KNOT ADJACENCY, GENUS AND ESSENTIAL TORI

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Abstract. A knot K is called n-adjacent to another knot K', if K admits a projection containing n "generalized crossings" such that changing any $0 < m \leq n$ of them yields a projection of K'. We apply techniques from the theory of sutured 3-manifolds, Dehn surgery and the theory of geometric structures of 3-manifolds to answer the question of the extent to which non-isotopic knots can be adjacent to each other. A consequence of our main result is that if K is n-adjacent to K' for all $n \in \mathbf{N}$, then K and K' are isotopic. This provides a partial verification of the conjecture of V. Vassiliev that the finite type knot invariants distinguish all knots. We also show that if no twist about a crossing circle L of a knot K changes the isotopy class of K, then L bounds a disc in the complement of K. This leads to a characterization of nugatory crossings on knots.

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Contents

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1. Introduction

A *crossing disc* for a knot $K \subset S^3$ is an embedded disc $D \subset S^3$ such that K intersects $\mathrm{int}(D)$ twice with zero algebraic number. Let $q \in \mathbf{Z}$. Performing $\frac{1}{q}$-surgery on $L_1 := \partial D_1$, changes K to another knot $K' \subset S^3$. We will say that K' is obtained from K by a generalized crossing change of order q (see Figure 1).

An *n-collection* for a knot K is a pair $(\mathcal{D}, \mathbf{q})$, such that:

i) $\mathcal{D} := \{D_1, \ldots, D_n\}$ is a set of disjoint crossing discs for K;

ii) $\mathbf{q} := \{\frac{1}{q_1}, \ldots, \frac{1}{q_n}\}$, with $q_i \in \mathbf{Z} - \{0\}$;

iii) the knots $L_1, \ldots, L_n$ are labeled by $\frac{1}{q_1}, \ldots, \frac{1}{q_n}$, respectively. Here, $L_i := \partial D_i$. The link $L := \cup_{i=1}^{n} L_i$ is called the *crossing link* associated to $(\mathcal{D}, \mathbf{q})$.

Given a knot K and an n-collection $(\mathcal{D}, \mathbf{q})$, for $j = 1, \ldots, n$, let $i_j \in \{0,\ 1\}$ and

$$\mathbf{i} := (i_1, \ldots, i_n).$$

We denote $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$.

For every $\mathbf{i}$, we will denote by $K(\mathbf{i})$ the knot obtained from K by a surgery modification of order q_i (resp. 0), along each L_j for which $i_j = 1$ (resp. $i_j = 0$).

Definition 1.1. *We will say that K is n-adjacent to K' if there exists an n-collection $(\mathcal{D}, \mathbf{q})$ for K, such that the knot $K(\mathbf{i})$ is isotopic to K' for every $\mathbf{i} \neq \mathbf{0}$. We will write $K \xrightarrow{n} K'$ and we will say that $(\mathcal{D}, \mathbf{q})$ transforms K to K'.*

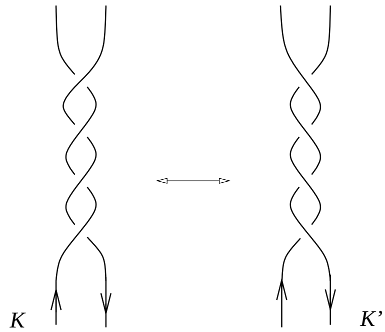

FIGURE 1. The knots K and K' differ by a generalized crossing change of order $q = -4$.

Our main result is the following:

Theorem 1.2. *Suppose that K and K' are non-isotopic knots. There exists a constant $C(K,\, K')$ such that if $K \xrightarrow{n} K'$, then $n \leq C(K,\, K')$.*

The quantity $C(K,\, K')$ can be expressed in terms of computable invariants of the knots K and K'. Let $g(K)$ and $g(K')$ denote the genera of K and K', respectively and let $g := \max\{\, g(K),\, g(K')\,\}$. The constant $C(K,\, K')$ encodes information about the relative size of $g(K)$, $g(K')$ and the behavior of the satellite structures of K and K' under the Dehn surgeries imposed by knot adjacency. In many cases $C(K,\, K')$ can be made explicit. For example, when $g(K) > g(K')$ we have $C(K,\, K') = 6g - 3$. Thus, in this case, Theorem 1.2 can be restated as follows:

Theorem 1.3. *Suppose that K, K' are knots with $g(K) > g(K')$. If $K \xrightarrow{n} K'$, then $n \leq 6g(K) - 3$.*

In the case that K' is the trivial knot Theorem 1.3 was proven by H. Howards and J. Luecke in [HL].

