Recall the Factor Theorem:

If a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ becomes zero when the number r is substituted for x, then $(x - r)$ is a factor of $p(x)$, and therefore $p(x) = (x - r)q(x)$ where $q(x)$ is a polynomial of degree $n - 1$.

For the Factor Theorem, we're assuming $a_n \neq 0$ and $n \geq 1$

Some consequences of the Factor Theorem:

(i) $f(x) = x^n - b^n$ is always divisible by $x - b$.

(ii) $f(x) = x^n - b^n$ is divisible by $x + b$ if and only if n is even.

(iii) $f(x) = x^n + b^n$ is divisible by $x + b$ if and only if n is odd.

In the above, we're assuming $b \neq 0$ and n is a (positive) integer.

Note that these statements were mentioned in a previous discussion, with a replacing x. For (ii) and (iii) r in the Factor Theorem is $-b$.

The following discussion and examples concern finding integral and rational roots of polynomials. Of course, not all polynomials have rational roots, even if the coefficients are integers. For example $f(x) = x^2 - 2$ has roots $\pm\sqrt{2}$.

Digression: Pythagoras was crushed when he realized that the square root of 2 is irrational. It ruined his philosophy based on the belief that all real numbers are rational. Search on the internet to read about Pythagoras, a very interesting historical figure.

For this discussion, from now on we assume the coefficients of a given polynomial are integers. If the leading coefficient of the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, (the coefficient a_n of the highest power of x) is 1, and if $p(x) = (x - r)q(x)$, then r is a factor of the constant term a_0. This observation is useful in factoring polynomials in certain cases.

Examples:

Factor $p(x) = x^4 - 2\,x^3 + x - 2$

Solution: Test the numbers $\{r = 1, -1, 2, -2\}$ as possible roots since they are the (integral) factors of $a_0 = 2$. Direct substitution shows that $p(-1) = 0$ and $p(2) = 0$. Thus $(x - 2)$ and $(x + 1)$ are factors of $p(x)$. Division of $p(x)$ by $(x-2)(x+1)$ shows $p(x) = (x-2)(x+1)(x^2 - x+1)$. The factor $q(x) = x^2 - x+1$ cannot be factored (without using complex numbers.)

If you had a good course in algebra in high school, perhaps you remember this useful fact:

Given a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where all the coefficients are integers, and if $x = c/d$ is a rational root of $p(x)$ (this means c and d are integers and $p(c/d) = 0$), then d is a factor of a_n and c is a factor of a_0.

Example: Find all roots of $p(x) = 3\,x^3 + 2\,x^2 - 3\,x - 2 = 0$.

Solution: If c/d is a rational root, the possibilities for c are $\{1, -1, 2, -2\}$. The possible values for d are $\{1, -1, 3, -3\}$. The possible values for c/d are then $\{1, -1, 2, -2, 1/3, -1/3, 2/3, -2/3\}$.

Direct substitution shows that $p(1) = 0$, so $p(x) = (x - 1)(3\,x^2 + 5\,x + 2)$. The other roots are -1 and $-2/3$ by direct substitution.

If you remember synthetic division, this is a fast way to evaluate $p(r)$. In fact, it is much faster than using a calculator. If you don't remember synthetic division, go to the previous topic for a quick review.

Find all rational roots and if the determination of these leads to a factor which is quadratic, find all of the roots (including complex roots).

1. $x^3 - 3x^2 - 33x + 35 = 0$

2. $x^3 + 3x^2 - 4x - 12 = 0$

3. $x^3 + 8x^2 + 24x + 27 = 0$

4. $x^4 + 2x^3 = x^2 + 4x + 2$

5. $x^4 - 4x^3 + 7x^2 - 12x + 12 = 0$

6. $5x^3 - 17x^2 - 4x + 4 = 0$

7. $2x^3 + 13x^2 + 17x + 3 = 0$

8. $x^4 + 4x^3 + 8x^2 + 16x + 16 = 0$

9. $3x^3 - 16x^2 + 25x - 11 = 0$

The following observation might save you some work:

When you are testing for a root r by long division or synthetic division, if you find that $p(x) = (x - r)q(x) + r_1$ and if all the coefficients of $q(x)$ are positive or zero and the remainder term r_1 is also nonnegative, then all real roots of $p(x)$ are less than or equal to r. In other words, r is an upper limit for the real roots of $p(x)$ in this special case. Do you see why this is true? The answer is at the end of the answers to the problems. To find a lower limit for the real roots, apply this observation to $p(-x) = 0$, or $-p(-x) = 0$, whose roots are the negatives of the roots of $p(x) = 0$.

Find all rational roots:

10. $x^3 - 8x^2 - 28x + 215 = 0$

11. $x^3 + 3x^2 + 4x + 240 = 0$

Answers:

1. $\{\pm 1, \pm 5, \pm 7, \pm 35\}$ are possible roots.

