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Futer, David (1-TMPL); **Kalfagianni, Efstratia** (1-MIS);
Purcell, Jessica (1-BYU)

★**Guts of surfaces and the colored Jones polynomial.**

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A relationship between the geometry of knot complements and the colored Jones polynomial is given in this monograph. The writing is well organized and comprehensive, and the book is accessible to both researchers and graduate students with some background in geometric topology and Jones-type invariants.

In the late 1970's, the Jaco-Shalen-Johannson (JSJ) decomposition provided a method of decomposing a 3-manifold along surfaces of small genus [W. H. Jaco and P. B. Shalen, Mem. Amer. Math. Soc. **21** (1979), no. 220, viii+192 pp.; MR0539411 (81c:57010); K. Johannson, *Homotopy equivalences of* 3*-manifolds with boundaries*, Lecture Notes in Mathematics, 761, Springer, Berlin, 1979; MR0551744 (82c:57005)]. Thurston's geometrization conjecture stated that the pieces of this decomposition should admit homogeneous geometric structures. In the mid 1980's, the Jones polynomial [V. F. R. Jones, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 103–111; MR0766964 (86e:57006)] was introduced, which can be computed using the combinatorics of the link diagram via Kauffman's bracket formulation. Subsequently, Witten's work on $(2+1)$-dimensional topological quantum field theory (TQFT) provided a framework relating the Jones polynomial and the geometric structures of a 3-manifold. However, Reshetikhin and Turaev's rigorous definition of a TQFT relied on representations of quantum groups and did not provide insight as to the interaction between the Jones polynomial and the geometry of the knot complement. This was followed by Kashaev's family of invariants of links in 3-manifolds, which led to the volume conjecture—which proposes that the volume of a hyperbolic knot is determined by the large N asymptotics of the colored Jones polynomial.

More recently, researchers have studied the relationship between the coefficients of the Jones (and colored Jones) polynomials and the volume of hyperbolic links. Work of A. Champanerkar, I. Kofman and E. Patterson provides numerical evidence of this relationship [J. Knot Theory Ramifications **13** (2004), no. 7, 965–987; MR2101238 (2005k:57010)] as does work of O. T. Dasbach and X.-S. Lin [Compos. Math. **142** (2006), no. 5, 1332–1342; MR2264669 (2007g:57018)]. The authors of this monograph have several articles—relating twist numbers to coefficients of the Jones polynomial and relating twist numbers to hyperbolic volume—that rely on properties of Turaev surfaces to relate twist numbers and coefficients of the Jones polynomial [J. Differential Geom. **78** (2008), no. 3, 429–464; MR2396249 (2009c:57010); Math. Res. Lett. **16** (2009), no. 2, 233–253; MR2496741 (2010k:57011); Int. Math. Res. Not. IMRN **2010**, no. 23, 4434–4497; MR2739802 (2011k:57027)].

In this monograph, the authors establish a relationship between the topology of incompressible surfaces in the knot complement and the colored Jones polynomial. Specifically, they measure how far the surface is from a fiber, using the reduced state graph $\mathbb{G}'_A$, by relating the negative Euler characteristic of the guts with the reduced state graph. In the conclusion, the authors propose some directions for future work. One possible direction is an analysis of essential surfaces in $S^3\backslash K$ using the decomposition as an approach to the cabling conjecture. They also propose a coarse volume conjecture.