A crossing of a knot K, with crossing disc D, is called *nugatory* iff ∂D bounds a disc that is disjoint from K. The techniques used in the proof of Theorem 1.2 have applications to the question of whether a crossing change that doesn't change the isotopy class of the underlying knot is nugatory (Problem 1.58, [GT]). As a corollary of the proof of Theorem 1.2 we obtain the following characterization of nugatory crossings:

Corollary 1.4. *For a crossing disc D of a knot K let $K(r)$ denote the knot obtained by a twist of order r along D. The crossing is nugatory if and only if $K(r)$ is isotopic to K for all $r \in \mathbf{Z}$.*

Definition 1.1 is equivalent to the definition of n-adjacency given in the abstract of this paper or in [KLi]. With this reformulation, it follows that n-adjacency implies n-similarity in the sense of [Oh], which in turn, as shown in [NS], implies n-equivalence. Gussarov showed that two knots are n-equivalent precisely when all of their finite type invariants of orders $< n$ are the same. Vassiliev ([V]) has conjectured that if two oriented knots have all of their finite type invariants the same then they are isotopic. In the light Gussarov's result, Vassiliev's conjecture can be reformulated as follows:

Conjecture 1.5. *Suppose that K and K' are knots that are n-equivalent for all $n \in \mathbf{N}$. Then K is isotopic to K'.*

To that respect, Theorem 1.2 implies the following corollary that provides a partial verification to Vassiliev's conjecture:

Corollary 1.6. *If $K \xrightarrow{n} K'$, for all $n \in \mathbf{N}$, then K and K' are isotopic.*

We now describe the contents of the paper and the idea of the proof of the main theorem. Let K be a knot and let $(\mathcal{D}, \mathbf{q})$ be a n-collection with associated crossing link L. Since the linking number of K and every component of L is zero, K bounds a Seifert surface in the complement of L. Thus, we can define the genus of K in the complement of L, say $g^n_L(K)$. In Section two we study the question of the extent to which a Seifert surface of K that is of minimal genus in the complement of L remains of minimal genus under various surgery modifications along the components of L. Using a result of of Gabai ([Ga]) we show that if $K \xrightarrow{n} K'$, and $(\mathcal{D}, \mathbf{q})$ is an n-collection that transfers K to K' then $g^n_L(K) = g := \max\{\, g(K),\, g(K')\,\}$, where $g(K), g(K')$ denotes the genus of K, K' respectively. This is done in Theorem 2.1.

In Section three, we prove Theorem 1.3. In Section four, we finish the proof of Theorem 1.2: We begin by defining a notion of m-adjacency between knots K, K' with respect to an one component crossing link

L_1 of K (see Definition 4.1). To describe our approach in more detail, set $N := S^3 \setminus \eta(K \cup L_1)$, and let $\tau(N)$ denote the number of disjoint, pairwise non-parallel, essential embedded tori in N. We employ results of Cooper and Lackenby ([CoLa]), Gordon ([Go]) and McCullough ([M]) and an induction argument on $\tau(N)$ to show the following: Given knots K, K', there exists a constant $b(K,\ K') \in \mathbf{N}$ such that if K is m-adjacent to K' with respect to a crossing link L_1 then either $m \leq b(K,\ K')$ or L_1 bounds an embedded disc in the complement of K. This is done in Theorem 4.3. Theorem 2.1 implies that if $K \xrightarrow{n} K'$ and $n > m(6g-3)$, then an n-collection that transforms K to K' gives rise to a crossing link L_1 such that K is m-adjacent to K' with respect to L_1. Combining this with Theorem 4.3 yields Theorem 1.2.

In Section five, we present some applications of the results of Section four and the methods used in their proofs. Also, for every $n \in \mathbf{N}$, we construct examples of non-isotopic knots K, K' such that $K \xrightarrow{n} K'$.

Throughout the entire paper we work in the PL or the smooth category. In [K1], the techniques of this paper are refined and used to study adjacency to fibered knots and the problem of nugatory crossings in fibered knots. In [KLi1] the results of this paper are used to obtain criteria for detecting non-fibered knots and for detecting the non-existence of symplectic structures on certain 4-manifolds. Further applications include constructions of 3-manifolds that are indistinguishable by certain Cochran-Melvin finite type invariants ([KLi1]), and constructions of hyperbolic knots with trivial Alexander polynomial and arbitrarily large volume ([K]).

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and careful review that has lead to a significant improvement of the exposition in this paper.

