ijk}\,E^k + 2\,B_{[i}\,n_{j]},$$

which entail further decompositions

$$H_k = P_k - n_k\,P, \quad H^*_k = -Q_k + n_k\,Q,$$

with

$$P = \mathcal{D}^i\,E_i + K^{ij}\,B^k\,\epsilon_{ijk} + 4\,\pi\,\rho, \quad Q = \mathcal{D}^i\,B_i - K^{ij}\,E^k\,\epsilon_{ijk},$$

$$P_i = \mathcal{L}_n\,E_i - \epsilon_i{}^{jk}\,\mathcal{D}_j\,B_k - E_j\,(K_i{}^j + K^j{}_i) + E_i\,K_j{}^j + a^j\,B^k\,\epsilon_{jki} - 4\,\pi\,j_i,$$

$$Q_i = \mathcal{L}_n\,B_i + \epsilon_i{}^{jk}\,\mathcal{D}_j\,E_k - B_j\,(K_i{}^j + K^j{}_i) + B_i\,K_j{}^j - a^j\,E^k\,\epsilon_{jki},$$

where we set $\rho = n^k\,J_k$, $j_l = h_l{}^k\,J_k$. Notice that the terms in the first two equations which involve K_{ij} drop out if n is hypersurface orthogonal. It holds $n^k\,P_k = 0$, $n^k\,Q_k = 0$.

For convenience we shall assume that the normals to S are tangent to T on Σ. The frame e_k is now chosen on M such that e_0 and e_3 coincide on S, respectively T, with the unit normals pointing into M. Set $x^0 = 0$ on S and let x^α, $\alpha = 1, 2, 3$, be coordinates on S with $x^3 = 0$ on Σ and $x^3 > 0$ on $S \setminus \Sigma$. These coordinates are extended into M such that $e_0(x^\mu) = \delta^\mu{}_0$ on M. We have

$$e^3{}_k = e_k(x^3) = e^3{}_3\,\delta^3{}_k, \quad e^3{}_3 > 0 \quad \text{on} \quad T, \quad e^\mu{}_0 = \delta^\mu{}_0 \quad \text{on} \quad M.$$

Choose now J_k on M such that the conservation law

$$\nabla^k\,J_k = 0,$$

holds on M and prescribe data E_i, B_i on S satisfying the constraints:

$$P = 0, \quad Q = 0 \quad \text{on} \quad S. \tag{3.2}$$

To study the time evolution we observe that by our formalism the equations $P_0 = 0$, $Q_0 = 0$ are trivially satisfied and we consider the propagation equations

$$P_r = 0, \quad Q_r = 0 \quad \text{on} \quad M, \quad r = 1, 2, 3. \tag{3.3}$$

If we write these equations in the form (3.1) with $u = {}^t(u_1, \ldots, u_6)$, where $u_r = E_r$, $u_{3+r} = B_r$ for $r = 1, 2, 3$, we find

$$A^\mu = I\,\delta^\mu{}_0 + F^\mu \quad \text{on} \quad M,$$

with

$$F^\mu = \begin{bmatrix} 0 & 0 & 0 & 0 & e^\mu{}_3 & -e^\mu{}_2 \\ 0 & 0 & 0 & -e^\mu{}_3 & 0 & e^\mu{}_1 \\ 0 & 0 & 0 & e^\mu{}_2 & -e^\mu{}_1 & 0 \\ 0 & -e^\mu{}_3 & e^\mu{}_2 & 0 & 0 & 0 \\ e^\mu{}_3 & 0 & -e^\mu{}_1 & 0 & 0 & 0 \\ -e^\mu{}_2 & e^\mu{}_1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $A^\mu\,g_{\mu\nu}\,e^\nu{}_0 = I$, the 6×6 unit matrix, and the matrices A^μ are symmetric, we see that Eqs. (3.3) form a symmetric hyperbolic system. On S we have $F^0 = 0$. We shall assume that A^0 is positive definite on M.

For the normal matrix on T we find

$$A^3 = e^3{}_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which tells us that T is a characteristic of constant multiplicity since $e^3{}_3 > 0$ on T. To study the maximally dissipative boundary condition we perform a transformation $u \to v = C\,u$ such that $A^3 = {}^tC\,D\,C$ with $D = e^3{}_3\ diag\,(-1,-1,0,0,1,1)$. Such a transformation is given by

$$\begin{aligned} v_1 &= \frac{1}{\sqrt{2}}(u_1 - u_5), & v_2 &= \frac{1}{\sqrt{2}}(u_2 + u_4), & v_3 &= u_3, \\ v_4 &= u_6, & v_5 &= \frac{1}{\sqrt{2}}(u_1 + u_5), & v_6 &= \frac{1}{\sqrt{2}}(u_2 - u_4). \end{aligned}$$

As discussed above we introduce now a real matrix valued function H on T,

$$H = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{with} \quad (a\,v_1 + b\,v_2)^2 + (c\,v_1 + d\,v_2)^2 \leq v_1^2 + v_2^2, \quad (v_1, v_2) \in \mathbb{R}^2, \tag{3.4}$$

to write down inhomogeneous maximally dissipative boundary conditions. Translated back into the original unknowns these conditions read

$$\begin{aligned} q_1 &= E_1 + B_2 - a\,(E_1 - B_2) - b\,(E_2 + B_1), \\ q_2 &= E_2 - B_1 - c\,(E_1 - B_2) - d\,(E_2 + B_1), \end{aligned}$$

where q_1 and q_2 are smooth functions, prescribed on T. In terms of the spin frame formalism of Newman and Penrose [11] this equation takes the form

$$q = \phi_2 + \alpha\,\phi_0 + \beta\,\overline{\phi}_0,$$

where we set

$$q = -\frac{1}{2}\,(q_1 + i\,q_2), \quad \alpha = \frac{1}{2}\,(a + d - i\,b + i\,c), \quad \beta = \frac{1}{2}\,(a - d + i\,b + i\,c), \tag{3.5}$$

$$l = \frac{1}{\sqrt{2}}\,(e_0 + e_3), \quad k = \frac{1}{\sqrt{2}}\,(e_0 - e_3), \quad m = \frac{1}{\sqrt{2}}\,(e_1 - i\,e_2), \tag{3.6}$$

$$\phi_0 = F_{ij}\,l^i\,m^j, \quad \phi_1 = F_{ij}\,(l^i\,k^j + \overline{m}^i\,m^j), \quad \phi_2 = F_{ij}\,\overline{m}^i\,k^j.$$

By picking the matrix H appropriately, we see that we could alternatively prescribe e.g. the components (E_1, E_2) or (B_1, B_2) or ϕ_2 freely on T. The function ϕ_2 can be interpreted as the component of the Maxwell field which is transverse to and travels in the direction of e_3. We note that all the prescriptions above depend on e_0, for which there exists no privileged choice on T.

The non-positivity condition (ii) implies for the Poynting vector $S^i = -\frac{1}{4\pi}\epsilon^{ijk}E_jB_k$ on T the relation

$$S^3 = \frac{1}{4\,\pi}\,(E_1\,B_2 - E_2\,B_1) = \frac{1}{8\,\pi\,e^3{}_3}\,{}^tu\,A^3\,u \leq 0, \tag{3.7}$$

with e_3 pointing towards M on T.

Given the field equations and the data on S, we can derive a formal expansion of the prospective solution u on S, in particular on Σ, in terms of x^0. If we want to ensure the smoothness of u, we need to give the boundary conditions such that they are consistent with the formal expansion of u on Σ. We shall not discuss these "consistency conditions" (cf. [8]) any further.

The initial boundary value problem which we have outlined here admits a unique smooth solution E_k, B_k of (3.3) on a suitably given neighbourhood of Σ in M. We still need to show that these fields satisfy also the constraints $P = 0$, $Q = 0$. In the case of Maxwell equations the argument is straightforward. A direct calculation shows that the fields H_k, H^*_k satisfy for arbitrary fields E_k, B_k the identities

$$\nabla^k H_k = -R_{jk}\, F^{jk} = 0, \quad \nabla^k H^*_k = -R_{jk}\, F^{jk} = 0.$$

On the other hand, observing the decompositions of H_k, H^*_k given above and the fact that our fields solve (3.3), we find for P and Q the "subsidiary equations"

$$0 = -\nabla^k H_k = \mathcal{L}_n\, P + P\, K_j{}^j, \quad 0 = \nabla^k H^*_k = \mathcal{L}_n\, Q + Q\, K_j{}^j.$$

Because of (3.2) it follows from these ODE's that P and Q also vanish off S.

In the following we shall reduce the "geometric" initial boundary value problem for Einstein's equation to a maximally dissipative boundary value problem for a suitably chosen reduced system. We have seen above that at least three important conditions have to be met by the gauge conditions and the reduced system: The system should be symmetric hyperbolic, the resulting problem should satisfy the condition of maximal dissipativity, and the problem should allow us to demonstrate the preservation of the constraints. Besides studying the Bianchi equation, which is similar to the Maxwell's equations, we need to take care of the equations for the frame and the connection coefficients and we need to characterize the boundary itself in terms of some data. The enormous freedom available here allows for reduced systems which satisfy the first two conditions but which lead to difficulties when it comes to verifying the third condition. This should be kept in mind when we study now the gauge conditions and then extract the reduced system.

4. The Gauge Conditions

Consider the 4-manifold $M = \mathbb{R}^+_0 \times S$, where S is a smooth orientable 3-manifold with boundary $\Sigma \equiv \partial S \neq \emptyset$. We write $\partial M = S \cup T$ and $S \cap T = \{0\} \times \Sigma \equiv \Sigma$, where we identify S in the obvious way with $\{0\} \times S \subset M$ and set $T = \mathbb{R}^+_0 \times \Sigma$. Let g be a smooth Lorentz metric on M for which S is space-like and T is time-like. Given a point $p \in \Sigma$ we want to construct in some appropriate neighbourhood U of p coordinates x^μ and an orthonormal frame field e_k which are conveniently adapted to $S \cap U$ and $T \cap U$. It will be seen that our construction works for a suitably chosen neighbourhood U of p.

Set $x^0 = 0$ on S and let x^α, $\alpha = 1, 2, 3$, be local coordinates on $S \cap U$ with $x^3 = 0$ on $\Sigma \cap U$ and $x^3 > 0$ elsewhere. Choose a time-like unit vector field e_0 on U which is tangent to $T \cap U$, orthogonal to the 2-surfaces $\{x^3 = c = \text{const.} > 0\}$ in $S \cap U$, and points towards M on $S \cap U$. We assume that the integral curves of e_0 starting on $S \cap U$ generate U. We extend the functions x^μ to U such that $e^\mu{}_0 \equiv e_0(x^\mu) = \delta^\mu{}_0$ on U, i.e. x^0 is the parameter on the integral curves of e_0 which vanishes on $S \cap U$ and the x^α are constant on these curves. The x^μ provide smooth coordinates on U. The sets $T_c = \{x^3 = c\}$ are smooth time-like hypersurfaces in U with $T_0 = T \cap U$. Let e_3 be the smooth unit vector field normal to T_c which points towards M on T_0. We denote by D the Levi–Civita connection defined by the metric induced on T_c. Choose vector fields e_A, $A = 1, 2$, on $S \cap U$ which are tangent to $T_c \cap S$ and which form with e_0, e_3 a smooth orthonormal frame field on $S \cap U$. Using the connection D, we extend the fields e_A to

T_c by Fermi transport on T_c in the direction of e_0 such that (in signature-independent form)

$$g(e_0, e_0)\, D_{e_0}\, e_A + g(e_A, D_{e_0}\, e_0)\, e_0 - g(e_A, e_0)\, D_{e_0}\, e_0 = 0 \quad \text{on} \quad U. \tag{4.1}$$

The e_k form a smooth orthonormal frame field on U. We shall refer the type of gauge considered above as an "adapted gauge".

In the further discussion we will have to consider three types of projections. Since our frame is well adapted to our geometrical situation we can avoid the introduction of corresponding projection formalisms by distinguishing four groups of indices. The latter are given, together with the values they take, as follows:

$$a, c, d, e, f = 0, 1, 2; \quad i, j, k, l, m, n = 0, 1, 2, 3;$$

$$p, q, r, s, t = 1, 2, 3; \quad A, B, C, D = 1, 2.$$

We assume the summation rule for each group.

The frame coefficients $e^\mu{}_k$ satisfy

$$e^\mu{}_0 = \delta^\mu{}_0, \quad e^3{}_a = 0, \quad e^3{}_3 > 0 \quad \text{on} \quad U. \tag{4.2}$$

A part of the connection coefficients defines the inner connection D on T_c, we have

$$D_a\, e_c \equiv D_{e_a}\, e_c = \Gamma_a{}^b{}_c\, e_b. \tag{4.3}$$

The condition (4.1) reads in terms of the connection coefficients

$$\Gamma_0{}^A{}_B = 0. \tag{4.4}$$

As a consequence the fields e_a satisfy on T_c the equations

$$D_{e_0}\, e_0 = \Gamma_0{}^A{}_0\, e_A, \quad D_{e_0}\, e_A = -g_{AB}\, \Gamma_0{}^B{}_0\, e_0. \tag{4.5}$$

Given the hypersurfaces T_c, the coefficients $\Gamma_0{}^A{}_0$ can be considered as gauge source functions (cf. [5]) which govern the evolution of the coordinates and the frame field off S.

Lemma 4.1. *Suppose that the hypersurfaces T_c are given on U and let the coordinates x^μ and the frame field e_k described above be given on $S \cap U$. If $F^{'A} = F^{'A}(x^{'\mu})$, $A = 1, 2$, are smooth functions on $\{x' \in \mathbb{R}^4 | x^{'0} \geq 0, x^{'3} \geq 0\}$, there exist unique coordinates $x^{'\mu}$ and unique frame vector fields e'_k on some neighbourhood U' of p in U which represent an adapted gauge and which are such that on $S \cap U'$ $x^{'\mu} = x^\mu$, $e'_k = e_k$ holds, and on U' $x^{'3} = x^3$, $\Gamma'_0{}^A{}_0(x^{'\mu}) = F^{'A}(x^{'\mu})$ holds, where $\Gamma'_i{}^j{}_k$ denote the connection coefficients with respect to e'_k.*

Proof. The new coordinates and frame vector fields would need to satisfy the equations

$$D_{e'_0}\, e'_0 = F^{'A}(x^{'\mu})\, e'_A, \quad D_{e'_0}\, e'_A = -g_{AB}\, F^{'B}(x^{'\mu})\, e'_0,$$
$$e'_0(x^{'\mu}) = \delta^\mu{}_0,$$

with $x^{'3} = c$ on the hypersurface $T_c \cap U'$, $c \geq 0$. Since the connection D on T_c can be considered as known, we can read the equations above as a system of ODE's on T_c for the coordinates $x^{'\alpha}(x^\beta, c)$ and the coefficients $e^{'\alpha}{}_a(x^\beta, c)$ of the vector fields e'_a in the coordinates x^α, where $\alpha, \beta = 0, 1, 2$. For the given data on $S \cap U$ this system of ODE's has a unique solution in some neighbourhood U' of p which depends smoothly on the initial data and the parameter c. We set $e'^3{}_a(x^\alpha, c) = 0$ and express the frame in the new coordinates. Equations (4.5) imply a system of ODE's for the quantities $g(e'_a, e'_b)$ which allows us to show that the frame is indeed orthonormal. $\square$

The second fundamental form of T_c in the frame e_a is given by

$$\chi_{ab} \equiv g(\nabla_{e_a} e_3, e_b) = \Gamma_a{}^j{}_3\, g_{jb} = \Gamma_a{}^3{}_b = \Gamma_{(a}{}^3{}_{b)}, \tag{4.6}$$

the mean extrinsic curvature of the hypersurfaces T_c is given by

$$\chi \equiv g^{ab}\,\chi_{ab} = g^{jk}\,\Gamma_j{}^3{}_k = \nabla_\mu\, e^\mu{}_3. \tag{4.7}$$

Lemma 4.2. *Consider the smooth functions $\chi(x^\alpha, 0)$, $\Gamma_0{}^A{}_0(x^\alpha, 0)$, $\alpha = 0, 1, 2$, which are implied on $T \cap U$ in the adapted gauge considered above. Let x^μ be smooth functions on $S \cap U$ with $x^0 = 0$ and such that x^1, x^2, x^3 are local coordinates on $S \cap U$ with $x^3 = 0$ on $\Sigma \cap U$ and $x^3 > 0$ elsewhere. Let $\{e_k\}_{k=0,\dots,3}$ be a smooth orthonormal frame field on S with the following properties. The vector field e_0 points towards M, it is tangent to T on $\Sigma \cap U$, and for given number c, $0 \le c \le \sup_U x^3$, it is orthogonal to the sets $S_c \equiv \{x^3 = c\} \subset S$. The vector fields e_A are tangent to the sets S_c and the vector field e_3 points towards M on $\Sigma \cap U$.*

If $f = f(x^\mu)$, $F^A = F^A(x^\mu)$, $A = 1, 2$, are smooth functions on $\{x \in \mathbb{R}^4 | x^0 \ge 0, x^3 \ge 0\}$ satisfying

$$f(x^\alpha, 0) = \chi(x^\alpha, 0), \quad F^A(x^\alpha, 0) = \Gamma_0{}^A{}_0(x^\alpha, 0),$$

then, if there exists a smooth extension of the functions x^μ and the vector fields e_k to some neighbourhood U' of p in U such that x^μ, e_k represent the coordinates and the frame field in an adapted gauge on U' for which $\chi(x^\mu) = f(x^\mu)$ and $\Gamma_0{}^A{}_0(x^\mu) = F^A(x^\mu)$ on U', the extension is unique. If $\chi(x^\alpha, 0) = \chi_0 = \text{const.}$ and f is chosen to be constant and equal to χ_0, there exists a smooth extension of x^μ and e_k with the properties listed above.