We provide an overview of the monograph. Basic definitions from low-dimensional topology are included in Chapter 1 along with a summary of the results. The main results of the monograph are presented in Chapters 5 and 9; technical lemmas used in the theorems are given in Chapters 3, 4, and 6–8. The main result in Chapter 5 is Theorem 5.11, which relates a reduced state graph of the Jones polynomial $\mathbb{G}'_A$ and the topology of the complement of the knot and its spanning surface (S_A). This result leads to a relation between $\chi_-(\text{guts}(S^3\backslash\backslash S_A))$ and $\chi_-(\mathbb{G}'_A)$ for the spanning surface S_A. Work by I. Agol, P. A. Storm and W. P. Thurston established that computation of lower bounds on $\chi_-(\text{guts})$ of an essential surface in $S^3\backslash K$ leads to a lower bound on the volume of $S^3\backslash K$ [J. Amer. Math. Soc. **20** (2007), no. 4, 1053–1077; MR2328715 (2008i:53086)]. Hence, Chapter 9 focuses on the relationship between the second-to-last coefficients of the Jones polynomial and the guts calculations to establish volume bounds.

In Chapter 1, terminology arising from the colored Jones polynomial and geometric topology is recalled. For a knot diagram K, $\mathbb{H}_A$ (respectively $\mathbb{H}_B$) is the graph obtained from the all-A state (respectively the all-B state). The notation $\mathbb{G}_A$ denotes the state graph obtained by collapsing the state circles of the state A to vertices; $\mathbb{G}'_A$ denotes the reduced state graph obtained by removing all multiple edges from $\mathbb{G}_A$. The nth colored Jones polynomial of K is written as

$$J^n_K(t) = \alpha_n t^{m_n} + \beta_n t^{m_n-1} \cdots \beta'_n t^{r_n+1} + \alpha'_n t^{r_n}$$

where r_n is the lowest degree and m_n is the highest. The authors consider semi-adequate links (links that contain no self-touching loops in either the all-A state or the all-B state) throughout the monograph. The notation $M = S^3\backslash K$ denotes the three-manifold obtained by removing a tubular neighborhood of K, S_A denotes the all-A state surface and $M\backslash\backslash S_A$ denotes the path-metric closure of $M\backslash S_A$. $S^3\backslash\backslash S_A$ (which is homeomorphic to $M\backslash\backslash S_A$) is denoted as M_A.

In Chapter 2, the authors describe how to cut the link complement along the state surface S_A, thereby forming M_A. The state surface is used to decompose the handlebody into topological balls with a checkerboard coloring. We state key points of Lemma 2.21. The decomposition of M_A along a collection of non-prime arcs $\alpha_1, \alpha_2, \ldots, \alpha_n$ consists of an upper ball and a collection of lower balls. The boundary of the balls consists of white faces and shared regions. Since the lower 3-balls are in one-to-one correspondence with complementary regions of $S_A \cup (\bigcup_{i=1}^n \alpha_i)$, the lower 3-balls are ideal polyhedrons. The upper 3-ball (with its faces and vertices) is projected onto the state graph H_A.

In Chapter 3, the authors prove that the shaded regions of the upper 3-ball are simply connected and conclude that the upper 3-ball is an ideal polyhedron. The main method of proof consists of "tentacle chasing"—the shaded faces of the upper polyhedron have tentacles extending along the graph H_A. A sequence of arguments about tentacles extending from the shaded faces proves that the shaded regions are simply connected. This leads to the conclusion that the upper polyhedron is a checkerboard colored, ideal polyhedron. The definitions of normal surfaces and compression disks are provided. In Proposition 3.18, the authors show that the polyhedra do not contain normal bigons and that all of the ideal polyhedra are prime. This result is used to give a new proof of the following result of Ozawa: Let D be a connected diagram of a link K. The surface S_A is essential in $S^3\backslash K$ if and only if D is A-adequate [M. Ozawa, J. Aust. Math. Soc. **91** (2011), no. 3, 391–404;MR2900614].

In Chapter 4, M_A is decomposed using the JSJ decomposition along essential annuli. This decomposition divides M_A into the characteristic submanifold and the guts, which admit a hyperbolic metric with totally geodesic boundary. The authors prove that the characteristic submanifold consists of I-bundles and Seifert fibered pieces. The pieces of the characteristic submanifold that affect Euler characteristic are I-bundles and these are spanned by essential product disks. (An essential product disk (EPD) is a properly embedded essential disk in M_A whose boundary meets the parabolic locus twice.) Each EPD embeds in a single polyhedra. The main result of the chapter is

Theorem 4.4: Let B be a nontrivial component of the characteristic submanifold of M_A. Then B is spanned by a collection of essential product disks $D_1, D_2, \ldots, D_n$ with the property that each D_i is embedded in a single polyhedron in the polyhedral decomposition of M_A.