2. Taut surfaces, knot genus and multiple crossing changes

Let K be a knot and $(\mathcal{D}, \mathbf{q})$ an n-collection for K with associated crossing link L. Since the linking number of K and every component of L is zero, K bounds a Seifert surface S in the complement of L. Define

$$g_n^L(K) := \min \{ \text{genus(S)} \,|\, S \text{ a Seifert surface of } K \text{ as above} \}.$$

Our main result in this section is the following:

Theorem 2.1. *Suppose that $K \xrightarrow{n} K'$, for some $n \geq 1$. Let $(\mathcal{D}, \mathbf{q})$ be an n-collection that transforms K to K' with associated crossing link L. We have*

$$g_n^L(K) = \max \{ g(K), g(K') \}.$$

In particular, $g_n^L(K)$ is independent of L and n.

Before we proceed with the proof of Theorem 2.1 we need some preparation: For a link $\bar{L} \subset S^3$ we will use $\eta(\bar{L})$ to denote a regular neighborhood of $\bar{L}$. For a knot $K \subset S^3$ and an n-collection $(\mathcal{D}, \mathbf{q})$, let

$$M_L := S^3 \setminus \eta(K \cup L),$$

where L is the crossing link associated to $(\mathcal{D}, \mathbf{q})$.

Lemma 2.2. *Suppose that K, K' are knots such that $K \xrightarrow{n} K'$, for some $n \geq 1$. Let $(\mathcal{D}, \mathbf{q})$ be an n-collection that transforms K to K'. If M_L is reducible then a component of L bounds an embedded disc in the complement of K. Thus, in particular, K is isotopic to K'.*

Proof: Let Σ be an essential 2-sphere in M_L. Assume that Σ has been isotoped so that the intersection $I := \Sigma \cap (\cup_{i=1}^n D_i)$ is minimal. Notice that we must have $I \neq \emptyset$ since otherwise Σ would bound a 3-ball in M_L. Let $c \in (\Sigma \cap D_i)$ denote a component of I that is innermost on Σ; that is c bounds a disc $E \subset \Sigma$ such that $\text{int}(E) \cap (\cup_{i=1}^n D_i) = \emptyset$. Since Σ is separating in M_L, E can't contain just one point of $K \cap D_i$. E can't be disjoint from K or c could be removed by isotopy. Hence

E contains both points of $K \cap D_i$ and so $c = \partial E$ is parallel to ∂D_i in $D_i \setminus K$. It follows that L_i bounds an embedded disc in the complement of K. Since $\frac{1}{q_i}$-surgery on L_i turns K into K', we conclude that K is isotopic to K'. □

To continue we recall the following definition:

Definition 2.3. ([T]) *Let M be a compact, oriented 3-manifold with boundary ∂M. For a compact, connected, oriented surface $(S,\ \partial S) \subset (M,\ \partial M)$, the complexity $\chi^{-}(S)$ is defined by*
$\chi^{-}(S) := \max\{0,\ -\chi(S)\}$, *where $\chi(S)$ denotes the Euler characteristic of S. If S is disconnected then $\chi^{-}(S)$ is defined to be the sum of the complexities of all the components of S. Let $\eta(\partial S)$ denote a regular neighborhood of ∂S in ∂M. The Thurston norm $x(z)$ of a homology class $z \in H_2(M,\ \eta(\partial S))$ is the minimal complexity over all oriented, embedded surfaces representing z. The surface S is called taut if it is incompressible and we have $x([S,\ \partial S]) = \chi^{-}(S)$; that is S is norm-minimizing.*

We will need the following lemma the proof of which follows from the definitions:

Lemma 2.4. *Let $(\mathcal{D}, \mathbf{q})$ be an n-collection for a knot K with associated crossing link L and $M_L := S^3 \setminus \eta(K \cup L)$. A compact, connected, oriented surface $(S,\ \partial S) \subset (M_L,\ \partial\eta(K))$, such that $\partial S = K$, is taut if and only if among all Seifert surfaces of K in the complement of L, S has the minimal genus.*

To continue, we need to introduce some more notation. For $\mathbf{i}$ as before the statement of Definition 1.1, let $M_L(\mathbf{i})$ denote the 3-manifold obtained from M_L by performing Dehn filling on ∂M_L as follows: The slope of the filling for the components $\partial\eta(L_j)$ for which $i_j = 1$ (resp. $i_j = 0$) is $\frac{1}{q_j}$ (resp. $\infty := \frac{1}{0}$). Clearly we have $M_L(\mathbf{i}) = S^3 \setminus \eta(K(\mathbf{i}))$, where $K(\mathbf{i})$ is as in Definition 1.1. Also let $M_L^{+}(\mathbf{i})$ (resp. $M_L^{-}(\mathbf{i})$) denote the 3-manifold obtained from M_L by only performing Dehn filling with slope $\frac{1}{q_j}$ (resp. ∞) on the components $\partial\eta(L_j)$ for which $i_j = 1$.