Remark 4.1. We shall in the following consider the functions F^A and, for $x^3 > 0$, the function f as gauge source functions which determine the foliation by hypersurfaces $\{x^3 = \text{const.}\}$ and the evolution of the field e_0 on these hypersurfaces. Therefore the existence of the extensions of the x^μ, e_k is important for us also in the case of general functions $\chi(x^\alpha, 0)$, $f(x^\mu)$ (cf. Sect. 8). Since the general existence proof appears to require techniques which are different from the ones used in this article we will make it a topic of separate investigation.

Proof. Let $x^{\mu'}$ be coordinates on U such that we have $x^{\mu'} = x^\mu$ for $\mu' = \mu$ as well as $e_0(x^{3'}) = 0$ on $S \cap U$ and $x^{3'} = 0$ on $T \cap U$. In the following indices $\alpha, \beta, \alpha', \beta'$ take values $0, 1, 2$. For given number c we wish to construct a hypersurface $T_c = \{x^3 = c\}$ such that $T_c \cap S = S_c$ and the mean extrinsic curvature of T_c satisfies $\chi(x^\alpha, c) = f(x^\alpha, c)$. The hypersurface will be given as the graph of a smooth function $\phi(x^{\alpha'}, c)$. We set $\Phi(x^{\mu'}, c) = x^{3'} - \phi(x^{\alpha'}, c)$ and require

$$p \in T_c \quad \text{iff} \quad \Phi(x^{\mu'}(p), c) = c. \tag{4.8}$$

In the following the dependence of the various quantities on the parameter c will not always be written out explicitly but it should be kept in mind.

The unit normal $N^{\mu'}$ to T_c with $N^{3'} > 0$ on S will be given by

$$N^{\mu'} = N\, \nabla^{\mu'}\, \Phi. \quad \text{with} \quad N = -(-\nabla_{\nu'}\, \Phi\, \nabla^{\nu'}\, \Phi)^{-\frac{1}{2}}.$$

It will be ensured by condition (4.11) that $N \neq 0$ close to $S \cap U$. The second fundamental form of T_c will be given by

$$\chi_{\mu'\nu'} = k_{\mu'}{}^{\rho'}\, k_{\nu'}{}^{\lambda'}\, \nabla_{\rho'}\, N_{\lambda'} = N\, k_{\mu'}{}^{\rho'}\, k_{\nu'}{}^{\lambda'}\, \nabla_{\rho'}\, \nabla_{\lambda'}\, \Phi. \tag{4.9}$$

Here indices are raised and lowered by using the metric g and we denote by

$$k_{\mu'\nu'} = g_{\mu'\nu'} + N_{\mu'}\, N_{\nu'}$$

the metric which will be induced on the hypersurface T_c.

The equation which relates the function f to the mean extrinsic curvature of T_c takes the form

$$\nabla_{\mu'}\, N^{\mu'} = N\, k^{\mu'\nu'}\, \nabla_{\mu'}\, \nabla_{\nu'}\, \Phi = -N\, k^{\alpha'\beta'}\, \partial_{\alpha'}\, \partial_{\beta'}\, \phi + h(x^{\alpha'}, \phi, \partial_{\beta'}\, \phi) = f(x^{\alpha}, c), \tag{4.10}$$

with some smooth function h. To ensure that $x^3 = c$ on $T_c \cap S$ and e_0 is tangent to T_c, we require

$$\phi = 0, \quad 0 = e_0(\Phi) = e^{3'}{}_0 - \phi_{,\alpha'}\, e^{\alpha'}{}_0 = -\phi_{,\alpha'}\, e^{\alpha'}{}_0 \quad \text{on} \quad S_c. \tag{4.11}$$

In (4.9), (4.10) and (4.11) it is assumed that $x^{3'} = c + \phi(x^{\alpha'})$ in the arguments of the background fields $g_{\mu'\nu'}$, $\Gamma_{\mu'}{}^{\rho'}{}_{\nu'}$, $e^{\mu'}{}_0$ entering the equations. It follows from (4.11) that $N_{\mu'} = -(-g^{3'3'})^{-\frac{1}{2}}\, \delta^{3'}{}_{\mu'} \neq 0$ at S_c. The metric induced by $k_{\mu'\nu'}$ on the tangent spaces of T_c at points of S_c is Lorentzian and this property will be preserved in some neighbourhood of S_c in T_c if ϕ is smooth.

The quasilinear wave equation (4.10) and the initial conditions (4.11) suggest to find T_c by solving a Cauchy problem for ϕ. In the particular case where $f = \text{const.} = \chi_0$, the existence of a unique smooth solution can be inferred from general theorems (cf. [9]) which also entail the smooth dependence of the solution from the initial data. This allows us to construct (sufficiently close to S) a family of hypersurface T_c with mean extrinsic curvature χ_0, which can be described as the set of level hypersurfaces of a smooth function x^3 with $d\,x^3 \neq 0$ and $x^3 = 0$ on the intersection of its domain of definition with T. In view of Lemma 4.1 this entails the last statement of the lemma above.

However, if $\partial_\alpha f \neq 0$, $\alpha = 0, 1, 2$, we cannot proceed in this way. While the left hand side of (4.10) is expressed in terms of the coordinates $x^{\alpha'}$, the function f on the far right hand side is given in terms of the coordinates x^{α} which still need to be determined as functions of the $x^{\mu'}$. This leads us to consider Eqs. (4.5) again.

We begin with a few basic remarks. A vector field s is tangent to T_c if and only if $s(\Phi) = 0$ or, equivalently, if

$$s^{3'} = \phi_{,\alpha'}\, s^{\alpha'}. \tag{4.12}$$

Thus we only need to determine ϕ and $s^{\alpha'}$ to find $s^{\mu'}$ on T_c.

We shall consider equations, for unknowns on T_c, in which vector fields e_a tangent to T_c act as operators. Any such unknown h will be thought of as being induced by a function

H defined on some neighbourhood of T_c. In our coordinates $x^{\mu'}$, which are not adapted to T_c, the usual expression $e_a(h) = h_{,\mu'}\, e^{\mu'}{}_a$ is not directly defined. By our procedure above, T_c is parametrized by the $x^{\alpha'}$ and we have $h = h(x^{\alpha'}) = H(x^{\alpha'}, c + \phi(x^{\alpha'}))$, which entails

$$h_{,\alpha'}\, e^{\alpha'}{}_a = H_{,\alpha'}\, e^{\alpha'}{}_a + H_{,3'}\, \phi_{,\alpha'}\, e^{\alpha'}{}_a = H_{,\alpha'}\, e^{\alpha'}{}_a + H_{,3'}\, e^{3'}{}_a = e_a(H).$$

Therefore, any expression $e_a(h)$ with h defined on T_c and e_a tangent to T_c will be interpreted in the following by

$$e_a(h) = h_{,\alpha'}\, e^{\alpha'}{}_a. \tag{4.13}$$

Since the connection D induced on T_c is not known yet, we express D in terms of the derivative operator ∇ and the second fundamental form $\chi_{\mu'\nu'}$ on T_c. For any vector fields e_0, s tangent to T_c we should have on T_c,

$$D_{e_0}\, s^{\mu'} = \nabla_{e_0}\, s^{\mu'} - N^{\mu'}\, \chi_{\rho'\lambda'}\, s^{\rho'}\, e^{\lambda'}{}_0.$$

Because of (4.12) it is sufficient to consider the α'-components of this equation, the $3'$ component will be a consequence.

Thus Eqs. (4.5) take the form

$$\nabla_{e_0}\, e^{\alpha'}{}_0 - N^{\alpha'}\, \chi_{00} = F^A(x^\alpha, c)\, e^{\alpha'}{}_A, \tag{4.14}$$

$$\nabla_{e_0}\, e^{\alpha'}{}_A - N^{\alpha'}\, \chi_{0A} = -g_{AB}\, F^B(x^\alpha, c)\, e^{\alpha'}{}_0, \tag{4.15}$$

with

$$e^{3'}{}_a = \phi_{,\alpha'}\, e^{\alpha'}{}_a \tag{4.16}$$

being used wherever $e^{3'}{}_a$ occurs in the equations. The transformation $x^\alpha = x^\alpha(x^{\alpha'})$ will be obtained as solution to

$$e_0(x^\alpha) = \delta^\alpha{}_0. \tag{4.17}$$

In Eqs. (4.14), (4.15) we set $N^{\mu'} = e^{\mu'}{}_3 = N\, \nabla^{\mu'}\, \Phi$. However, writing $\chi_{ab} = \chi_{\mu'\nu'}\, e^{\mu'}{}_a\, e^{\nu'}{}_b$ with the expression (4.9) of the second fundamental form, would introduce terms of second order in ϕ which would spoil the hyperbolicity of the system. We shall derive instead propagation equations for χ_{ab}.

On the hypersurface T_c we will have Codazzi's and Gauss' equations which will take in the frame e_a on T_c the form

$$D_b\, \chi_{ca} - D_c\, \chi_{ba} = R^3{}_{abc}, \tag{4.18}$$

$$\begin{aligned} &e_c(\Gamma_d{}^a{}_b) - e_d(\Gamma_c{}^a{}_b) + \Gamma_c{}^a{}_e\, \Gamma_d{}^e{}_b - \Gamma_d{}^a{}_e\, \Gamma_c{}^e{}_b \\ &-\Gamma_e{}^a{}_b\, (\Gamma_c{}^e{}_d - \Gamma_d{}^e{}_c) + \chi_c{}^a\, \chi_{db} - \chi_d{}^a\, \chi_{cb} = R^a{}_{bcd}, \end{aligned} \tag{4.19}$$

respectively, with $\Gamma_a{}^c{}_b$ denoting the connection coefficients of D in the frame e_a. We write as usual $D_b\, \chi_{ca} = e_b(\chi_{ca}) - \Gamma_b{}^e{}_c\, \chi_{ea} - \Gamma_b{}^e{}_a\, \chi_{ce}$ and assume the interpretation (4.13) of directional derivatives.

Equation (4.18) implies the system

$$D_0\, \chi_{01} - D_1\, \chi_{11} - D_2\, \chi_{12} = e_1(f) + g^{ab}\, R^3{}_{ab1}, \tag{4.20}$$

$$D_0\, \chi_{02} - D_1\, \chi_{12} - D_2\, \chi_{22} = e_2(f) + g^{ab}\, R^3{}_{ab2}, \tag{4.21}$$

$$D_0\,\chi_{11} - D_1\,\chi_{01} = R^3{}_{101}, \tag{4.22}$$

$$2\,D_0\,\chi_{12} - D_1\,\chi_{02} - D_2\,\chi_{01} = R^3{}_{102} + R^3{}_{201}, \tag{4.23}$$

$$D_0\,\chi_{22} - D_2\,\chi_{02} = R^3{}_{202}, \tag{4.24}$$

where we set $f = g^{ab}\,\chi_{ab}$ and assume that the function χ_{00}, of which no derivative is taken in the equations, is given by $\chi_{00} = f + \chi_{11} + \chi_{22}$. We write $R_{jabc} = R_{\mu'\nu'\lambda'\rho'}\,e^{\mu'}{}_j\,e^{\nu'}{}_a\,e^{\lambda'}{}_b\,e^{\rho'}{}_c$ and use (4.12) to express $e^{3'}{}_a$ in terms of $\phi_{,\alpha}$ and $e^{\alpha'}{}_a$.

With the gauge conditions $\Gamma_0{}^A{}_B = 0$, $\Gamma_0{}^A{}_0 = F^A$, Eq. (4.19) implies the system

$$e_0(\Gamma_A{}^B{}_0) = e_A(F^B) - \Gamma_C{}^B{}_0\,\Gamma_A{}^C{}_0$$
$$-\Gamma_A{}^B{}_C\,F^C - F^B\,F^C\,g_{AC} - \chi_0{}^B\,\chi_{A0} + \chi_A{}^B\,\chi_{00} + R^B{}_{00A}, \tag{4.25}$$

$$e_0(\Gamma_A{}^B{}_C) = -F^B\,\Gamma_A{}^0{}_C - \Gamma_A{}^B{}_0\,F^D\,g_{CD}$$
$$-\Gamma_D{}^B{}_C\,\Gamma_A{}^D{}_0 - \chi_0{}^B\,\chi_{AC} + \chi_A{}^B\,\chi_{0C} + R^B{}_{C0A}. \tag{4.26}$$

It remains to explain the meaning of the expressions $e_A\,(f)$, $e_A(F^B)$. We should have

$$e_a(f) = f_{,\mu}\,\frac{\partial\,x^\mu}{\partial\,x^{\mu'}}\,e^{\mu'}{}_a = f_{,\mu}\,e^\mu{}_a = f_{,\alpha}\,e^\alpha{}_a,$$

where $e^\mu{}_a$ denotes the coefficients of the frame field e_a in the coordinates x^μ. We derive equations for the quantities $e^\alpha{}_A$.

Because the intrinsic connection on T_c will be torsion free we should have

$$0 = D_{e_a}\,D_{e_b}x^\alpha - D_{e_b}\,D_{e_a}x^\alpha = D_{e_a}\,e^\alpha{}_b - D_{e_b}\,e^\alpha{}_a,$$

where $e^\alpha{}_a$ is considered for given α as the expression of $d\,x^\alpha$ in our frame. Observing our gauge conditions, in particular their implication $e^\alpha{}_0 = \delta^\alpha{}_0$, we get the equation

$$e_0(e^\alpha{}_A) = -g_{AB}\,F^B\,\delta^\alpha{}_0 - \Gamma_A{}^B{}_0\,e^\alpha{}_B. \tag{4.27}$$

In the equations above we set now

$$e_A\,(f) = f_{,\alpha}\,e^\alpha{}_A, \quad e_A(F^B) = F^B_{,\alpha}\,e^\alpha{}_A. \tag{4.28}$$

With the interpretations and gauge conditions given above Eqs. (4.10), (4.14), (4.15), (4.17), (4.20) to (4.27), form a quasi-linear system of equations for the unknowns ϕ, $e^{\alpha'}{}_a$, $x^\alpha(x^{\alpha'})$, χ_{ab}, $\Gamma_a{}^b{}_c$, $e^\alpha{}_a$.

