Guts of surfaces and the colored Jones polynomial

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Preface

Around 1980, W. Thurston proved that every knot complement satisfies the geometrization conjecture: it decomposes into pieces that admit locally homogeneous geometric structures. In addition, he proved that the complement of any non-torus, non-satellite knot admits a complete hyperbolic metric which, by the Mostow–Prasad rigidity theorem, is is necessarily unique up to isometry. As a result, geometric information about a knot complement, such as its volume, gives topological invariants of the knot.

Since the mid-1980's, knot theory has also been invigorated by ideas from quantum physics, which have led to powerful and subtle knot invariants, including the Jones polynomial and its relatives, the *colored* Jones polynomials. Topological quantum field theory predicts that these quantum invariants are very closely connected to geometric structures on knot complements, and particularly to hyperbolic geometry. The *volume conjecture* of R. Kashaev, H. Murakami, and J. Murakami, which asserts that the volume of a hyperbolic knot is determined by certain asymptotics of colored Jones polynomials, fits into the context of these predictions. Despite compelling experimental evidence, these conjectures are currently verified for only a few examples of hyperbolic knots.

This monograph initiates a systematic study of relations between quantum and geometric knot invariants. Under mild diagrammatic hypotheses that arise naturally in the study of knot polynomial invariants (A– or B–adequacy), we derive direct and concrete relations between colored Jones polynomials and the topology of incompressible spanning surfaces in knot and link complements. We prove that the growth of the degree of the colored Jones polynomials is a boundary slope of an essential surface in the knot complement, and that certain coefficients of the polynomial measure how far this surface is from being a fiber in the knot complement. In particular, the surface is a fiber if and only if a certain coefficient vanishes.

Our results also yield concrete relations between hyperbolic geometry and colored Jones polynomials: for certain families of links, coefficients of the polynomials determine the hyperbolic volume to within a factor of 4. Our methods here provide a deeper and more intrinsic explanation for similar connections that have been previously observed.

Our approach is to generalize the checkerboard decompositions of alternating knots and links. For A– or B–adequate diagrams, we show that the checkerboard knot surfaces are incompressible, and obtain an ideal polyhedral decomposition of their complement. We use normal surface theory to establish a dictionary between the pieces of the JSJ decomposition of the surface complement and the combinatorial structure of certain spines of the checkerboard surface (state graphs). In particular, we give a combinatorial formula for the complexity of the hyperbolic part of the JSJ decomposition (the guts) of the surface complement in terms of the diagram of the knot, and use this to give lower bounds on volumes of several classes of knots. Since state graphs have previously appeared in the study of Jones polynomials, our setting and methods create a bridge between quantum invariants and geometries of knot complements.

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