Lemma 2.5. *Let $(\mathcal{D}, \mathbf{q})$ be an n-collection for a knot K such that M_L is irreducible. Let $(S,\ \partial S) \subset (M_L,\ \partial\eta(K))$ be an oriented surface with $\partial S = K$ that is taut. For $j = 1, \ldots, n$, define $\mathbf{i}_j := (0, \ldots, 0, 1, 0, \ldots, 0)$ where the unique entry 1 appears at the j-th place. Then, at least one of $M_L^+(\mathbf{i}_j)$, $M_L^-(\mathbf{i}_j)$ is irreducible and S remains taut in that 3-manifold.*

Proof: The proof uses a result of [Ga] in the spirit of [ScT]: For $j \in \{1, \ldots, n\}$ set $M^+ := M_L^+(\mathbf{i}_j)$ and $M^- := M_L^-(\mathbf{i}_j)$. Also set $L^j := L \setminus L_j$ and $T_j := \partial\eta(L_j)$. We distinguish two cases:

Case 1: Suppose that every embedded torus that is incompressible in M_L and it separates $L^j \cup S$ from L_j, is parallel to T_j. Then, M_L is S_{L_j}-atoroidal (see Definition 1.6 of [Ga]). By Corollary 2.4 of [Ga], there is at most one Dehn filling along T_j that yields a 3-manifold which is either reducible or in which S doesn't remain taut. Thus the desired conclusion follows.

Case 2: There exists an embedded torus $T \subset M_L$ such that i) T is incompressible in M_L; ii) T separates $L^j \cup S$ from L_j; and iii) T is not parallel to T_j. In S^3, T bounds a solid torus V, with $\partial V = T$. Suppose, for a moment, that L_j lies in $\text{int}(V)$ and $L^j \cup S$ lies in $S^3 \setminus V$. If V is knotted in S^3 then, since L_j is unknotted, L_j is homotopically inessential in V. But then T compresses in V and thus in M_L; a contradiction. If V is unknotted in S^3 then the longitude of V bounds a disc E in $S^3 \setminus V$. Since S is disjoint from T, K intersects E at least twice. On the other hand, since T is incompressible in M_L and K intersects D_j twice, L_j is isotopic to the core of V. Hence, T is parallel to T_j in M_L; a contradiction. Hence $L^j \cup S$ lies in $\text{int}(V)$ while L_j lies in $S^3 \setminus V$. We will show that M^+, M^- are irreducible and that S remains taut in both of these 3-manifolds.

Among all tori in M_L that have properties (i)-(iii) stated above, choose T to be one that minimizes $|T \cap D_j|$. Then, that $D_j \cap T$ consists of a single curve which bounds a disc $D^* \subset \text{int}(D_j)$, such that $(K \cap D_j) \subset \text{int}(D^*)$ and D^* is a meridian disc of V. See Figure 2 below. Since T is not parallel to T_j, V must be knotted. For $r \in \mathbf{Z}$, let $M(r)$ denote the 3-manifold obtained from M_L by performing Dehn filling along $\partial\eta(L_j)$ with slope $\frac{1}{r}$. Since the core of V intersects D_j once, the Dehn filling

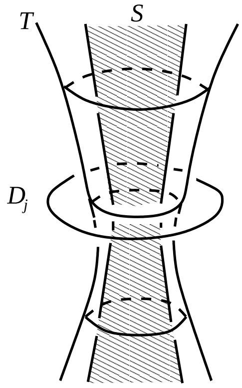

FIGURE 2. The intersection of T and S with D_j.

doesn't unknot V and $T = \partial V$ remains incompressible in $M(r) \setminus V$. On the other hand, T is incompressible in $V \setminus (K \cup L^j)$ by definition. Notice that both $M(r) \setminus V$ and $V \setminus (K \cup L^j)$ are irreducible and

$$M(r) = (M(r) \setminus V) \bigcup_T (V \setminus (K \cup L^j)).$$

We conclude that T remains incompressible in $M(r)$ and $M(r)$ is irreducible. In particular M^+ and M^- are both irreducible.