The initial data for the coordinates $x^\alpha(x^{\alpha'})$ and the frame coefficients $e^{\alpha'}{}_a$ are given in the statement of the lemma. The initial data for ϕ are given by (4.11). Using Eq. (4.10), we can calculate ϕ to second order on S_c, which allows us to obtain the initial data for χ_{ab} from (4.9) and the data for the frame. From Eqs. (4.14), (4.15) we can determine the frame coefficients to first order on S_c which allows us to calculate the coefficents $\Gamma_a{}^b{}_c$. Finally, we get from (4.17) the coordinate transformation to first order, which allows us to determine the coefficients $e^\alpha{}_a$ on S_c.

Equation (4.10) is of wave equation type while the remaining equations form a symmetric hyperbolic system if ϕ is thought to as being given. The coupled system can be dealt with either directly or by using the well known procedure to write the wave equation as a symmetric hyperbolic system. Then the whole system will be symmetric hyperbolic and the existence and uniqueness of smooth solutions to this system for the given data follows from known results [9]. This implies in particular the first assertion of the lemma. □

We comment on the open problem. The data as well as the coefficients of our differential system depend smoothly on the coordinates $x^{\alpha'}$ and the parameter c. Thus the solutions will be jointly smooth in $x^{\alpha'}$ and c in some neighbourhood U' of p. Choosing U' small enough, we can define the hypersurfaces $T_c \cap U'$ to be the level hypersurfaces $\{x^3 = c\}$ of a smooth function x^3 which together with the functions x^α on $T_c \cap U'$ provides a smooth coordinate system x^μ on U'.

To solve the existence problem with this type of argument we would need to show e.g. that $\chi_{ab} = \hat{\chi}_{ab}$ on T_c, where χ_{ab} denotes the symmetric tensor obtained as solution to Eqs. (4.20) to (4.24), while $\hat{\chi}_{ab} = \chi_{\mu'\nu'}\, e^{\mu'}{}_a\, e^{\nu'}{}_b$ with $\chi_{\mu'\nu'}$ denoting the second fundamental form on T_c given by (4.9). This will be discussed elsewhere.

5. The Reduced Equations

While the constraints induced by the Einstein equations on space- or time-like hypersurfaces are defined uniquely, there are many ways to extract evolution equations from the Einstein equations. Our choice of "reduced equations" or "propagation equations" (and in fact also the representation of the field equations and the gauge conditions introduced in the previous sections) is motivated by the following observations:

(i) Our propagation equations are symmetric hyperbolic and allow us to formulate a maximally dissipative boundary value problem.

(ii) The constraints are preserved by our propagation equations irrespective of the chosen maximal dissipative boundary condition.

While the requirements in (i) are met by many systems, property (ii) imposes strong restriction on the choice of propagation equations. There appears to be no systematic way to derive such equations and a priori there appears to be no reason why propagation equations satisfying (ii) should exist at all.

Observing (2.2) and (4.2), we obtain for the coefficients $e^\mu{}_p$ the propagation equations

$$0 = -T_0{}^k{}_p\, e^\mu{}_k = \partial_{x^0}\, e^\mu{}_p - (\Gamma_0{}^q{}_p - \Gamma_p{}^q{}_0)\, e^\mu{}_q - \Gamma_0{}^0{}_p\, \delta^\mu{}_0. \tag{5.1}$$

The functions $F^A(x^\mu) = \Gamma_0{}^A{}_0(x^\mu)$ and, for $x^3 > 0$, $f(x^\mu) = \chi(x^\mu)$ will be considered in the following as gauge source functions. They are to be prescribed in accordance with the boundary conditions but are free otherwise. Observing (4.4), (4.6), and the symmetries of the connection coefficients, we have to derive equations for the connection coefficients $\Gamma_A{}^b{}_c$, $\chi_{ab} = \Gamma_a{}^3{}_b$, and $\Gamma_3{}^j{}_b$. The Gauss equations with respect to the hypersurfaces T_c provide the equations

$$\begin{aligned} 0 = \Delta^B{}_{00A} &= e_0(\Gamma_A{}^B{}_0) - e_A(F^B) + \Gamma_C{}^B{}_0\, \Gamma_A{}^C{}_0 \\ &-\Gamma_A{}^B{}_C\, F^C + F^B\, F^C\, g_{AC} + \chi_0{}^B\, \chi_{A0} - \chi_A{}^B\, \chi_{00} - C^B{}_{00A}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} 0 = \Delta^B{}_{C0A} &= e_0(\Gamma_A{}^B{}_C) + F^B\, \Gamma_A{}^0{}_C + \Gamma_A{}^B{}_0\, F^D\, g_{CD} \\ &+\Gamma_D{}^B{}_C\, \Gamma_A{}^D{}_0 + \chi_0{}^B\, \chi_{AC} - \chi_A{}^B\, \chi_{0C} - C^B{}_{C0A}. \end{aligned} \tag{5.3}$$

Codazzi's equations, $0 = \Delta^3{}_{abc} = D_b\, \chi_{ca} - D_c\, \chi_{ba} - C^3{}_{abc}$, imply propagation equations

$$0 = g^{ab}\, \Delta^3{}_{ab1} = D_0\, \chi_{01} - D_1\, \chi_{11} - D_2\, \chi_{12} - D_1(f), \tag{5.4}$$

$$0 = g^{ab}\,\Delta^3{}_{ab2} = D_0\,\chi_{02} - D_1\,\chi_{12} - D_2\,\chi_{22} - D_2(f), \tag{5.5}$$

$$0 = \Delta^3{}_{101} = D_0\,\chi_{11} - D_1\,\chi_{01} - C^3{}_{101}, \tag{5.6}$$

$$0 = \Delta^3{}_{201} + \Delta^3{}_{102} = 2\,D_0\,\chi_{12} - D_1\,\chi_{02} - D_2\,\chi_{01} - C^3{}_{201} - C^3{}_{102}, \tag{5.7}$$

$$0 = \Delta^3{}_{202} = D_0\,\chi_{22} - D_2\,\chi_{02} - C^3{}_{202}. \tag{5.8}$$

In these equations it is understood that the component χ_{00}, which appears only in undifferentiated form, is given by $\chi_{00} = \chi_{11} + \chi_{22} + f$. Using again the relation $g^{ab}\Gamma_a{}^3{}_b = f$, we get for the coefficients $\Gamma_3{}^j{}_b$ the equations

$$\begin{aligned} 0 = \Delta^A{}_{B03} &= e_0(\Gamma_3{}^A{}_B) + F^A\,\Gamma_3{}^0{}_B + \Gamma_3{}^A{}_0\,F^C\,g_{BC} + \Gamma_C{}^A{}_B\,\Gamma_3{}^C{}_0 \\ &+ \Gamma_3{}^A{}_B\,\Gamma_3{}^3{}_0 + \chi_0{}^A\,\Gamma_3{}^3{}_B - \Gamma_3{}^A{}_3\,\chi_{0B} - \Gamma_C{}^A{}_B\,\chi_0{}^C - C^A{}_{B03}, \end{aligned} \tag{5.9}$$

$$\begin{aligned} 0 = \Delta^A{}_{003} &= e_0(\Gamma_3{}^A{}_0) - e_3(F^A) + \chi_0{}^A\,\Gamma_3{}^3{}_0 - \Gamma_3{}^A{}_B\,F^B + \Gamma_B{}^A{}_0\,\Gamma_3{}^B{}_0 \\ &+ \Gamma_3{}^A{}_0\,\Gamma_3{}^3{}_0 - \Gamma_B{}^A{}_0\,\chi_0{}^B - \Gamma_3{}^3{}_B\,g^{BA}\,\chi_{00} - F^A\,\chi_{00} - C^A{}_{003}, \end{aligned} \tag{5.10}$$

$$\begin{aligned} 0 = \Delta^3{}_{A03} + \Delta^3{}_{03A} &= e_0(\Gamma_3{}^3{}_A) - e_A(\Gamma_3{}^3{}_0) \\ &+ \Gamma_3{}^3{}_0\,F^B\,g_{BA} + \Gamma_3{}^3{}_C\,\Gamma_A{}^C{}_0, \end{aligned} \tag{5.11}$$

$$\begin{aligned} 0 = g^{ab}\,\Delta^3{}_{ab3} &= e_0(\Gamma_3{}^3{}_0) + g^{AB}\,e_A(\Gamma_3{}^3{}_B) - e_3(f) \\ &- g^{ab}\,\Gamma_3{}^3{}_k\Gamma_b{}^k{}_a + g^{ab}\,\Gamma_b{}^3{}_k\Gamma_3{}^k{}_a + g^{ab}\,\Gamma_m{}^3{}_a(\Gamma_3{}^m{}_b - \Gamma_b{}^m{}_3). \end{aligned} \tag{5.12}$$

There are various ways to extract symmetric hyperbolic propagation equations from the overdetermined Bianchi equation. We shall use the boundary adapted system introduced in [4] because this is particularly well suited for the discussion of initial boundary value problems. We denote by $N = e_3$ the vector orthogonal to the family of hypersurfaces T_c and write again $n = e_0$, $\epsilon_{ijk} = n^l\,\epsilon_{lijk}$.

Using the fact that the electric and magnetic parts of the conformal Weyl tensor are symmetric and trace-free, the boundary adapted system is written as a system for the unknowns

$$E_{ij}, \quad B_{ij}, \quad 1 \le i \le j, \quad i < 3.$$

It is understood that the relations $g^{ij}\,E_{ij} = 0$ and $g^{ij}\,B_{ij} = 0$ are used everywhere in the following equations to replace the fields E_{33} and B_{33} by our unknowns. The boundary adapted system is then given by

$$P_{ij} + N_{(i}\,\epsilon_{j)}{}^{kl}\,N_k\,Q_l = 0, \quad Q_{ij} - N_{(i}\,\epsilon_{j)}{}^{kl}\,N_k\,P_l = 0, \quad 1 \le i \le j,\ i < 3. \tag{5.13}$$

Writing it in the equivalent form

$$\begin{array}{rlrl} P_{11} - P_{22} &= 0, & Q_{11} - Q_{22} &= 0, \\ 2\,P_{12} &= 0, & 2\,Q_{12} &= 0, \\ P_{11} + P_{22} &= 0, & Q_{11} + Q_{22} &= 0, \\ P_{13} &= \frac{1}{2}Q_2, & Q_{13} &= -\frac{1}{2}P_2, \\ P_{23} &= -\frac{1}{2}Q_1, & Q_{23} &= \frac{1}{2}P_1, \end{array} \tag{5.14}$$

as a system for the unknown "vector" u which is the transpose of

$$((E_-, 2E_{12}, E_+, E_{13}, E_{23}), (B_-, 2B_{12}, B_+, B_{13}, B_{23})),$$

where $E_\pm = E_{11} \pm E_{22}$, and $B_\pm = B_{11} \pm B_{22}$, it takes the form

$$(\mathbf{I}^\mu + \mathbf{A}^\mu)\,\partial_\mu u = b, \tag{5.15}$$

with

$$\mathbf{I}^\mu = \begin{bmatrix} I^\mu & 0 \\ 0 & I^\mu \end{bmatrix}, \qquad \mathbf{A}^\mu = \begin{bmatrix} 0 & A^\mu \\ {}^t A^\mu & 0 \end{bmatrix},$$

where

$$I^\mu = \delta^\mu{}_0 \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}, \qquad A^\mu = \begin{bmatrix} 0 & -e^\mu{}_3 & 0 & e^\mu{}_2 & e^\mu{}_1 \\ e^\mu{}_3 & 0 & 0 & -e^\mu{}_1 & e^\mu{}_2 \\ 0 & 0 & 0 & e^\mu{}_2 & -e^\mu{}_1 \\ -e^\mu{}_2 & e^\mu{}_1 & -e^\mu{}_2 & 0 & 0 \\ -e^\mu{}_1 & -e^\mu{}_2 & e^\mu{}_1 & 0 & 0 \end{bmatrix}. \tag{5.16}$$

The reduced equations consisting of (5.1) to (5.13), in which our gauge conditions, in particular (4.2), (4.4), are assumed, is thus seen to form a symmetric hyperbolic system. The following specific feature of the system (5.13) should be noticed here. As discussed in [4], we could have taken the system $P_{jk} = 0$, $Q_{jk} = 0$, suitably interpreted, as propagation equations. This would also have resulted in a symmetric hyperbolic system of propagation equations. However, in that case the rank of the matrix A^3, and with it the freedom to prescribe boundary data for the reduced system, would have been larger than in the present case. Another important reason for the choice of (5.13) will be pointed out in our discussion of the subsidiary system.

6. The Subsidiary System

We show now that solutions to the reduced system which satisfy the constraints on S are indeed solutions to the full Einstein equations. Let g' be a time-oriented Lorentz metric on M for which T is time-like and S is space-like and such that it is in the past of $M \setminus S$. For a given subset V of $T \cup S$ and an open subset U of M we define the domain of dependence of V in U with respect to g' as the set of points $p \in U$ such that (i) $I^-(p)$, the chronological past of p in (M, g'), is contained in U, (ii) every past inextendible g'-causal curve through p meets $V \cap U$.

Theorem 6.1. *Suppose that the fields $e^\mu{}_k$, $\Gamma_i{}^j{}_k$, $C^i{}_{jkl}$, with $\chi_{ab} \equiv \Gamma_a{}^3{}_b$ symmetric, are smooth on some open neighbourhood U of $p \in \Sigma$ in M and satisfy the gauge conditions (4.2), (4.4) as well as the reduced equations (5.1) to (5.13) on U. Let g be the metric for which the frame e_k is orthonormal and denote by D the domain of dependence of $(S \cup T) \cap U$ in U with respect to g. Then the Einstein equations (2.5) will be satisfied on D if they are satisfied on $S \cap U$.*

Remark 6.1. It is a remarkable feature of Einstein's equation that it admits a hyperbolic reduced system which allows us to draw such a conclusion without any reference to the behaviour of the fields on T.