Chapter 5 focuses on the problem of computing the Euler characteristic of the guts of M_A, and the computation of the Euler characteristic is reduced to the problem of counting how many complex EPDs are required to span the I-bundle. The main results of this chapter are given in Theorem 5.14 and Theorem 5.11. We introduce some terminology used in the chapter. A collection of EPDs span the I-bundle of M_A if their complement is a collection of prisms and tori. An EPD is simple if it is the boundary of a regular neighborhood of a white bigon face. An EPD is semi-simple if it can be parabolically compressed to a union of simple disks. An EPD is complex if it is neither simple nor semi-simple. Theorem 5.11 states that if $D(K)$ is any link diagram, and S_A is the spanning surface determined by the all-A state of this diagram, then the following are equivalent: (1) The reduced graph $\mathbb{G}'_A$ is a tree. (G'_A is obtained from the state graph by removing multiple edges between pairs of vertices.) (2) $S^3 \backslash K$ fibers over S^1 with fiber S_A. (3) M_A is an I-bundle over S_A.

The authors prove that the EPDs embedded in the lower polyhedra are in one-to-one correspondence with 2-edge loops in $\mathbb{G}_A$. Theorem 5.14 states that if $D(K)$ is an A-adequate diagram and S_A is the essential spanning surface determined by the diagram, then $\chi_-(\text{guts}(M_A)) = \chi_-(\mathbb{G}'_A) - \|E_c\|$.

The key result in Chapter 6 states that normal squares (EPDs) correspond to 2-edge loops in $\mathbb{G}_A$. The technical theorems rely on tentacle chasing arguments. The authors prove that the EPDs correspond to only 7 subgraphs of H_A, which allows them to count complex EPDs for large classes of link complements. In Chapter 7, attention is restricted to the special case of A-adequate diagrams $D(K)$ with no non-prime arcs or switches. This special case simplifies the estimate on the guts of M_A. The majority of Chapter 7 is devoted to proving the following theorem: Let $D(K)$ be a prime, A-adequate diagram with essential spanning surface S_A determined by the diagram. Suppose that the polyhedral decomposition of M_A contains no non-prime arcs. Then $\chi_-(\mathbb{G}'_A) - 8m_A \leq \chi_-(\text{guts}(M_A)) \leq \chi_-(\mathbb{G}'_A)$ where the lower bound is an equality if and only if $m_A = 0$.

Chapters 8 and 9 contain applications of the guts calculations. Montesinos links are described along with rational tangles. In Theorem 8.6, the authors prove that if K is a Montesinos link with a reduced admissible diagram $D(K)$, and at least three tangles of positive slope, then $D(K)$ is A-adequate and $\chi_-(\mathrm{guts}(M_A)) = \chi_-(\mathbb{G}'_A)$. In Chapter 9, the calculation of $\chi_-(\mathrm{guts}(M_A))$ leads to a relationship between the geometry of A-adequate links and diagrammatic quantities and the Jones polynomial and the construction of diagrammatic estimates for the volume of Montesinos links. Next, the quantity $\chi_-(\mathrm{guts}(M_A))$ is related to coefficients of the Jones and colored Jones polynomials. As an application, the authors show that the second lowest coefficient of the Jones polynomial (β'_K) detects whether a state surface is a fiber in the knot complement. In Section 9.4, the authors obtain relations between the Jones polynomial and volume and give new, stronger, two-sided bounds on both the positive braids and the Montesinos links.

Heather A. Dye (Lebanon, IL)