Next we show that S remains taut in M^+ and M^-. By Lemma 2.4, we must show that S is a minimal genus surface for K in M^+ and in M^-. To that end, let S_1 be a minimal genus surface for K in M^+ or in M^-. We may isotope so that $S_1 \cap T$ is a collection of parallel essential curves on T. Since the linking number of K and L_j is zero, $S_1 \cap T$ is homologically trivial in T. Thus, we may attach annuli along the components of $S_1 \cap T$ and then isotope off T in $\text{int}(V)$, to obtain a Seifert surface S_1' for K that is disjoint from L_j. Thus S_1' is a surface in the complement of L. Since T is incompressible, no component of $S_1 \setminus V$ is a disc. Thus, $\text{genus}(S_1') \leq \text{genus}(S_1)$. On the other hand, by definition of S, $\text{genus}(S) \leq \text{genus}(S_1')$ and thus $\text{genus}(S) \leq \text{genus}(S_1)$. □

Lemma 2.6. *Let* $(\mathcal{D}, \mathbf{q})$ *be an* n*-collection for a knot* K *such that* M_L *is irreducible. Let* $(S,\ \partial S) \subset (M_L,\ \partial\eta(K))$ *be an oriented surface with* $\partial S = K$ *that is taut. There exists at least one sequence* $\mathbf{i} := (i_1, \ldots i_n)$, *with* $i_j \in \{1,0\}$*, such that* S *remains taut in* $M_L(\mathbf{i})$*. Thus we have,* $g(K(\mathbf{i})) = \text{genus}(S)$ *.*

Proof: The proof is by induction on n. For $n = 1$, the conclusion follows from Lemma 2.5. Suppose inductively that for every $m < n$ and every m-collection $(\mathcal{D}_1, \mathbf{q}_1)$ of a knot K_1 such that M_{L^1} is irreducible, the conclusion of the lemma is true. Here, L^1 denotes the crossing link associated to $\mathcal{D}_1$ and $M_{L^1} := S^3 \setminus \eta(K_1 \cup L^1)$.

Now let K, $(\mathcal{D}, \mathbf{q})$ and S be as in the statement of the lemma and let $\mathbf{i}_1 := (1, 0, \ldots, 0)$. By Lemma 2.5 at least one of $M_L^{\pm}(\mathbf{i}_1)$, say $M_L^{-}(\mathbf{i}_1)$, is irreducible and S remains taut in that 3-manifold. Let

$$\mathcal{D}_1 := \{D_2, \ldots, D_n\} \text{ and } \mathbf{q}_1 := \{q_2, \ldots, q_n\}.$$

Let $L^1 := L \setminus L_1$ and let K_1 denote the image of K in $M_L^{-}(\mathbf{i}_1)$. Clearly, $M_{L^1} = M_L^{-}(\mathbf{i}_1)$ and thus M_{L^1} is irreducible. By the induction hypothesis, applied to K_1 and the $(n-1)$-collection $(\mathcal{D}_1, \mathbf{q}_1)$, it follows that there is at least one sequence $\mathbf{i}_0 := (i_{02}, \ldots i_{0n})$, with $i_{0j} \in \{1, 0\}$, such that S remains taut in $M_{L^1}(\mathbf{i}_0)$. Since $M_{L^1}(\mathbf{i}_0) = M_L(\mathbf{i})$, where $\mathbf{i} := (0, i_{02}, \ldots i_{0n})$, the desired conclusion follows. □

Proof: [Proof of Theorem 2.1] Let $K \xrightarrow{n} K'$, L and M_L be as in the statement of the theorem. Let S be a Seifert surface for K in the complement of L such that $\text{genus}(S) = g_n^L(K)$. First, assume that M_L is irreducible. By Lemma 2.4, S gives rise to a surface $(S,\ \partial S) \subset (M_L,\ \eta(\partial S))$ that is *taut*. By Lemma 2.6, there exists at least one sequence $\mathbf{i} := (i_1, \ldots i_n)$, with $i_j \in \{1, 0\}$, such that S remains taut in $M_L(\mathbf{i})$. There are three cases to consider:

(1) $g(K) > g(K')$,

(2) $g(K) < g(K')$,

(3) $g(K) = g(K')$.

In case (1), for every $\mathbf{i} \neq \mathbf{0}$, we have

$$g(K') = g(K(\mathbf{i})) < g(K) \leq \text{genus}(S).$$

Therefore S doesn't remain taut in $M_L(\mathbf{i}) = S^3 \setminus \eta(K(\mathbf{i}))$. Hence S must remain taut in $M_L(\mathbf{0}) = S^3 \setminus \eta(K)$ and we have $g_n^L(K) = g(K)$. In case (2), notice that we have a n-collection $(\mathcal{D}', \mathbf{q}')$ for K' where $\mathcal{D}' = \mathcal{D}$ and

$\mathbf{q}' = -\mathbf{q}$, such that $K'(\mathbf{i}) = K'$ for all $\mathbf{i} \neq \mathbf{1}$ and $K'(\mathbf{1}) = K$. So we may argue similarly as in case (1) that $g_n^L(K) = g(K')$. In fact, in case (2), S must remain taut in $M_L(\mathbf{i})$ for all $\mathbf{i} \neq \mathbf{0}$. Finally in case (3), S remains taut in $M_L(\mathbf{i})$ for all $\mathbf{i}$, and it follows that $g_n^L(K) = g(K') = g(K)$.