Proof. Since we have to show that the tensor fields defined by the left-hand sides of Eqs. (2.2), (2.3), (2.4) vanish on D, we shall refer to these fields as to the "zero quantities". The reduced equations are equivalent to the equations

$$T_0{}^k{}_j = 0, \tag{6.1}$$

$$\Delta^a{}_{b0k} = 0, \quad \Delta^3{}_{A0A} = 0, \quad g^{ab}\,\Delta^3{}_{ab3} = 0, \tag{6.2}$$

$$\Delta^3{}_{A03} + \Delta^3{}_{03A} = 0, \quad \Delta^3{}_{201} + \Delta^3{}_{102} = 0, \quad g^{ab}\,\Delta^3{}_{abA} = 0,$$

$$P_{ij} + N_{(i}\,\epsilon_{j)}{}^{kl}\,N_k\,Q_l = 0, \quad Q_{ij} - N_{(i}\,\epsilon_{j)}{}^{kl}\,N_k\,P_l = 0. \tag{6.3}$$

We get slightly more information on the torsion tensor. Observing the assumed symmetry of χ_{ab}, we get

$$-T_1{}^i{}_2\,e^\mu{}_i = e^\nu{}_1\,\partial_\nu\,e^\mu{}_2 - e^\nu{}_2\,\partial_\nu\,e^\mu{}_1 - (\Gamma_1{}^a{}_2 - \Gamma_2{}^a{}_1)\,e^\mu{}_a.$$

Evaluating this expression for $\mu = 3$ we get $T_1{}^3{}_2\,e^3{}_3 = 0$, from which we conclude by (4.2) that

$$T_A{}^3{}_B = 0. \tag{6.4}$$

Since $\Gamma_{ijk} = -\Gamma_{ikj}$, we know that the connection defined by the connection coefficients is metric. However, it is not clear at this stage whether it is torsion free, since so far we only know that (6.1) holds. For this reason the curvature tensor defined by the connection coefficients is not given by $r^i{}_{jkl}$ but by

$$R^i{}_{jkl} = e_k\,(\Gamma_l{}^i{}_j) - e_l\,(\Gamma_k{}^i{}_j) + \Gamma_k{}^i{}_m\,\Gamma_l{}^m{}_j$$
$$-\Gamma_l{}^i{}_m\,\Gamma_k{}^m{}_j - \Gamma_m{}^i{}_j\,(\Gamma_k{}^m{}_l - \Gamma_l{}^m{}_k - T^m{}_{kl}),$$

which is equivalent to

$$R^i{}_{jkl} = \Delta^i{}_{jkl} + C^i{}_{jkl} + \Gamma_m{}^i{}_j\,T_k{}^m{}_l.$$

Furthermore, it is not known at this stage whether the tensor $C^i{}_{jkl}$ is indeed the conformal Weyl tensor of the metric defined by the frame coefficients $e^\mu{}_k$. Together with the torsion tensor the curvature tensor satisfies the Bianchi identities

$$\sum_{(jkl)} \nabla_j\,T_k{}^i{}_l = \sum_{(jkl)} (R^i{}_{jkl} + T_j{}^m{}_k\,T_l{}^i{}_m),$$
$$\sum_{(jkl)} \nabla_j\,R^i{}_{mkl} = -\sum_{(jkl)} R^i{}_{mnj}\,T_k{}^n{}_l,$$

where $\sum_{(jkl)}$ denotes the sum over the cyclic permutation of the indices jkl. Observing that we assumed the symmetry $\sum_{(jkl)} C^i{}_{jkl} = 0$, the first identity can be written in the form

$$\sum_{(jkl)} \nabla_j T_k{}^i{}_l = \sum_{(jkl)} (\Delta^i{}_{jkl} + \Gamma_m{}^i{}_j\,T_k{}^m{}_l + T_j{}^m{}_k\,T_l{}^i{}_m). \tag{6.5}$$

Again, observing this equation and that the tensor $C^i{}_{jkl}$ has the algebraic properties of a conformal Weyl tensor, we get the relation

$$\sum_{(jkl)} \nabla_j C^i{}_{mkl} = -\frac{1}{2}\,\epsilon^{k'l'i}{}_m \epsilon_{jkl}{}^{m'}\,\nabla_{i'}\,C^{i'}{}_{m'k'l'}.$$

This allows us to write the second Bianchi identity in the form

$$\sum_{(jkl)} \nabla_j \Delta^i{}_{mkl} = L^i{}_{mjkl}, \tag{6.6}$$

where we set

$$L^i{}_{mjkl} = \frac{1}{2}\,\epsilon^{k'l'i}{}_m\,\epsilon_{jkl}{}^{j'}\,H_{j'k'l'} - \sum_{(jkl)}\Big\{\Gamma_n{}^i{}_m\,\Delta^n{}_{jkl} \tag{6.7}$$
$$+(R^i{}_{mnj} + \nabla_j\,\Gamma_n{}^i{}_m + \Gamma_{n'}{}^i{}_m\,\Gamma_n{}^{n'}{}_j)\,T_k{}^n{}_l + \Gamma_n{}^i{}_m\,T_j{}^{n'}{}_k\,T_l{}^n{}_{n'}\Big\},$$

with $\nabla_j\,\Gamma_n{}^i{}_m = e_j(\Gamma_n{}^i{}_m) - \Gamma_j{}^l{}_n\,\Gamma_l{}^i{}_m + \Gamma_j{}^i{}_l\,\Gamma_n{}^l{}_m - \Gamma_j{}^l{}_m\,\Gamma_n{}^i{}_l$. Notice that the field $L^i{}_{mjkl}$ is a polynomial in the zero quantities which vanishes if the zero quantities vanish.

The identities above will be used to derive certain systems of differential equations, the "subsidiary systems", which are satisfied by the zero quantities. In view of (6.1), we get from (6.5) the equation

$$\nabla_0\,T_k{}^i{}_l + \nabla_l\,T_0{}^i{}_k + \nabla_k\,T_l{}^i{}_0 = \Delta^i{}_{0kl} + \Delta^i{}_{l0k} + \Delta^i{}_{kl0} + \Gamma_m{}^i{}_0\,T_k{}^m{}_l, \tag{6.8}$$

which can be rewritten

$$e_0(T_k{}^i{}_l) = \Delta^i{}_{0kl} + \Delta^i{}_{l0k} + \Delta^i{}_{kl0} + (\Gamma_m{}^i{}_0 - \Gamma_0{}^i{}_m)\,T_k{}^m{}_l \tag{6.9}$$
$$+ (\Gamma_l{}^m{}_0 - \Gamma_0{}^m{}_l)\,T_m{}^i{}_k + (\Gamma_k{}^m{}_0 - \Gamma_0{}^m{}_k)\,T_l{}^i{}_m.$$

With (6.2), (6.4) this implies in our gauge

$$e_0(T_A{}^3{}_3) = -\Gamma_A{}^B{}_0\,T_B{}^3{}_3,$$

from which we conclude that $T_A{}^3{}_3 = 0$. Combined with (6.1), (6.4) this gives

$$T_i{}^3{}_j = 0. \tag{6.10}$$

Using this equation in (6.5) we get the relation

$$\sum_{(jkl)} \Delta^3{}_{jkl} = \sum_{(jkl)} (\Gamma_j{}^3{}_a - \Gamma_a{}^3{}_j)\,T_k{}^a{}_l, \tag{6.11}$$

which implies

$$\Delta^3{}_{012} + \Delta^3{}_{201} + \Delta^3{}_{120} = 0, \tag{6.12}$$
$$\Delta^3{}_{123} + \Delta^3{}_{231} = \Gamma_3{}^3{}_a\,T_1{}^a{}_2. \tag{6.13}$$

We shall show now that the "vector" v which is the transpose of

$$(T_1{}^a{}_2,\ \Delta^A{}_{012},\ \Delta^1{}_{221},\ \Delta^3{}_{001},\ \Delta^3{}_{002},\ \Delta^3{}_{012}),$$

vanishes on D. For this purpose we derive a homogeneous symmetric hyperbolic system for v as follows.

The equations for $T_1{}^a{}_2$, obtained from (6.9), are given by

$$e_0(T_1{}^a{}_2) = \Delta^a{}_{012} + (\Gamma_A{}^a{}_0 - \Gamma_0{}^a{}_A)T_1{}^A{}_2 - (\Gamma_1{}^1{}_0 + \Gamma_2{}^2{}_0)T_1{}^a{}_2. \quad (6.14)$$

The equations for the remaining components of v are obtained by observing (6.2), the symmetry of $\Gamma_a{}^3{}_b$ as well as the results obtained so far, and by writing out in detail the six equations of (6.6) where the quantities

$$L^1{}_{0012}, \quad L^2{}_{0012}, \quad L^1{}_{2021}, \quad L^3{}_{0012}, \quad L^3{}_{2210}, \quad L^3{}_{1120},$$

occur on the left-hand sides. It is important to note that there is one component of the tensor H_{jkl} for each component of the tensor $L^i{}_{mjkl}$ coming from $\epsilon^{k'l'i}{}_m\,\epsilon_{jkl}{}^{j'}\,H_{j'k'l'}$. For the quantities mentioned above these components are respectively,

$$H_{323} \quad H_{331} \quad H_{330} \quad H_{310} \quad H_{302} \quad H_{312}.$$

However all these quantities vanish due to the reduced equations (6.3). A straightforward calculation shows us that

$$\begin{aligned} H_{323} &= Q_{13} + \frac{1}{2}P_2 = 0, & H_{310} &= P_{13} - \frac{1}{2}Q_2 = 0,\\ H_{331} &= Q_{23} - \frac{1}{2}P_1 = 0, & H_{302} &= P_{23} + \frac{1}{2}Q_1 = 0,\\ H_{330} &= P_{33} = 0, & H_{312} &= Q_{33} = 0, \end{aligned}$$

where the last two equations follow from the reduced equations since P_{ij} and Q_{ij} are trace free. After a somewhat lengthy though straightforward calculation we get

$$\begin{aligned} e_0(\Delta^1{}_{012}) = {}& \frac{3}{2}\Gamma_0{}^3{}_1\,\Delta^3{}_{012} - \left(\Gamma_0{}^3{}_0 + \Gamma_1{}^3{}_1\right)\Delta^3{}_{002} + \Gamma_2{}^3{}_1\,\Delta^3{}_{001} \\ & - \Gamma_0{}^2{}_0\,\Delta^1{}_{221} - \left(2\Gamma_1{}^1{}_0 + \Gamma_2{}^2{}_0\right)\Delta^1{}_{012} - \Gamma_2{}^1{}_0\,\Delta^2{}_{012} \\ & - \left(R^1{}_{0a0} + \nabla_0\Gamma_a{}^1{}_0 + \Gamma_k{}^1{}_0\,\Gamma_a{}^k{}_0\right)T_1{}^a{}_2, \end{aligned} \quad (6.15)$$

$$\begin{aligned} e_0(\Delta^2{}_{012}) = {}& \frac{3}{2}\Gamma_0{}^3{}_2\,\Delta^3{}_{012} + \left(\Gamma_0{}^3{}_0 + \Gamma_2{}^3{}_2\right)\Delta^3{}_{001} - \Gamma_1{}^3{}_2\,\Delta^3{}_{002} \\ & + \Gamma_0{}^1{}_0\,\Delta^1{}_{221} - \left(\Gamma_1{}^1{}_0 + 2\Gamma_2{}^2{}_0\right)\Delta^2{}_{012} - \Gamma_1{}^2{}_0\,\Delta^1{}_{012} \\ & - \left(R^2{}_{0a0} + \nabla_0\Gamma_a{}^2{}_0 + \Gamma_k{}^2{}_0\,\Gamma_a{}^k{}_0\right)T_1{}^a{}_2, \end{aligned} \quad (6.16)$$

$$\begin{aligned} e_0(\Delta^1{}_{221}) = {}& \Gamma_0{}^3{}_1\,\Delta^3{}_{001} + \Gamma_0{}^3{}_2\,\Delta^3{}_{002} - \left(\Gamma_0{}^2{}_0 - \Gamma_1{}^1{}_2\right)\Delta^1{}_{012} \\ & + \left(\Gamma_0{}^1{}_0 + \Gamma_2{}^1{}_2\right)\Delta^2{}_{012} - \left(\Gamma_1{}^1{}_0 + \Gamma_2{}^2{}_0\right)\Delta^1{}_{221} \\ & + \left(R^1{}_{2a0} + \nabla_0\Gamma_a{}^1{}_2 + \Gamma_k{}^1{}_2\,\Gamma_a{}^k{}_0\right)T_1{}^a{}_2, \end{aligned} \quad (6.17)$$

$$\begin{aligned} & e_0(\Delta^3{}_{012}) - e_1(\Delta^3{}_{002}) + e_2(\Delta^3{}_{001}) = -\left(2\Gamma_0{}^2{}_0 + \Gamma_1{}^1{}_2\right)\Delta^3{}_{001} \\ & + \left(2\Gamma_0{}^1{}_0 + \Gamma_2{}^2{}_1\right)\Delta^3{}_{002} - 2\Gamma_0{}^3{}_A\,\Delta^A{}_{012} - \frac{3}{2}\left(\Gamma_1{}^1{}_0 + \Gamma_2{}^2{}_0\right)\Delta^3{}_{012} \\ & - \left(R^3{}_{0a0} + \nabla_0\Gamma_a{}^3{}_0 + \Gamma_k{}^3{}_0\,\Gamma_a{}^k{}_0\right)T_1{}^a{}_2, \end{aligned} \quad (6.18)$$

$$e_0(\Delta^3{}_{001}) + \frac{1}{2}e_2(\Delta^3{}_{012}) = -\left(\Gamma_1{}^1{}_0 + 2\Gamma_2{}^2{}_0\right)\Delta^3{}_{001} + \Gamma_1{}^2{}_0\,\Delta^3{}_{002} \tag{6.19}$$
$$-\frac{3}{2}\Gamma_0{}^2{}_0\,\Delta^3{}_{012} + \Gamma_0{}^3{}_0\,\Delta^2{}_{012} + \Gamma_A{}^3{}_2\,\Delta^A{}_{012} + \Gamma_0{}^3{}_1\,\Delta^1{}_{221}$$
$$+\left(R^3{}_{2a0} + \nabla_0\Gamma_a{}^3{}_2 + \Gamma_k{}^3{}_2\,\Gamma_a{}^k{}_0\right)T_1{}^a{}_2,$$
$$e_0(\Delta^3{}_{002}) - \frac{1}{2}e_1(\Delta^3{}_{012}) = \Gamma_2{}^1{}_0\,\Delta^3{}_{001} - \left(2\Gamma_1{}^1{}_0 + \Gamma_2{}^2{}_0\right)\Delta^3{}_{002} \tag{6.20}$$
$$+\frac{3}{2}\Gamma_0{}^1{}_0\,\Delta^3{}_{012} - \left(\Gamma_0{}^3{}_0 + \Gamma_1{}^3{}_1\right)\Delta^1{}_{012} - \Gamma_2{}^3{}_1\Delta^2{}_{012} - \Gamma_0{}^3{}_2\,\Delta^1{}_{221}$$
$$-\left(R^3{}_{1a0} + \nabla_0\Gamma_a{}^3{}_1 + \Gamma_k{}^3{}_1\,\Gamma_a{}^k{}_0\right)T_1{}^a{}_2.$$

Multiplying the last two equations by 2 we obtain a system for v which is symmetric hyperbolic. A calculation shows that its characteristics are non-space-like for g. Moreover, it does not contain the directional derivative operator e_3. As pointed out in our discussion of maximally dissipative boundary value problems this allows us to obtain energy estimates for our solution regardless of its behaviour on T. Since $v = 0$ on S by assumption, it follows that v vanishes in D. Thus we have

$$T_1{}^a{}_2 = 0,$$
$$\Delta^A{}_{012} = 0, \quad \Delta^1{}_{221} = 0, \quad \Delta^3{}_{001} = 0, \quad \Delta^3{}_{002} = 0, \quad \Delta^3{}_{012} = 0,$$

whence, by (6.2), (6.12), (6.13),

$$\Delta^3{}_{201} = 0, \quad \Delta^3{}_{102} = 0, \quad \Delta^3{}_{123} + \Delta^3{}_{231} = 0.$$

To write out the equations for the "vector" u which is the transpose of

$$(\Delta^3{}_{103}, \Delta^3{}_{203}, \Delta^3{}_{123}, \Delta^3{}_{113}, \Delta^3{}_{223}),$$

it will be convenient to use the following notation. For any tensor field $T^{ij\cdots}{}_{kl\ldots}$ we write

$$\Gamma_m\,T^{i\ldots j}{}_{k\ldots l} \equiv (\nabla_m - e_m)\,T^{i\ldots j}{}_{k\ldots l} = \Gamma_m{}^i{}_n\,T^{n\ldots j}{}_{k\ldots l} + \ldots - \Gamma_m{}^n{}_l\,T^{i\ldots j}{}_{k\ldots n},$$

which is a bi-linear expression in the components of the tensor field and the connection coefficients.