By inspection or by synthetic division, $f(1) = 0$ so $f(x) = (x - 1)q(x)$.
Scratchwork showing this:

1	1	-3	-33	35
		1	-2	-35
	1	-2	-35	0

$f(x) = (x - 1)(x^2 - 2x - 35) = (x - 1)(x - 7)(x + 5)$ so the roots are $\{1, 7, -5\}$

2. $\{-3, -2, 2\}$

3. $\{\pm 1, \pm 3, \pm 9, \pm 27\}$ are possible integral roots according to the discussion. However, notice that since all the coefficients are positive, any positive value of x will yield a value greater than 27. In other words, any real roots must be negative. So you only need to check $\{-1, -3, -9, -27\}$.

By inspection or by synthetic division, $f(-3) = 0$ so $f(x) = (x + 3)q(x)$.
Scratchwork showing this:

-3	1	8	24	27
		-3	-15	-27
	1	5	9	0

$f(x) = (x+3)(x^2+5x+9)$

$q(x) = x^2+5x+9$ has no real roots since $b^2-4ac = 25-36 = -11$ is negative. By the quadratic formula, the roots of $q(x)$ are

$$\frac{-5 \pm \sqrt{-11}}{2} = \frac{-5 \pm \sqrt{11}\,i}{2}$$

Final answer: the roots are

$$\{-3, \frac{-5 \pm \sqrt{11}\,i}{2}\}$$

4. The roots are $\{-1, -1, -\sqrt{2}, \sqrt{2}\}$.

5. The roots are $\{2, 2, -\sqrt{3}\,i, \sqrt{3}\,i\}$

6. If $r = c/d$ is a rational root of $f(x) = 5x^3 - 17x^2 - 4x + 4$ then possibilities for c and d are $\{\pm 1, \pm 2, \pm 4\}$ and $\{\pm 1, \pm 5\}$. By trial and error you finally find that $2/5$ is a root and that $5x^3 - 17x^2 - 4x + 4 = (x - 2/5)(5x^2 - 15x - 10) = (5x-2)(x^2-3x-2)$.
Scratchwork showing this:

2/5	5	-17	-4	4
		2	-6	-4
	5	-15	-10	0

The roots of x^2-3x-2 are $\frac{3-\sqrt{17}}{2}, \frac{3+\sqrt{17}}{2}$ by the quadratic formula.

Final Answer: The roots are $\{\frac{2}{5}, \frac{3-\sqrt{17}}{2}, \frac{3+\sqrt{17}}{2}\}$

7. Like the third problem, all coefficients are positive so you only check negative values for possible roots:

$$\{-\left(\frac{3}{2}\right), \frac{-5-\sqrt{21}}{2}, \frac{-5+\sqrt{21}}{2}\}$$

8. The roots are $\{-2, -2, -2\,i, 2\,i\}$

9. If r is a rational root, then it is one of the following: $\{\pm 1, \pm 11, \pm 1/3, \pm 11/3\}$ A quick check using synthetic division shows that none is a root.

10. If r is a rational root, then it is one of the following: $\{\pm 1, \pm 5, \pm 43, \pm 215\}$.

Synthetic division or long division gives $x^3-8x^2-28x+215=(x-5)(x^2-3x-43)$.

$x^2-3x-43$ has no rational roots so 5 is the only rational root.

11. Since all the coefficients and the constant term are positive, we know $x^3+3x^2+4x+240$ has no real positive roots. In order to use the hint, look for the positive rational roots of $f(-x)=-x^3+3x^2-4x+240=0$. To use the observation, look for the roots of $-f(-x)=x^3-3x^2+4x-240=0$. A rational root would be of the form

$$3^m 5^n 2^k$$

where $m=\{0,1\}, \quad n=\{0,1\}, \quad, k=\{0,1,2,3,4\}$ since $240=2^4\cdot 3\cdot 5$
There are 20 such candidates. However, synthetic division by ten gives all positive numbers in the bottom row, hence you only need to check those candidates less than 10. None from the list $\{1,2,3,4,5,6,8\}$ is a root, hence $x^3+3x^2+4x+240$ has no rational roots.

Answer to the question before problem 10:

If $z>r$, all the numbers in the third (bottom) row when using synthetic division for $(x-z)$ will be greater than or equal to those entries obtained by synthetic division for $(x-r)$, with at least one entry greater, hence the last number, which gives $f(z)$, cannot be zero. In other words, r is an upper bound for the real roots. We have assumed that at least one coefficient of $q(x)$ is positive, for otherwise $p(x)=(x-r)q(x)+r_1=(x-r)\times 0+r_1=r_1$ is a constant polynomial and we would know the roots, if any, by inspection.