Suppose, now, that M_L is reducible. By Lemma 2.2, there is at least one component of L that bounds an embedded disc in the complement of K. Let L^1 denote the union of the components of L that bound disjoint discs in the complement of K and let $L^2 := L \backslash L^1$. We may isotope S so that it is disjoint from the discs bounded by the components of L^1. Now S can be viewed as taut surface in $M_{L^2} := S^3 \setminus \eta(K \cup L^1)$. If $L^2 = \emptyset$, the conclusion is clearly true. Otherwise M_{L^2} is irreducible and the argument described above applies. □

3. Genus reducing n-collections

The purpose of this section is to prove Theorem 1.2 in the case that $g(K) > g(K')$. The argument is essentially that in the proof of the main result of [HL].

Proof: [Proof of Theorem 1.3] Let K, K' be as in the statement of the theorem. Let $(\mathcal{D}, \mathbf{q})$ be an n-collection that transforms K to K' with associated crossing link L. Let S be a Seifert surface for K that is of minimum genus among all surfaces bounded by K in the complement of L. By Theorem 2.1 we have $\text{genus}(S) = g(K)$. Since S is incompressible, after an isotopy, we can arrange so that for $i = 1, \ldots, n$, each closed component of $S \cap \text{int}(D_i)$ is essential in $D_i \setminus K$ and thus parallel to $L_i = \partial D_i$ on D_i. Then , after an isotopy of L_i in the complement of K, we may assume that $S \cap \text{int}(D_i)$ consists of a single properly embedded arc $(\alpha_i, \partial\alpha_i) \subset (S, \partial S)$ (see Figure 3). Notice that α_i is essential on S. For, otherwise, D_i would bound a disc in the complement of K and thus the genus of K could not be lowered by surgery on L_i.

We claim that no two of the arcs $\alpha_1, \ldots \alpha_n$, can be parallel on S. For, suppose on the contrary, that the arcs $\alpha_i := \text{int}(D_i) \cap S$ and $\alpha_j := \text{int}(D_j) \cap S$ are parallel on S. Then the crossing circles L_i and

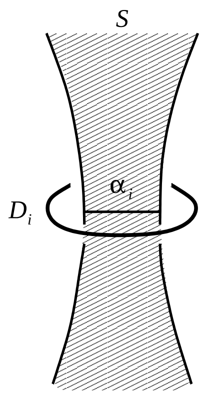

FIGURE 3. The intersection of S with $\text{int}(D_i)$.

L_j cobound an embedded annulus that is disjoint from K. Let

$$M := S^3 \setminus \eta(K \cup L_i) \text{ and } M_1 := S^3 \setminus \eta(K \cup L_i \cup L_j).$$

For $r, s \in \mathbf{Z}$ let $M(r)$ (resp. $M_1(r,s)$) denote the 3-manifold obtained from M (resp. M_1) by filling in $\partial\eta(L_i)$ (resp. $\partial\eta(L_i \cup L_j)$) with slope $\frac{1}{r}$ (resp. slopes $\frac{1}{r}, \frac{1}{s}$). By assumption, S doesn't remain taut in any of $M(q_i)$, $M_1(q_i, q_j)$. Since L_i, L_j are coannular we see that $M_1(q_i, q_j) = M(q_i + q_j)$. Notice that $q_i + q_j \neq q_i$ since otherwise we would conclude that a twist of order q_j along L_j cannot reduce the genus of K. Hence we would have two distinct Dehn fillings of M along $\partial\eta(L_i)$ under which S doesn't remain taut, contradicting Corollary 2.4 of [Ga]. Therefore, we conclude that no two of the arcs $\alpha_1, \ldots \alpha_n$, can be parallel on S. Now the conclusion follows since a Seifert surface of genus g contains $6g - 3$ essential arcs no pair of which is parallel. □

4. KNOT ADJACENCY AND ESSENTIAL TORI

In this section we will complete the proof of Theorem 1.2. For this we need to study the case of n-adjacent knots $K \xrightarrow{n} K'$ in the special situation where all the crossing changes from K to K' are supported on a single crossing circle of K. Using Theorem 2.1, we will see that the general case is reduced to this special one.