Observing the results obtained so far, we get, again from (6.6), the equations

$$e_0(\Delta^3{}_{103}) - e_1(\Delta^3{}_{113}) - e_2(\Delta^3{}_{123}) \tag{6.21}$$
$$= g^{ij}\left[L^3{}_{ij13} - \left(\Gamma_j\,\Delta^3{}_{i13} + \Gamma_3\,\Delta^3{}_{ij1} + \Gamma_1\,\Delta^3{}_{i3j}\right)\right],$$

$$e_0(\Delta^3{}_{203}) - e_1(\Delta^3{}_{123}) - e_2(\Delta^3{}_{223}) \tag{6.22}$$
$$= g^{ij}\left[L^3{}_{ij23} - \left(\Gamma_j\,\Delta^3{}_{i23} + \Gamma_3\,\Delta^3{}_{ij2} + \Gamma_2\,\Delta^3{}_{i3j}\right)\right],$$

$$2\,e_0(\Delta^3{}_{123}) - e_1(\Delta^3{}_{203}) - e_2(\Delta^3{}_{103}) = L^3{}_{1023} + L^3{}_{2130} \tag{6.23}$$
$$-\left(\Gamma_0\,\Delta^3{}_{123} + \Gamma_3\,\Delta^3{}_{102} + \Gamma_2\,\Delta^3{}_{130}\right) - \left(\Gamma_1\,\Delta^3{}_{230} + \Gamma_0\,\Delta^3{}_{213} + \Gamma_3\,\Delta^3{}_{201}\right),$$

$$e_0(\Delta^3{}_{113}) - e_1(\Delta^3{}_{103}) = L^3{}_{1013} \tag{6.24}$$
$$- \left(\Gamma_0\, \Delta^3{}_{113} + \Gamma_3\, \Delta^3{}_{101} + \Gamma_1\, \Delta^3{}_{130}\right),$$

$$e_0(\Delta^3{}_{223}) - e_2(\Delta^3{}_{203}) = L^3{}_{2023} \tag{6.25}$$
$$- \left(\Gamma_0\, \Delta^3{}_{223} + \Gamma_3\, \Delta^3{}_{202} + \Gamma_2\, \Delta^3{}_{230}\right).$$

The system (6.21) to (6.25) for u takes the form $A^\mu\, \partial_\mu\, u = b$ with

$$A^\mu = \begin{bmatrix} e^\mu{}_0 & 0 & -e^\mu{}_2 & -e^\mu{}_1 & 0 \\ 0 & e^\mu{}_0 & -e^\mu{}_1 & 0 & -e^\mu{}_2 \\ -e^\mu{}_2 & -e^\mu{}_1 & 2\, e^\mu{}_0 & 0 & 0 \\ -e^\mu{}_1 & 0 & 0 & e^\mu{}_0 & 0 \\ 0 & -e^\mu{}_2 & 0 & 0 & e^\mu{}_0 \end{bmatrix}.$$

Using finally the definition of H_{jkl} we set

$$N_{kl} = \nabla^j H_{jkl} = \nabla^i\, \nabla^j\, C_{ijkl}. \tag{6.26}$$

Observing that our solution C_{ijkl} has by definition all the symmetries of a conformal Weyl tensor, i.e. $C_{ijkl} = C_{[ij][kl]}$, $C_{ijkl} = C_{klij}$, $C_{ijk}{}^j = 0$, $C_{i[jkl]} = 0$, which imply $C_{ij[k}{}^m C^{ij}{}_{l]m} = 0$, we find

$$N_{kl} = R_{mi}{}^m{}_j\, C^{ij}{}_{kl} - R_{ij[k}{}^m\, C^{ij}{}_{l]m} - \frac{1}{2} T_i{}^m{}_j\, \nabla_m\, C^{ij}{}_{kl} \tag{6.27}$$
$$= -\Delta_{mi}{}^m{}_j\, C^{ij}{}_{kl} - \Gamma_m{}^n{}_i\, T_n{}^m{}_j\, C^{ij}{}_{kl} + \Delta_{ij[k}{}^m\, C^{ij}{}_{l]m}$$
$$-\Gamma_{mij}\, C^{ijn}{}_{[l} T_{k]}{}^m{}_n + \frac{1}{2}\, T_i{}^m{}_j\, \nabla_m\, C^{ij}{}_{kl}.$$

We note that the expression on the far right-hand side is linear in the zero quantities.

On the other hand we obtain from the identity (2.6) and the reduced equations (5.13) a relation

$$H_{jkl} = P_i\, u^i{}_{jkl} + Q_i\, v^i{}_{jkl}, \tag{6.28}$$

with

$$u^i{}_{jkl} = -2\, n_j\, n_{[k}\, h_{l]}{}^i + h_{j[k}\, h_{l]}{}^i - N_{(j}\, \epsilon_{n)}{}^{mi}\, N_m\, \epsilon^n{}_{kl},$$
$$v^i{}_{jkl} = n_j\, \epsilon^i{}_{kl} - \epsilon^i{}_{j[k}\, n_{l]} + n_l\, N_{(j}\, \epsilon_{k)}{}^{mi}\, N_m - n_k\, N_{(j}\, \epsilon_{l)}{}^{mi}\, N_m.$$

Contracting (6.28) with $2\, h_q{}^k\, n^l$ and $-\epsilon_p{}^{kl}$ respectively, we finally get

$$2\, \mathcal{L}_n\, P_q + (\epsilon_{qji} + 2\, N_{(j}\, \epsilon_{q)mi}\, N^m)\, \mathcal{D}^j\, Q^i \tag{6.29}$$
$$= 2\, K_q{}^i\, P_i - 2\, P_i\, h_q{}^k\, n^l\, \nabla^j\, u^i{}_{jkl} - 2\, Q_i\, h_q{}^k\, n^l\, \nabla^j\, v^i{}_{jkl} + 2\, h_q{}^k\, n^l\, N_{kl},$$

$$2\, \mathcal{L}_n\, Q_p - (\epsilon_{pji} + 2\, N_{(j}\, \epsilon_{p)mi}\, N^m)\, \mathcal{D}^j\, P^i \tag{6.30}$$
$$= 2\, K_q{}^i\, Q_i + P_i\, \epsilon_p{}^{kl}\, \nabla^j\, u^i{}_{jkl} + Q_i\, \epsilon_p{}^{kl} \nabla^j\, v^i{}_{jkl} - \epsilon_p{}^{kl}\, N_{kl}.$$

If we write these equations as a system for the unknown w which is the transpose of $((P_1, P_2, P_3), (Q_1, Q_2, Q_3))$, they take the form

$$\mathbf{I}^\mu\,\partial_\mu\,w + \mathbf{A}^\mu\,\partial_\mu\,w = b,$$

with

$$\mathbf{I}^\mu = \begin{bmatrix} I^\mu & 0 \\ 0 & I^\mu \end{bmatrix}, \qquad \mathbf{A}^\mu = \begin{bmatrix} 0 & A^\mu \\ {}^tA^\mu & 0 \end{bmatrix},$$

$$I^\mu = \begin{bmatrix} 2\,e^\mu{}_0 & 0 & 0 \\ 0 & 2\,e^\mu{}_0 & 0 \\ 0 & 0 & e^\mu{}_0 \end{bmatrix}, \qquad A^\mu = \begin{bmatrix} 0 & 0 & e^\mu{}_2 \\ 0 & 0 & -\,e^\mu{}_1 \\ -\,e^\mu{}_2 & e^\mu{}_1 & 0 \end{bmatrix}.$$

Equations (6.9) for the remaining components of the torsion tensor, Eqs. (6.21) to (6.25), and Eqs. (6.29), (6.30) provide the subsidiary system for those zero quantities of which we do not know yet whether they vanish. The system is symmetric hyperbolic and a calculation shows that its characteristics are non-space-like for $g_{\mu\nu}$. The derivative operator e_3 does not occur in the system. Since the zero quantities vanish on S, we conclude that they vanish on D. $\square$

The requirement that the operator e_3 does not occur in the subsidiary systems was one of our main criteria for choosing the reduced system. Otherwise we would have been confronted with the task to analyse in detail the structure of boundary data for the subsidiary systems which are determined by the reduced system from the data on S as well as on T.

7. Initial and Boundary Data

In the following we discuss how to prepare initial and boundary data for the reduced equations. The discussion of the initial data is somewhat complicated by the fact that we do not require the unit normal e_0 of Σ in T to be orthogonal to S. Without this generality our results would be of rather restricted applicability.

7.1. The construction of initial data. Experience with the standard Cauchy problem for Einstein's vacuum field equation tells us that we have to assume as initial data on S a smooth (negative) Riemannian metric $\gamma_{\alpha\beta}$ and a smooth symmetric tensor field $\kappa_{\alpha\beta}$ satisfying the Hamiltonian and the momentum constraint

$$R' - (\kappa^\alpha{}_\alpha)^2 + \kappa^{\alpha\beta}\,\kappa_{\alpha\beta} = 0, \qquad \delta'^\alpha\,\kappa_{\alpha\beta} - \delta'_\beta\,\kappa^\alpha{}_\alpha = 0, \tag{7.1}$$

on S. Here δ' denotes the Levi–Civita connection and R' the Ricci scalar of the metric γ. To derive initial data for the reduced equations we shall first determine data in terms of coordinates x'^μ and a frame e'_k which are suitably adapted to the initial hypersurface S and shall then express these data in the coordinates x^μ and the frame field e_k which satisfy the conditions described in Sect. 4.

Let x'^μ be functions on $S \cap U$ with $x'^0 = 0$ on $S \cap U$, $x'^3 = 0$ on $\Sigma \cap U$, $x'^3 > 0$ on $(S \setminus \Sigma) \cap U$, such that the x'^α, $\alpha = 1, 2, 3$, define a smooth coordinate system on $S \cap U$ and the x'^α, $\alpha = 1, 2$, are constant along the integral curves of the gradient of x'^3. For numbers $c \geq 0$ in the range of x'^3 we set $S_c = \{x'^3 = c\}$. Let $\{e'_p\}_{p=1,2,3}$ be a smooth frame field on $S \cap U$ such that e'_3 is orthogonal to the surfaces S_c, pointing towards S on $S_0 = \Sigma \cap U$, and such that

$$\gamma(e'_p, e'_q) = g'_{pq} = diag\,(-1, -1, -1).$$

The information on the metric γ is contained in the coefficients $e'^{\alpha}{}_p = e'_p(x'^{\alpha})$. We write $\kappa'_{pq} = \kappa_{\alpha\beta}\, e'^{\alpha}{}_p\, e'^{\beta}{}_q$.

Imagine now the initial data set (S, γ, κ) as being isometrically embedded into a solution $(M, g_{\mu\nu})$ of the field equations and denote by ∇ the connection defined by g. Let e'_0 be the (future directed) unit normal of S. We assume that the orthonormal frame field $\{e'_k\}_{k=0,\ldots,3}$ and the functions x'^{μ} on S are extended off S such that the frame is parallely propagated in the direction of e'_0 and that the coordinates x'^{α}, $\alpha = 1, 2, 3$, are constant on the integral curves of e'_0, while x'^0 is a natural parameter on these curves. The connection coefficients defined by $\nabla_{e'_k}\, e'_j = \Gamma'_k{}^i{}_j\, e'_i$ then satisfy on S (cf. condition (7.27) added below)

$$\delta'_{e'_p}\, e'_q = \Gamma'_p{}^r{}_q\, e'_r, \quad \Gamma'_0{}^i{}_j = 0, \quad \Gamma'_p{}^0{}_q = -\kappa'_{pq}, \tag{7.2}$$

and we have for the coefficients $e'^{\mu}{}_k = e'_k(x'^{\mu})$,

$$e'^{\mu}{}_0 = \delta^{\mu}{}_0, \quad e'^0{}_p = 0, \quad e'^{\mu}{}_3 = \delta^{\mu}{}_3\, e'^3{}_3, \quad e'^3{}_3 > 0.$$

The electric part of the conformal Weyl tensor with respect to $\nu = e'_0$ then follows under our assumptions from the Gauss equation on S. It is given by

$$E'_{pq} = C'_{p0q0} = R'_{pq} - \frac{1}{4}\, R'\, g'_{pq} - \{\kappa'_r{}^r\, (\kappa'_{pq} - \frac{1}{4}\, \kappa'_s{}^s\, g'_{pq}) - \kappa'_{sp}\, \kappa'_q{}^s + \frac{1}{4}\, \kappa'_{rs}\, \kappa'^{rs}\, g'_{pq}\},$$

where R'_{pq} denotes the Ricci tensor of the metric γ in the frame e'_p and R' is the Ricci scalar of γ. The tensor above is obviously symmetric, the Hamiltonian constraint ensures that it is trace free.

The magnetic part follows under our assumptions from the Codazzi equation. It is given by

$$B'_{pq} = \frac{1}{2}\, C'_{p0ik}\, \epsilon'_{q0}{}^{ik} = -\epsilon'_q{}^{rs}\, \delta'_{e'_r}\, \kappa'_{sp},$$

where ϵ'_{ijkl} is totally antisymmetric, $\epsilon'_{0123} = 1$, and $\epsilon'_{pqr} = \nu'^i\, \epsilon'_{ipqr}$. The symmetry of the tensor above is a consequence of the momentum constraint, it is trace free because of the symmetry of κ'_{pq}. These fields together determine the conformal Weyl tensor C'_{ijkl} in the frame e'_j by the formula

$$C'_{ijkl} = 2\left(l'_{j[k}\, E'_{l]i} - l'_{i[k}\, E'_{l]j} - \nu'_{[k}\, B'_{l]m}\, \epsilon'^m{}_{ij} - \nu'_{[i}\, B'_{j]m}\, \epsilon'^m{}_{kl}\right),$$

where we set $l'_{ij} = g'_{ij} - 2\, \nu'_i\, \nu'_j$.

Only projections into S of expressions (2.1) to (2.4) can be determined from our data $e'^{\mu}{}_k$, $\Gamma'_i{}^j{}_k$, E'_{pq}, B'_{pq}, C'_{ijkl}. Using the projector $\hat{h}'_i{}^j = g'_i{}^j - \nu'_i\, \nu'^j$ and the fact that in 3 dimensions the Riemann tensor is given in terms of the Ricci tensor, we find by the way we derived our data from γ and κ that

$$\hat{h}'_p{}^i\, \hat{h}'_q{}^j\, T'_i{}^k{}_j = 0, \tag{7.3}$$

$$\hat{h}'_p{}^k\, \hat{h}'_q{}^l\, \Delta'^i{}_{jkl} = 0, \tag{7.4}$$

$$P'_s = \nu'^j\, \hat{h}'_s{}^l\, \nu'^m\, \nabla_{e'_i}\, C'^i{}_{jlm} = 0, \quad Q'_s = -\frac{1}{2}\, \nu'^j\, \epsilon'_s{}^{lm}\, \nabla_{e'_i}\, C'^i{}_{jlm} = 0, \tag{7.5}$$

i.e. the constraints induced by Eqs. (2.5) on $S \cap U$ are satisfied by our data.

If the normal n of $\Sigma \cap U$ in T were orthogonal to S, we could set $e_k = e'_k$ on $S \cap U$ and use the data determined above as initial data for the reduced field equations. To include the case where n does not necessarily coincide with e'_0 on Σ we proceed as follows. We choose functions x^μ such that $x^\mu = x'^\mu$ on $S \cap U$. We set $e_A = e'_A$, $A = 1, 2$, and

$$e_0 = \cosh(\theta)\, e'_0 + \sinh(\theta)\, e'_3, \quad e_3 = \sinh(\theta)\, e'_0 + \cosh(\theta)\, e'_3, \tag{7.6}$$

with $\theta \in C^\infty(S \cap U)$ chosen such that $e_0 = n$ on $\Sigma \cap U$. We write the relations above in the form $e_k = \Lambda^j{}_k\, e'_j$ with a Lorentz transformation $\Lambda^j{}_k$.

We note here that $\Theta \equiv \theta|_{\Sigma \cap U}$ is a free datum which determines in part the geometry of the space-time we wish to construct, while on the remaining part of $S \cap U$ the function θ must be regarded as a gauge source function.