4.1. **Knot adjacency with respect to a crossing circle.** We begin with the following definition that provides a refined version of knot adjacency:

Definition 4.1. *Let K, K' be knots and let D_1 be a crossing disc for K. We will say that K is m-adjacent to K' with respect to the crossing circle $L_1 := \partial D_1$, if there exist non-zero integers $s_1, \ldots, s_m$ such that the following is true: For every $\emptyset \neq J \subset \{1, \ldots, m\}$, the knot obtained from K by a surgery modification of order $s_J := \sum_{j \in J} s_j$ along L_1 is isotopic to K'. We will write $K \overset{(m,L_1)}{\longrightarrow} K'$.*

Suppose that $K \overset{(m,L_1)}{\longrightarrow} K'$ and consider the m-collection obtained by taking m parallel copies of D_1 and labeling the i-th copy of L_1 by $\frac{1}{s_i}$. As it follows immediately from the definitions, this m-collection transforms K to K' in the sense of Definition 1.1; thus $K \overset{m}{\longrightarrow} K'$. The following lemma provides a converse statement that is needed for the proof of Theorem 1.2:

Lemma 4.2. *Let K, K' be knots and set $g := \max\{\, g(K), g(K')\,\}$. Suppose that $K \overset{n}{\longrightarrow} K'$. If $n > m(6g-3)$ for some $m > 0$, then there exists a crossing link L_1 for K such that $K \overset{(m+1,L_1)}{\longrightarrow} K'$.*

Proof: Let $(\mathcal{D}, \mathbf{q})$ be an n-collection that transforms K to K' and let L denote the associated crossing link. Let S be a Seifert surface for K that is of minimal genus among all surfaces bounded by K in the complement of L. Isotope so that, for $i = 1, \ldots, n$, the intersection $S \cap \mathrm{int}(D_i)$ is an arc α_i that is properly embedded and essential on S. By Theorem 2.1, we have genus$(S) = g$. Since $n > m(6g-3)$, the set $\{\alpha_i \mid i = 1, \ldots, n\}$ contains at least $m+1$ arcs that are parallel on S. Suppose, without loss of generality, that these are the arcs α_i, $i = 1, \ldots, m+1$. It follows that the components $L_1, \ldots, L_{m+1}$ of L are isotopic in the complement of K; thus any surgery along any of these components can be realized as surgery on L_1. It now follows from Definitions 1.1 and 4.1 that $K \overset{(m+1,L_1)}{\longrightarrow} K'$. □

The main ingredient needed to complete the proof of Theorem 1.2 is provided by the following theorem:

Theorem 4.3. *Given knots K, K', there exists a constant $b(K,\ K') \in \mathbf{N}$, that depends only on K and K', such that if L_1 is a crossing circle of K and $K \stackrel{(m,L_1)}{\longrightarrow} K'$, then either $m \leq b(K,\ K')$ or L_1 bounds an embedded disc in the complement of K.*

Proof: [Proof of Theorem 1.2 assuming Theorem 4.3] Suppose that K, K' are non-isotopic knots with $K \stackrel{n}{\longrightarrow} K'$. If $g(K) > g(K')$ the conclusion follows from Theorem 1.3 by simply taking $C(K,\ K') := 6g-3$. In general, let $C(K,\ K') := b(K,\ K')\ (6g-3)$, where $b := b(K,\ K')$ is the constant of Theorem 4.3. We claim that we must have $n \leq C(K,\ K')$. Suppose, on the contrary, that $n > C(K,\ K')$. By Lemma 4.2, there exists a crossing circle L_1 for K, such that $K \stackrel{(b+1,L_1)}{\longrightarrow} K'$. By Theorem 4.3, L_1 bounds an embedded disc in the complement of K. But this implies that K is isotopic to K' contrary to our assumption. □

The rest of this section will be devoted to the proof of Theorem 4.3. For that we need to study whether the complement of $K \cup L_1$ contains essential tori and how these tori behave under the crossing changes from K to K'. Given K, K' and L_1 such that $K \stackrel{(m,L_1)}{\longrightarrow} K'$, set $N := S^3 \setminus \eta(K \cup L_1)$ and $N' := S^3 \setminus \eta(K')$. By assumption, N' is obtained by Dehn filling along the torus $T_1 := \partial\eta(L_1)$. If N is reducible, Lemma 2.2 implies that L_1 bounds a disc in the complement of K; thus Theorem 4.3 holds. For irreducible N, as it turns out, there are three basic cases to consider:

(a) K' is a composite knot.
(b) N is atoroidal.
(c) N is toroidal and K' is not a composite knot.

By Thurston ([T1]), if N is atoroidal then it is either hyperbolic (it admits a complete hyperbolic metric of finite volume) or it is a Seifert fibered space. To handle the hyperbolic case we use a result of Cooper and Lackenby ([CoLa]). The Seifert fibered spaces that occur are known to be very special and this case is handled by a case-by-case analysis. Case (c) is handled by induction on the number of essential tori contained in N. To set up this induction one needs to study the behavior of these essential tori under the Dehn fillings from N to N'. In particular, one needs to know the circumstances under which these Dehn fillings *create* essential tori in N'. For this step, we employ a result of Gordon ([Go]).