Near S the coordinates x^μ will be chosen such that $e^\mu{}_0 = e_0(x^\mu) = \delta^\mu{}_0$. This implies on S the relation

$$\delta^\mu{}_0 = e_0(x^\mu) = \cosh(\theta)\, e'_0(x^\mu) + \sinh(\theta)\, e'_3(x^\mu) = \cosh(\theta)\, \frac{\partial x^\mu}{\partial x'^0} + \sinh(\theta)\, e'^\mu{}_3,$$

which allows us to determine $\frac{\partial x^\mu}{\partial x'^\nu}$ and thus the frame coefficients $e^\mu{}_k = e_k(x^\mu)$ on $S \cap U$.

The transformation law between the connection coefficients defined by $\nabla_{e_i} e_k = \Gamma_i{}^j{}_k\, e_j$ and the connection coefficients $\Gamma'_i{}^j{}_k$ reads

$$\Gamma_i{}^k{}_j = e_i(\Lambda^m{}_j)\, \Lambda_m{}^k + \Lambda^l{}_i\, \Lambda^n{}_j\, \Gamma'_l{}^m{}_n\, \Lambda_m{}^k, \tag{7.7}$$

with $\Lambda_i{}^k = g_{ji}\, \Lambda^j{}_l\, g^{lk}$, which satisfies $\Lambda_i{}^k\, \Lambda^i{}_l = \delta^k{}_l$. To determine the left hand side of (7.7) we need to determine the derivatives of $\Lambda^i{}_k$. The requirement that the latter is a Lorentz transformation implies $e_i(\Lambda^m{}_j)\, \Lambda_{mk} + e_i(\Lambda^m{}_k)\, \Lambda_{mj} = 0$, which translates into the equivalent conditions

$$e_i(\Lambda^A{}_B) = -e_i(\Lambda^B{}_A), \tag{7.8}$$

$$\begin{aligned} e_i(\Lambda^A{}_0) &= \cosh(\theta)\, e_i(\Lambda^0{}_A) - \sinh(\theta)\, e_i(\Lambda^3{}_A), \\ e_i(\Lambda^A{}_3) &= \sinh(\theta)\, e_i(\Lambda^0{}_A) - \cosh(\theta)\, e_i(\Lambda^3{}_A), \\ e_i(\Lambda^0{}_0) &= e_i(\Lambda^3{}_3), \quad e_i(\Lambda^3{}_0) = e_i(\Lambda^0{}_3), \\ \cosh(\theta)\, e_i(\Lambda^0{}_0) &= \sinh(\theta)\, e_i(\Lambda^3{}_0). \end{aligned} \tag{7.9}$$

Observing that $e_0(\Lambda^i{}_j) = \cosh(\theta)\, e'_0(\Lambda^i{}_j) + \sinh(\theta)\, e'_3(\Lambda^i{}_j)$ and that we can calculate the tangential derivatives $e'_p(\Lambda^i{}_j)$ for $p = 1, 2, 3$, on $S \cap U$, we find from the gauge condition $\Gamma_0{}^A{}_B = 0$ and (7.7),

$$e'_0(\Lambda^A{}_B) = -\Gamma'_0{}^A{}_B - \tanh(\theta)\, \Gamma'_3{}^A{}_B.$$

The gauge condition $\Gamma_0{}^0{}_A = -g_{AB}\, F^B$ gives with (7.8),

$$\cosh(\theta)\, e'_0(\Lambda^A{}_0) = -g_{AB}\, F^B - \Lambda^k{}_0\, \Gamma'_k{}^l{}_A\, \Lambda_l{}^0.$$

The requirement that e_3 is hypersurface orthogonal at S_c, i.e. $\chi_{0A} = \chi_{A0}$, implies with (7.8)

$$\cosh(\theta)\, e'_0(\Lambda^A{}_3) = -e'_A(\theta) - \Lambda^k{}_0\, (\Gamma'_A{}^l{}_k - \Gamma'_k{}^l{}_A)\, \Lambda_l{}^3.$$

Using again (7.8) we obtain from these relations the quantities $e'_0(\Lambda^i{}_j)$ with $i = A = 1, 2$, or $j = B = 1, 2$. The gauge condition $g^{ab}\,\Gamma_a{}^3{}_b = f$ implies on $S \cap U$,

$$e'_0(\Lambda^3{}_0) = -\sinh(\theta)\, e'_3(\theta) + f - g^{ab}\,\Lambda^k{}_a\,\Lambda^l{}_b\,\Gamma'_k{}^i{}_l\,\Lambda_i{}^3,$$

from which we determine the quantities $e'_0(\Lambda^i{}_j)$ with $i, j = 0, 3$, by using (7.9). We note that the quantity $\Gamma_A{}^3{}_B$ is indeed symmetric because of the relation

$$\Gamma_A{}^3{}_B = -\sinh(\theta)\,\Gamma'_A{}^0{}_B + \cosh(\theta)\,\Gamma'_A{}^3{}_B,$$

implied by (7.7). The terms on the right hand side are symmetric because e'_0 and e'_3 are orthogonal to $S \cap U$ and S_c respectively.

Finally, the conformal Weyl tensor is given in our gauge by

$$C_{ijkl} = C'_{i'j'k'l'}\,\Lambda^{i'}{}_i\,\Lambda^{j'}{}_j\,\Lambda^{k'}{}_k\,\Lambda^{l'}{}_l,$$

where the primed indices take values $0, \ldots, 3$.

The data so obtained are useful for our purpose because we have

Lemma 7.1. *Suppose $e^\mu{}_k$, $\Gamma_i{}^j{}_k$, C_{ijkl} coincide on $S \cap U$ with the data determined above and satisfy the reduced field equations (5.1) to (5.13) as well as our gauge conditions in some neighbourhood of $S \cap U \simeq \{0\} \times (S \cap U) \subset \mathbb{R} \times (S \cap U)$. Then $e^\mu{}_k$, $\Gamma_i{}^j{}_k$, C_{ijkl} satisfy Eqs. (2.5) on S.*

Proof. We have to show that the tensor fields $T_i{}^k{}_j$, $\Delta^i{}_{jkl}$, H_{jkl} vanish on $S \cap U$. Given the metric for which $e^\mu{}_k$ is orthonormal, we can extend the coordinates x'^μ and the frame e'_k off $S \cap U$ as described above and express the tensor fields in terms of this gauge. We have $n = n'^i\, e'_i$ with $n'^i = \cosh(\theta)\,\nu'^i + \sinh(\theta)\,\delta^i{}_3$. In terms of $T'_i{}^j{}_k = T'_i{}^j{}_k[e', \Gamma']$ Eq. (5.1) reads $0 = n'^i\, T'_i{}^j{}_k = \cosh(\theta)\,\nu'^i\, T'_i{}^j{}_k + \sinh(\theta)\, T'_3{}^j{}_k$. Using (7.3) we obtain from this relation that $\nu'^i\, T'_i{}^j{}_k = 0$ on $S \cap U$. This equation and (7.3) imply by the tensorial nature of T that $T_i{}^j{}_k = 0$ on $S \cap U$.

We have a decomposition

$$\Delta^i{}_{jkl} = \mathcal{D}^i{}_{jkl} + 2\,\mathcal{D}^i{}_{j[l}\, n_{k]},$$

with fields

$$\mathcal{D}_{ijkl} = \Delta_{ijmn}\, h^m{}_k\, h^n{}_l, \quad \mathcal{D}_{ijl} = \Delta_{ijkn}\, n^k\, h^n{}_l,$$

which are anti-symmetric in the indices i, j.

If the torsion tensor vanished to first order on $S \cap U$ we could use the first Bianchi identity to deduce the identity

$$2\,\mathcal{D}^3{}_{[AB]} = \mathcal{D}^3{}_{0AB}.$$

However, observing the assumed symmetry of $\chi_{ab} = \Gamma_a{}^3{}_b$, this relation can be verified in our case by a direct calculation. The reduced equations (6.2) can then be rewritten in the form

$$\mathcal{D}^a{}_{bp} = 0, \quad \mathcal{D}^3{}_{0p} = g^{ab}\,\mathcal{D}^3{}_{apb}, \quad \mathcal{D}^3{}_{AB} = \frac{1}{2}\,\mathcal{D}^3{}_{0AB}, \quad \mathcal{D}^3{}_{A3} = \mathcal{D}^3{}_{0A3}. \tag{7.10}$$

Equation (7.4) reads $\Delta^i{}_{jmn}\,\hat{h}^m{}_k\,\hat{h}^n{}_l = 0$, where $\hat{h}^j{}_k = g^j{}_k - \nu^j\,\nu_k$ with $\nu^i = \cosh(\theta)\, n^i - \sinh(\theta)\, N^i$. Transvecting this equation suitably with $h^j{}_k$ and n^i we find that it is equivalent to

$$\mathcal{D}^i{}_{jAB} = 0, \quad \mathcal{D}^i{}_{j3A} = \tanh(\theta)\, \mathcal{D}^i{}_{jA}. \tag{7.11}$$

It is now a matter of straightforward algebra to show that Eqs. (7.10), (7.11) imply $\mathcal{D}^i{}_{jp} = 0$, $\mathcal{D}^i{}_{jpq} = 0$.

Consider the field $H'_{ijk} = H'_{ijk}[e', \Gamma', C']$ decomposed with respect to ν according to the rule (2.6). The fact that the constraints (7.5) are satisfied on $S \cap U$ is expressed equivalently by the equation $\nu'^i\, H'_{ijk} = 0$, which is in turn equivalent to

$$0 = \nu^i\, H_{ijk} = \cosh(\theta)\, n^i\, H_{ijk} - \sinh(\theta)\, N^i\, H_{ijk},$$

on $S \cap U$. On the other hand we have by our assumptions the relation (6.28). Together these two equations imply

$$\begin{aligned} 0 &= \cosh(\theta)\, (P_i\, n^j\, u^i{}_{jkl} + Q_i\, n^j\, v^i{}_{jkl}) - \sinh(\theta)\, (P_i\, N^j\, u^i{}_{jkl} + Q_i\, N^j\, v^i{}_{jkl}) \\ &= \cosh(\theta)\, (-2\, n_{[k}\, P_{l]} + Q^i\, \epsilon_{ikl}) - \sinh(\theta)\, 2\, N_{[k}\, P_{l]}, \end{aligned}$$

which entails $P_l = 0$ and $Q_k = 0$ on $S \cap U$. □

7.2. The boundary conditions. The boundary conditions for the reduced system are determined by the rules described in Sect. 3. In the reduced system the only contribution to the normal matrix comes from (5.15) and the boundary conditions thus only involve the conformal Weyl tensor. By (5.16) we find

$$\begin{aligned} {}^t u\, A^3\, u = 4\, B_-\, E_{12} - 4\, E_-\, B_{12} &= -\{\frac{1}{\sqrt{2}}\, (E_- + 2\, B_{12})\}^2 - \{\frac{1}{\sqrt{2}}\, (B_- - 2\, E_{12})\}^2 \\ &+\{\frac{1}{\sqrt{2}}\, (E_- - 2\, B_{12})\}^2 + \{\frac{1}{\sqrt{2}}\, (B_- + 2\, E_{12})\}^2. \end{aligned}$$

Choosing a smooth matrix-valued function H on T as in (3.4), we can thus write the boundary conditions in the form

$$q_1 = E_{11} - E_{22} - 2\, B_{12} - a\, (E_{11} - E_{22} + 2\, B_{12}) - b\, (B_{11} - B_{22} - 2\, E_{12}), \tag{7.12}$$

$$q_2 = B_{11} - B_{22} + 2\, E_{12} - c\, (E_{11} - E_{22} + 2\, B_{12}) - d\, (B_{11} - B_{22} - 2\, E_{12}), \tag{7.13}$$

with some given smooth functions q_1, q_2 on T.

The components of the conformal Weyl tensor which enter these conditions are obtained by projecting its e_0-electric and e_0-magnetic parts into the plane orthogonal to e_3 and by taking then the trace-free parts. The resulting tensors are given in our notation by

$$\eta_{AB} = E_{AB} - \frac{1}{2}\, g_{AB}\, g^{CD}\, E_{CD}, \quad \beta_{AB} = B_{AB} - \frac{1}{2}\, g_{AB}\, g^{CD}\, B_{CD}.$$

In terms of the null frame defined by (3.6), the relevant components of the conformal Weyl tensor are given in NP notation by

$$\begin{aligned} \Psi_0 &= C_{\mu\nu\sigma\pi}\, l^\mu\, m^\nu\, l^\sigma\, m^\pi = \eta_{11} + \beta_{12} + i\, (\beta_{11} - \eta_{12}), \\ \Psi_4 &= C_{\mu\nu\sigma\pi}\, \bar{m}^\mu\, k^\nu\, \bar{m}^\sigma\, k^\pi = \eta_{11} - \beta_{12} + i\, (\beta_{11} + \eta_{12}), \end{aligned}$$

and the boundary conditions (7.12), (7.13) take the form

$$q = -\Psi_4 + \alpha\, \Psi_0 + \beta\, \bar{\Psi}_0, \tag{7.14}$$

where q, α, β are defined as in (3.5).

The form of (7.14) can be understood as follows. In our frame the components Ψ_0, Ψ_4 of the conformal Weyl tensor can be interpreted as parts of the field transverse to e_3, traveling into the directions $-e_3$, e_3 respectively (cf. also [5]). Assume there were a family of outgoing null hypersurfaces tangent to the vector field k on T. Then the field equations would imply on these hypersurfaces propagation equations of the form

$$\Psi_{0,\mu}\, k^\mu - \Psi_{1,\mu}\, m^\mu = L(\Gamma_i{}^j{}_k, \Psi_l).$$

This shows clearly that the values of Ψ_0 will be determined on T by the evolution equations once the other fields are given. This is consistent with the fact that the conditions on α, β prevent us from prescribing Ψ_0 on T. On the other hand, if there were a family of ingoing null hypersurfaces tangent to l on T, the field equations would imply on these hypersurfaces propagation equations of the form

$$\Psi_{4,\mu}\, l^\mu - \Psi_{3,\mu}\, \bar{m}^\mu = L'(\Gamma_i{}^j{}_k, \Psi_l),$$

and the quantity Ψ_0 would in fact represent the null datum on these hypersurfaces. Therefore it is natural that we can prescribe the value of Ψ_4 freely on T and couple parts of Ψ_0, $\bar{\Psi}_0$ back to it as it is realized in (7.14).

In trying to give along these lines any explanation of (7.14) in terms of "ingoing/outgoing gravitational radiation" it should be observed that our gauge conditions, in particular the components Ψ_0, Ψ_4 of the conformal Weyl tensor, depend on the choice of the vector field e_0 on T, which so far is rather arbitrary. This situation should be compared with that at null infinity, where one causal direction is singled out by the causal nature of the boundary and a natural concept of "radiation field" is obtained.

The condition on the coefficients of H in (3.4) can be expressed in terms of α, β, and $v = (v_1, v_2)$ in the form

$$^t v\, B\, v \leq \frac{1 - |\alpha|^2 - |\beta|^2}{2}\, {}^t v\, v, \quad v \in \mathbb{R}^2, \tag{7.15}$$

where the symmetric bi-linear form in v on the left hand side is defined by the matrix

$$B = B(\alpha, \beta) = \begin{bmatrix} Re(\bar{\alpha}\,\beta) & Im(\bar{\alpha}\,\beta) \\ Im(\bar{\alpha}\,\beta) & -Re(\bar{\alpha}\,\beta) \end{bmatrix}.$$

Since $v \neq 0$ can always be chosen such that the term on the left-hand side of (7.15) is non-negative, it follows that $|\alpha|^2 + |\beta|^2 \leq 1$. Moreover, since v is arbitrary in (7.15), it follows then that $|\alpha|^2 + |\beta|^2 = 1$ if and only if $\alpha = 0$ or $\beta = 0$.

We take this opportunity to correct a mistake in [4] (which is of no consequence in that article). Equation (5.49) in [4] should be replaced (observing the different notation) by (7.15).

The closest analogue to (3.7) appears to be the following. Observing that the Bel–Robinson tensor is given in spinor notation by $T_{aa'bb'cc'dd'} = \Psi_{abcd}\, \bar{\Psi}_{a'b'c'd'}$, we find that the non-positivity condition (ii) in Sect. 3 takes the form

$$^t u\, A^3\, u = -\Psi_0\, \bar{\Psi}_0 + \Psi_4\, \bar{\Psi}_4 = -2\,(T_{0333} + T_{3000}) \leq 0,$$

where we assume the Bel–Robinson tensor to be given in the frame e_k.

7.3. Boundary conditions and gauge conditions. So far our considerations were based on a fixed choice of a local gauge. If we want to go beyond the study of local solutions, we have to glue together different local solutions and therefore need to discuss the transformation behaviour of the initial and boundary conditions under changes of the gauge. The transformation behaviour of the initial data is obvious and will not be considered any further. The transformation behaviour of the boundary conditions is more complicated.

Since the time-like unit vector field e_0 is assumed to be given on T and e_3 is by definition the inward pointing unit normal of T, the remaining gauge freedom on T consists of smooth coordinate transformations

$$x^\alpha \to x'^\beta(x^\alpha), \quad \alpha, \beta = 1, 2, \tag{7.16}$$

and rotations of the frame

$$e_A \to e'_A = \Lambda^B{}_A \, e_B, \quad (\Lambda^B{}_A) = \Lambda(\Phi) = \begin{bmatrix} \cos\Phi & -\sin\Phi \\ \sin\Phi & \cos\Phi \end{bmatrix}, \tag{7.17}$$

or, in terms of the null frame (3.6),

$$m \to m' = e^{i\,\Phi}\, m. \tag{7.18}$$

Since we assume the vectors e_A to be Fermi-propagated in the direction of e_0 with respect to the intrinsic connection on T, the function Φ in (7.17) is independent of x^0. Further, the coordinates on Σ are dragged along with e_0. Thus the remaining gauge transformation can be specified on T completely in terms of their behaviour on Σ.

The connection coefficients $\Gamma_0{}^A{}_0$, which are specified in terms of the gauge source functions F^A, transform under (7.17) according to

$$\Gamma_0{}^A{}_0 \to \Gamma'_0{}^A{}_0 = \Lambda(\Phi)_B{}^A\, \Gamma_0{}^B{}_0.$$

It will be convenient to give F^A in terms of the complex function

$$F = F^1 + i\, F^2. \tag{7.19}$$

The transformation behaviour above is then reflected by

$$F \to F' = e^{i\,\Phi}\, F. \tag{7.20}$$

The components of the conformal Weyl tensor transform under (7.18) according to

$$\Psi_0 \to \Psi'_0 = e^{2\,i\,\Phi}\, \Psi_0, \quad \Psi_4 \to \Psi'_4 = e^{-2\,i\,\Phi}\, \Psi_4.$$

To make sense of the boundary condition (7.14) in a covariant way we require that the functions on T which enter this condition transform under (7.18) as

$$q \to q' = e^{-2\,i\,\Phi}\, q, \quad \alpha \to \alpha' = e^{-4\,i\,\Phi}\, \alpha, \quad \beta \to \beta' = \beta. \tag{7.21}$$

It is important to note that (7.15) is invariant under (7.21). This follows from the facts that B transforms according to

$$B(\alpha', \beta') = {}^t\Lambda(-\Phi)\, B(\alpha, \beta)\, \Lambda(-\Phi),$$

under (7.21) and that the quadratic expressions on the right-hand side of (7.15) are invariant under (7.21) and rotations of v.

To describe the boundary conditions in a covariant way we introduce some tensor fields which contract to zero with any vector orthogonal to e_A, $A = 1, 2$. We refer to such tensors as to "e_A-tensors". The symmetric trace-free e_A-tensors of rank 2 are generated by tensors t_{ij}, s_{ij} with non-trivial (i.e. not necessarily vanishing) components

$$t_{AB} = \delta^1{}_A\,\delta^1{}_B - \delta^2{}_A\,\delta^2{}_B, \quad s_{AB} = \delta^1{}_A\,\delta^2{}_B + \delta^2{}_A\,\delta^1{}_B,$$

respectively. The e_A-tensor J_{ijkl} with non-trivial components

$$J_{ABCD} = \frac{Re(\alpha)}{2}\,(t_{AB}\,t_{CD} - s_{AB}\,s_{CD}) + \frac{Im(\alpha)}{2}\,(t_{AB}\,s_{CD} + s_{AB}\,t_{CD}), \tag{7.22}$$

is completely symmetric and trace-free. In fact, any symmetric trace-free e_A-tensor of rank 4 has this form with certain coefficients $Re(\alpha)$, $Im(\alpha)$. Therefore the form is necessarily preserved under the transformations (7.17) and it turns out that the coefficients transform under (7.17) into $Re(e^{-4i\Phi}\,\alpha)$, $Im(e^{-4i\Phi}\,\alpha)$, in accordance with (7.21). We finally need the tensor $\epsilon_{jk} = N^i\,\epsilon_{ijk}$ and the induced metric on the subspaces orthogonal to e_0, e_3, which is represented by g_{AB}. We use these tensors and the function β, which transforms under (7.17) according to (7.21), to define the e_A-tensor I_{ijkl} with nontrivial components

$$I_{AB}{}^{CD} = Re(\beta)\,g_{(A}{}^C\,g_{B)}{}^D + Im(\beta)\,g_{(A}{}^C\,\epsilon_{B)}{}^D. \tag{7.23}$$

It is invariant under (7.17) and contracts with a symmetric trace-free e_A-tensor of rank 2 to yield another such tensor. Setting now

$$\rho^{\pm}_{AB} = \eta_{AB} \pm \beta_{AC}\,\epsilon^C{}_B, \tag{7.24}$$

and introducing as the free datum on T the symmetric trace-free e_A-tensor q_{ij} with non-trivial components

$$q_{AB} = Re(q)\,t_{AB} + Im(q)\,s_{AB}, \tag{7.25}$$

we find that the boundary conditions (7.12), (7.13) can be written as a tensor equation on T which has non-trivial components

$$q_{AB} = -\rho^-_{AB} + I_{AB}{}^{CD}\,\rho^+_{CD} + J_{AB}{}^{CD}\,\rho^+_{CD}. \tag{7.26}$$

The main property of the tensor fields J_{ijkl}, I_{ijkl}, q_{ij} is that the form of their expressions in the frame e_k is universal, that they do not depend on the vectors e_0, e_3 orthogonal to e_A, and that they are uniquely determined by the functions α, β, q.

Consider a neighbourhood W of Σ in T. Assume a fixed orientation of Σ, and denote by $O_+(\Sigma)$ the bundle of oriented orthonormal frames on Σ. Because of the transformation laws (7.20), (7.21) the complex-valued functions F, α, β, q should at Σ not be considered as functions on Σ but as spin weighted functions on $O_+(\Sigma)$.

Suppose that the time-like vector field e_0 is given on W, the flow lines of e_0 generate W, and x^0 maps each flow line onto the interval $[0, x^0_*[$, where x^0_* is a fixed positive number or infinity. For $p \in W$ denote by $p^* \in \Sigma$ the point at which the flow line on e_0 passing through p meets Σ. For given e_A at p denote by e^*_A the frame of Σ at p^* which is transported into e_A by T-intrinsic Fermi transport along the flow line.

With the values of F, α, β, q in the 2-frame e_A at p we associate the same complex numbers $F(x^0(p))$, $\alpha(x^0(p))$, $\beta(x^0(p))$, $q(x^0(p))$ in the 2-frame e^*_A at p^*. For given $x^0 \in [0, x^0_*[$ we thus get a set of smooth complex-valued functions F, α, β, q on $[0, x^0_*[\times O_+(\Sigma)$ of spin weight 1, -4, 0, -2 respectively. Giving these functions is equivalent to giving the tensor fields J_{ijkl}, I_{ijkl}, q_{ij} on W because of the universal form of the local expressions (7.22), (7.23), (7.25).

We will need to extend the gauge source function F into a neighbourhood of T in the prospective solution space-time.

Definition 7.1. *We call a smooth function x^3 defined on some open neighbourhood U of Σ in S a boundary defining function on S if*

$$d\,x^3 \neq 0, \quad x^3|_\Sigma = 0, \quad x^3|_{S\setminus\Sigma} > 0,$$

and if the sets $S_c = \{x^3 = c\} \subset S$ are diffeomorphic to Σ for $0 \leq c < x^3_* \equiv \sup_U x^3$ and are obtained by pushing forward Σ with the flow of the vector field $-(|d\,x^3|_\gamma)^{-2}\, grad_\gamma x^3$.

Given a boundary defining function x^3, we denote by e'_3 the smooth unit vector field which is orthogonal to S_c on U and points towards S on $S_0 = \Sigma$. Using the flow lines of e'_3 we can map S_c diffeomorphically onto Σ and get the representation $U = \Sigma \times [0, x^3_*[$. Following our discussion in Sect. 4, we consider time-like hypersurfaces T_c having intersection S_c with S. The hypersurfaces T_c and the coordinate x^0 on them are generated by a time-like vector field e_0. Repeating the discussion above, we find that for given c the gauge source functions F^A on T_c can be represented by a smooth complex-valued function on $[0, x^0_*[\times O_+(S_c)$ of spin weight 1. Using S-intrinsic Fermi-transport of local frames e_A on Σ in the direction of e'_3 and parametrizing the integral curves by x^3, we obtain bundle morphisms of the $O_+(S_c)$ onto $O_+(\Sigma)$. This allows us to specify the information about the gauge source functions F^A in a neighbourhood of T in M in terms of a smooth complex-valued function F on $[0, x^0_*[\times O_+(\Sigma) \times [0, x^3_*[$ of spin-weight 1. We shall say that F is based on the boundary defining function x^3 on S.

Conversely, given F as above and a local section of $O_+(\Sigma)$, i.e. an oriented orthonormal frame e_A on some open subset V of Σ, we can use Fermi-transport of e_A in the directions of e'_3 and e_0 to obtain the gauge source function F^A in the frame e_A (resp. e_k).

The requirement that the fields e_A be Fermi-transported along e'_3 should be added to (7.2) in the form

$$\Gamma'_3{}^A{}_B = 0 \quad \text{on} \quad U. \tag{7.27}$$

The following (where we set $\mathbb{R}^+_0 = [0, \infty[$) summarizes our main observations about the initial and boundary data and the gauge source functions.

Definition 7.2. *A smooth initial boundary data set for Einstein's vacuum field equation consists of the following.*

A smooth, orientable, compact, 3-dimensional initial manifold S with boundary $\Sigma \neq \emptyset$ and the boundary manifold $T = \mathbb{R}^+_0 \times \Sigma$ (with this product structure distinguished). The boundaries Σ and $\{0\} \times \Sigma$ of these manifolds are identified in the natural way.

A smooth (negative) Riemannian metric $\gamma_{\alpha\beta}$ and a smooth symmetric tensor field $\kappa_{\alpha\beta}$ on S which satisfy the constraints (7.1).

A smooth real function Θ on Σ.

A smooth real function $T \ni (x^0, p) \to \chi(x^0, p) \in \mathbb{R}$.

Smooth complex functions F, α, β, q on $\mathbb{R}^+_0 \times O_+(\Sigma)$ of spin weight 1, -4, 0, -2 respectively, such that the functions α, β satisfy condition (7.15).

It may be surprising that the function F is listed as part of the initial data set. The somewhat complicated situation concerning the pair χ, F will be discussed in the next section.

Definition 7.3. *Given an initial boundary data set as in Definition 7.2, an associated set of gauge source functions consists of:*

A smooth real function θ *on* S *with* $\theta|_\Sigma = \Theta$.

A smooth real function $\mathbb{R}^+_0 \times \Sigma \times [0, x^3_*[\ni (x^0, p, x^3) \to f(x^0, p, x^3) \in \mathbb{R}$ *such that* $f(x^0, p, 0) = \chi(x^0, p)$, $p \in \Sigma$. *Here we use a boundary defining function* x^3 *on* S *to represent a neighbourhood* U *of* Σ *in* S *in the form* $U = \Sigma \times [0, x^3_*[$ *with some* $x^3_* > 0$.

A smooth complex function

$$F : [0, x^0_*[\times O_+(\Sigma) \times [0, x^3_*[\ni (x^0, p, x^3) \to F(x^0, p, x^3) \in C,$$

of spin-weight 1, *based on the boundary defining function* x^3 *above, which coincides on* $[0, x^0_*[\times O_+(\Sigma) \times \{0\}$ *with the function* F *given in Definition 7.2.*

7.4. The consistency condition. Given the data γ, κ in Definition 7.2 and the gauge source function f, θ, F in Definition 7.3, we can, by taking formal derivatives of the reduced field equations, determine a formal expansion in terms of x^0 on a neighbourhood of Σ in S for the fields $e^\mu{}_k$, $\Gamma_i{}^j{}_k$, C_{ijkl}, and thus in particular for η_{AB}, β_{AB}. On the other hand, we also have at Σ a formal expansion of the functions α, β, q on T in terms of x^0. Therefore, to obtain a smooth solution, the formal expansions obtained for the quantities entering the two sides of the boundary condition (7.26) should coincide at any order, i.e. the data need to satisfy a certain "consistency condition". To meet this condition we may e.g. choose all fields except q, determine the formal expansion of the expression on the right-hand side of (7.26), and choose q on T such that it has the same formal expansion at Σ.

8. The Existence Result

Given an initial boundary data set as in Definition 7.2, we set $M = \mathbb{R}^+_0 \times S$ such that S and T, identified along their boundary Σ, can be considered in a natural way as the boundary of M. We define the function x^0 on M such that it induces the natural coordinate on the factor $\mathbb{R}^+_0$. Then $S = \{x^0 = 0\}$. We can now formulate our main result.

Theorem 8.1. *Suppose we are given a smooth initial boundary data set as in Definition 7.2 and an associated set of smooth gauge source functions as in Definition 7.3 such that the consistency conditions on* Σ *are satisfied at any order. Then we can find some open set* M' *in* M, *with* $\{p \in M \,|\, x^0(p) < \tau\} \subset M'$ *for some* $\tau > 0$, *and on* M' *a solution* g *to Einstein's field equation* $\mathrm{Ric}[g] = 0$ *such that* M' *coincides with the domain of dependence of* $S \cup T'$ *in* (M', g), *where* $T' = T \cap M'$, *and the following properties hold.*

(i) S *is space-like and* T' *is time-like for* g. *The first and second fundamental form induced by* g *on* S *is given (up to a common diffeomeorphism) by* γ *and* κ *respectively. The mean extrinsic curvature induced by* g *on* T' *is given by* χ.

(ii) The curves $\mathbb{R}^+_0 \ni x^0 \to (x^0, p) \in T$, $p \in \Sigma$, *induce curves on* T' *whose tangent vectors define a smooth time-like unit vector field* e_0 *on* T' *orthogonal to* Σ. *If* e'_3 *denotes the unit normal of* Σ *in* S *pointing towards* S, *we have* $g(e_0, e'_3) = -\sinh(\Theta)$.

(iii) Denote by e_3 *the inward pointing* g*-unit normal of* T'. *Let* e_A, $A = 1, 2$, *be an oriented frame on some open subset* V *of* Σ *such that* e_k, $k = 0, 1, 2, 3$, *defines an orthonormal frame for* M' *on* V. *Extend* e_A *into* T' *by* T'*-intrinsic Fermi-transport in the direction of* e_0. *For the components* F^A *of* F *in the frame* e_A *we have then* $F^A = g^{AB}\, g(\nabla_{e_0} e_0, e_B)$. *If* α, β, q *are given in the frame* e_A *the associated* e_A*-tensors (cf. (7.22), (7.23), (7.25)) satisfy the boundary condition (7.26).*

(iv) In the particular case where $\chi = \chi_0$ is constant and $F = 0$ on T the solution is locally (geometrically) unique near S.

Remark 8.1. (i) That the solution is (geometrically) unique in the domain of dependence of the set S is well known from the study of the standard Cauchy problem. To demonstrate in general the uniqueness of the solution locally in time of our initial-boundary value problem, we would have to show that the solution is independent of the choice of gauge source functions. To show this we would need the existence statement which is missing in Lemma 4.2.

(ii) We have included F as a datum in Definition 7.2. Given a solution, we can according to Lemma 4.1 always redefine the vector field e_0 and the associated coordinates on T close to Σ to achieve a transition

$$(\chi(x^\alpha),\ F(x^\alpha)) \to (\chi'(x^{\alpha'}),\ F' \equiv 0). \tag{8.1}$$

This shows that locally the freedom encoded in the pair χ, F corresponds to that of one real-valued function and Theorem 8.1 tells us that this function is not restricted by any condition if questions concerning the life-time of the solutions are ignored.

(iii) If we could perform the transition (8.1) globally on T', irrespective of the life time of the solution, it would be natural to use the particular gauge with $F' = 0$ and specify χ' as the part of the data which characterizes the nature of the boundary. However, the integral curves of the vector field e_0 will then be T'-intrinsic geodesics. In general, we can therefore expect that the gauge with $F' = 0$ will, due to focussing phenomena of the geodesics, have a lifetime much shorter than the lifetime of the solution which was specified in terms of χ and F.

(iv) This suggests to consider as a datum equivalence classes of pairs (χ, F) to characterize the boundary. However, which pairs are equivalent in this sense does depend also on the other data (which, incidentally, are related on the boundary to (χ, F) by the vector field e_0 which is specified implicitly in terms of F) and can only be decided after the solutions are available. There appears to be no way to compare different pairs by calculations on T solely in terms of the data prescribed on T. For the same reason it is not possible to determined which pairs (χ, F) are particularly "good" for specifying a space-time and which pairs are locally equivalent but not particularly useful because they refer to a gauge which breaks down quickly.

(v) These difficulties, which are intrinsic to the initial-boundary value problem and do not represent a peculiar feature of our specific type of analysis, arise because the coordinates on the boundary in which χ is given are related in a direct way to the evolution of the fields.

(vi) In the case of the Anti-de Sitter-type space-times studied in [4] boundary data are prescribed on the boundary at infinity which is singled out in a geometric way. There the difficulties pointed out above do not arise due to the special geometric features of the boundary.

(vii) For convenience we assume all data to be smooth and we obtain smooth solutions. If weaker smoothness requirements are imposed on the data, a loss of smoothness along the boundary may occur for the solution to the reduced equations. We do not analyse whether due to particular features of the Einstein equations (such as the presence of constraints) more smoothness will be preserved than suggested by the general results (cf. [8, 14]).

(viii) From the following proof it can be seen immediately that a result similar to Theorem 8.1 is obtained in the case where S has only inner boundaries and asymptotically flat ends or asymptotically hyperboloidal (cf. [3]) ends with smooth asymptotics.

(ix) We finally remark that all smooth solutions to Einstein's vacuum field equations on a region bounded by a space-like and a time-like hypersurface as considered in the introduction can be characterized in terms of data as considered above. Furthermore, if the boundary T and the boundary conditions are extended suitably "backward in time", the existence of a solution characterized by such data also follows from our result.

Proof. Since we are dealing with a hyperbolic problem, we can show the existence of a solution by patching together local solutions. A basic step consists in solving the initial boundary value problem in some neighbourhood U of a given point $p \in \Sigma$ in M. Let x^1, x^2 be coordinates and e_A a smooth oriented γ-orthonormal frame field on some open neighbourhood V of p in Σ. Let e'_3 be the smooth unit normal to the surfaces S_c in S defined by the boundary defining function x^3 such that e'_3 points towards S on Σ. We extend the coordinates x^1, x^2 into S such that they are constant on the integral curves of e'_3 and form together with x^3 a coordinate system on some neighbourhood of p in S which is denoted in our notation by $V \times [0, x^3_*[$. We assume the frame e_A to be extended to $V \times [0, x^3_*[$ by S-intrinsic Fermi-transport along the integral curves of e'_3. The vector fields e_A are then tangent to S_c.

Observing now the gauge conditions in Sect. 4 and the meaning of the gauge source functions, we use the gauge source functions in the gauge determined by the coordinates x^1, x^2 and the frame e_A on V to obtain the reduced equations described in Sect. 5. The initial data on $V \times [0, x^3_*[\subset S$ and the boundary conditions on $\mathbb{R}^+_0 \times V \subset T$ for the reduced equations are determined as described in Sect. 7.

With the help of suitable cut-off functions we can put the initial boundary value problem so obtained into the setting considered in [8] (cf. [4] for the details of such a procedure). The results in [8] then imply the existence of some neighbourhood U of p in M, with $S \cap U \subset V \times [0, x^3_*[$ and $T \cap U \subset \mathbb{R}^+_0 \times V$, on which there exists a unique smooth solution $u = (e^\mu{}_k, \Gamma_i{}^j{}_k, C^i{}_{jkl})$ of the reduced field equations which satisfies our gauge conditions on U and the initial and boundary conditions on $S \cap U$ and $T \cap U$ respectively. We assume that the neighbourhood U is chosen such that it coincides with the domain of dependence of the set $(S \cup T) \cap U$ in U with respect to the metric g for which the frame e_k is orthonormal. By Theorem 6.1 and Lemma 7.1 we concluded that u satisfies indeed Eqs. (2.5) and thus $\mathrm{Ric}[g] = 0$.

The local solutions can be patched together to yield a solution on some neighbourhood of Σ in M. Consider $p, q \in \Sigma$ and solutions u_p, u_q to (2.5) on neighbourhoods U_p, U_q of these points respectively which are obtained as described above. If $U_p \cap U_q \cap \Sigma = \emptyset$ we have also $U_p \cap U_q = \emptyset$. If $U_p \cap U_q \cap \Sigma \neq \emptyset$, the initial data given on $U_p \cap S$ and $U_q \cap S$ can be related on their intersection $U_p \cap U_q \cap S$ by the explicitly known simple gauge transformations (7.16), (7.17) which also relate on $U_p \cap U_q \cap S$ the boundary conditions given on $U_p \cap T$ and $U_q \cap T$. These transformations imply also transformations of the gauge source functions. Using the uniqueness property for the solution of the initial boundary value problem for the reduced equations (which is an immediate consequence of the energy estimates) we can thus show that the solution induced by u_p on the domain of dependence D_p (with respect to u_p) of $U_p \cap U_q \cap (S \cup T)$ in U_p is related by a gauge transformation to the solution induced by u_q on the domain of dependence D_q (with respect to u_q) of $U_q \cap U_p \cap (S \cup T)$ in U_q. Thus we can identify (U_p, u_p), (U_q, u_q) on $U_p \cap U_q$ via the gauge tranformation to obtain a solution on $U_p \cup U_q$. Since the time-like frame vectors on U_p and U_q are not affected by the gauge transformations (7.16), (7.17) they are also identified on $U_p \cap U_q$ and we obtain a unique time-like vector field e_0 on $U_p \cup U_q$. Proceeding along these lines we can construct a neighbourhood Z of Σ in M on which there exists a smooth solution u of (2.5) such that the initial and boundary conditions are satisfied on $Z \cap S$ and $Z \cap T$ respectively and Z coincides with the domain of dependence of $Z \cap (S \cup T)$ with respect to u. Furthermore we get on Z a unique time-like unit vector field e_0 which is in particular tangent to $Z \cap T$.

It is well known from the study of the Cauchy problem for Einstein's field equation that the data γ, κ on $S \setminus \Sigma$ determine a (geometrically) unique, smooth, maximal, globally hyperbolic solution (M_S, g_S) to the vacuum field equations. Denote by D_Z the domain of dependence of $(S \setminus \Sigma) \cap Z$ in (Z, g), with g the metric determined from u, and by D the domain of dependence of $(S \setminus \Sigma) \cap Z$ in (M_S, g_S). The results on the Cauchy problem then allow us to conclude that there must exist an isometric embedding ψ of D_Z into D. Using ψ to identify D_Z with $\psi(D_Z)$, we obtain a solution (M', g) to the vacuum field equations. We can, possibly after shrinking M' slightly, extend the vector field e_0 given in a neighbourhood of $Z \cap T$ to a time-like unit vector field e_0 in (M', g) and define a smooth function x^0 which vanishes on S and satisfies $< e_0, d\,x^0 > = 1$ on M'. Choosing $\tau > 0$ small enough, the integral curves of e_0 starting on S will have length not smaller than τ. This proves assertions (i)–(iii) of the theorem.

The proof of assertion (iv) relies on the fact that we can bring the solutions into a standard form near Σ if the mean extrinsic curvature is constant on the boundary.

Assume that $\chi = \chi_0 =$ const. and that (M', g), $(\hat{M}', \hat{g})$ are solutions of the vacuum equations satisfying conditions (i)–(iii). Denote by D, $\hat{D}$ the domain of dependence of $S \setminus \Sigma$ in (M', g), $(\hat{M}', \hat{g})$ respectively. We can assume, possibly after shrinking D, $\hat{D}$ in time, that there exists an isometry ψ of $(D, g|_D)$ onto $(\hat{D}, \hat{g}|_{\hat{D}})$ which induces the identity on $S \setminus \Sigma$.

Let x^3 be a boundary defining function on S with level sets S_c and θ a smooth function on S with $\theta|_\Sigma = \Theta$. Denote by e_3' the normalized gradient of x^3 pointing towards S on Σ. Notice thatit does not matter whether we use g or $\hat{g}$ here.

The following constructions will be done on (M', g). Let e_0 be the time-like unit vector field on S_c which is orthogonal to S_c and satisfies $g(e_0, e_3') = -\sinh(\theta)$. Following the discussion in Sect. 4 we can construct a slicing of a neighbourhood R' of Σ in M' by hypersurfaces T_c, $0 \leq c < \sup x^3$, such that $T_c \cap S = S_c$, e_0 is tangent to T_c on S_c, T_c has constant mean extrinsic curvature χ_0, the vector field e_0 on S_c can be extended to a T_c-intrinsic geodesics vector field e_0 on T_c with connected integral curves. We denote by x^0 the function on R' which vanishes on S and induces the natural (affine) parameter on the integral curves of e_0. Let $h \in C^\infty(\mathbb{R}, \mathbb{R})$ be a decreasing function with $h(0) > 0$, $h(a) = 0$ for some a, $0 < a < \sup x^3$ such that the set R which is bounded by T', S and $\{p \in M' |\, x^0 < h(x^3)\}$ is relative compact in M' and coincides with the domain of dependence of $R \cap (S \cup T')$ in M'.

We can repeat this discussion with $(\hat{M}', \hat{g})$ replacing (M', g) to obtain analogous sets $\hat{T}_c$ (with mean extrinsic curvature equal to χ_0), $\hat{R}$, vector field $\hat{e}_0$, and function $\hat{x}^0$ based on x^3 and θ. By a suitable choice of h we can assume that the same functions are used to define R and $\hat{R}$.

We define now a map $\bar{\psi}$ from R onto $\hat{R}$. If $p \in R$, there is a unique number c and a unique $q \in S_c$ such the T_c-intrinsic geodesic on T_c with tangent vector e_0 at q meets p. We define $\bar{\psi}(p)$ to be the unique point on the $\hat{T}_c$-intrinsic geodesic through q for which $\hat{x}^0(\bar{\psi}(p)) = x^0(p)$. The map $\bar{\psi}$ then defines a bijection which implies the identity on $R \cap S = \hat{R} \cap S$.

By Lemma 4.2 we can express the solutions on R, $\hat{R}$ locally in terms of a gauge as described in Sect. 4 with the gauge source function being in both cases given by θ, $F = 0$, $f = \chi_0$. In terms of such a gauge the data related by $\bar{\psi}$ are identical and the reduced field equations take the same form. The uniqueness of the local solutions, implied by the energy estimates, allows us to conclude that $\bar{\psi}$ is in fact an isometry.

We show that the restrictions of $\overline{\psi}$ and ψ to $R \cap D$ define identical maps from $R \cap D$ onto $\hat{R} \cap \hat{D}$. Since ψ is an isometry which leaves $(S \setminus \Sigma) \cap R$ pointwise invariant, the sets $\hat{T}_c = \psi(T_c \cap D \cap R)$ have constant mean extrinsic curvature equal to χ_0, satisfy $\hat{T}_c \cap S = S_c$, and are tangent to $\hat{e}_0$ on S_c, because $\hat{g}(T(\psi)\, e_0, e'_3) = \hat{g}(T(\psi)\, e_0, T(\psi)\, e'_3) = g(e_0, e'_3) = -\sinh(\theta)$ entails $\hat{e}_0 = T(\psi)\, e_0$. This implies that $\hat{T}_c \subset \hat{T}_c$. Since isometries map geodesic vector fields again onto such vector fields, it follows that $T(\psi)\, e_0 = \hat{e}_0$ on $\hat{T}_c$. Since isometries preserve affine parameters, we have $x^0 = \hat{x}^0 \circ \psi$ on $T_c \cap D \cap R$. This implies our assertion.

Defining the map Ψ from $M'' = R \cup D$ onto $\hat{M}'' = \hat{R} \cup \hat{D}$ to be equal to ψ on D and equal to $\overline{\psi}$ elsewhere, we get an isometry for the metrics induced by g and $\hat{g}$ on M'' and $\hat{M}''$ respectively. This proves assertion (iv). $\square$

References

1. Bartnik, R.: Einstein equations in the null quasispherical gauge. Class. Quantum Grav. **14**, 2185–2194 (1997)
2. Ellis, G.F.R.: Relativistic cosmology: Its nature, aims and problems. In: *General Relativity and Gravitation*, B. Bertotti et al. (eds.), Dordrecht: Reidel, 1984
3. Friedrich, H. Cauchy Problems for the Conformal Vacuum Field Equations in General Relativity. Commun. Math. Phys. **91**, 445–472 (1983)
4. Friedrich, H.: Einstein Equations and Conformal Structure: Existence of Anti-de Sitter-Type Space-Times. J. Geom. Phys. **17**, 125–184 (1995)
5. Friedrich, H.: Hyperbolic reductions for Einstein's field equations. Class. Quantum Gravity **13**, 1451–1469 (1996)
6. Friedrich, H.: Evolution equations for gravitating ideal fluid bodies in general relativity. Phys. Rev. D **57**, 2317–2322 (1998)
7. Friedrich, H.: Einstein's equation and geometric asymptotics. AEI–Preprint 059, 1998, *gr-qc* 9804009
8. Guès, O.: Problème mixte hyperbolique quasi-linéaire charactéristique. Commun. Part. Diff. Eq. **15**, 595–645 (1990)
9. Kato, T.: The Cauchy problem for quasi-linear symmetric hyperbolic systems. Arch. Ration. Mech. Anal. **58**, 181–205 (1975)
10. Kijowski, J.: A simple derivation of canonical structure and quasi-local Hamiltonians in reneral relativity. Gen. Rel. Grav. **29**, 307–343 (1997)
11. Newman, E.T., Penrose, R.: An Approach to Gravitational Radiation by a Method of Spin Coefficients. J. Math. Phys. **3**, 566–578 (1992)
12. Rauch, J.: Symmetric positive systems with boundary characteristics of constant multiplicity. Trans. Am. Math. Soc. **291**, 167–187 (1985)
13. Secchi, P.: The initial boundary value problem for linear symmetric hyperbolic systems with characteristic boundary of constant multiplicity. Diff. Int. Eqs **9**, 671–700 (1996)
14. Secchi, P.: Well-Posedness of Characteristic Symmetric Hyperbolic Systems. Arch. Rational Mech. Anal. **134**, 155–197 (1996)
15. Tamburino, L.A., Winicour, J.H.: Gravitational Fields in Finite and Conformal Bondi Frames. Phys. Rev. **150**, 1039–1053 (1966)

Communicated by H. Nicolai