4.2. **Composite knots.** Here we examine the circumstances under which a knot K is n-adjacent to a composite knot K'. We will need the following theorem.

Theorem 4.4. (Torisu, [To]) *Let $K' := K'_1 \# K'_2$ be a composite knot and K'' a knot that is obtained from K' by a generalized crossing change with corresponding crossing disc D. If K'' is isotopic to K' then either ∂D bounds a disc in the complement of K' or the crossing change occurs within one of K'_1, K'_2.*

Proof: For an ordinary crossing the result is given as Theorem 2.1 in [To]. The proof given in there works for generalized crossings. □

The next lemma handles possibility (a) above as it reduces Theorem 4.3 to the case that K' is a prime knot.

Lemma 4.5. *Let K, K' be knots such that $K \overset{(m,L_1)}{\longrightarrow} K'$, where L_1 is a crossing circle for K. Suppose that $K' := K'_1 \# K'_2$ is a composite knot. Then, either L_1 bounds a disc in the complement of K or K is a connect sum $K = K_1 \# K_2$ and there exist $J \in \{K_1, K_2\}$ and $J' \in \{K'_1, K'_2\}$ such that $J \overset{(m,L_1)}{\longrightarrow} J'$.*

Proof: By assumption there is an integer $r \neq 0$ so that the knot K'' obtained from K' by a generalized crossing change of order r is isotopic to K'. By Theorem 4.4, either L_1 bounds a disc in the complement of K' or the crossing change occurs on one of K'_1, K'_2; say on K'_1. Thus, in particular, in the latter case L_1 is a crossing link for K'_1. Since K is obtained from K' by twisting along L_1, K is a, not necessarily non-trivial, connect sum of the form $K_1 \# K'_2$. By the uniqueness of knot decompositions it follows that $K_1 \overset{(m,L_1)}{\longrightarrow} K'_1$. □

4.3. **Dehn surgeries that create essential tori.** Let M be a compact orientable 3-manifold. For a collection $\mathcal{T}$ of disjointly embedded, pairwise non-parallel, essential tori in M we will use $|\mathcal{T}|$ to denote the number of components of $\mathcal{T}$. By Haken's finiteness theorem ([H], Lemma 13.2), the number

$$\tau(M) = \max \{ \, |\mathcal{T}| \, | \, \mathcal{T} \text{ a collection of tori as above} \}$$

is well defined. A collection $\mathcal{T}$ for which $\tau(M) = |\mathcal{T}|$ will be called a *Haken system.*

In this section we will study the behavior of essential tori under the various Dehn fillings from $N := S^3 \backslash \eta(K \cup L_1)$ to $N' := S^3 \backslash \eta(K')$. Since N' is obtained from N by Dehn filling along $T_1 := \partial\eta(L_1)$, essential tori in N' occur in the following two ways:

Type I: An essential torus $T' \subset N'$ that can be isotoped in $N \subset N'$; thus such a torus is the image of an essential torus $T \subset N$.

Type II: An essential torus $T' \subset N'$ that is the image of an essential punctured torus $(P, \partial P) \subset (N, T_1)$, such that each component of ∂P is parallel on T_1 to the curve along which the Dehn filling from N to N' is done.

We begin with the following lemma that examines circumstances under which twisting a knot that is geometrically essential inside a knotted solid torus V yields a knot that is geometrically inessential inside V. In the notation of Definition 4.1, the lemma implies that an essential torus in N either remains essential in $N(s_J)$, for all $\emptyset \neq J \subset \{1, \ldots, m\}$, or it becomes inessential in all $N(s_J)$.

Lemma 4.6. *Let $V \subset S^3$ be a knotted solid torus and let $K_1 \subset V$ be a knot that is geometrically essential in V. Let $D \subset \operatorname{int}(V)$ be a crossing disc for K_1 and let K_2 be a knot obtained from K_1 by a non-trivial twist along D. Suppose that K_1 is isotopic to K_2 in S^3. Then, K_2 is geometrically essential in V. Furthermore, if K_1 is not the core of V then K_2 is not the core of V.*

Proof: Suppose that K_2 is not geometrically essential in V. Then there is an embedded 3-ball $B \subset \operatorname{int}(V)$ that contains K_2. Since making crossing changes on K_2 doesn't change the homology class it represents in V, the winding number of K_1 in V must be zero. Set $L := \partial D$ and $N := S^3 \setminus \eta(K \cup L)$. Let S be a Seifert surface for K_1 such that among all the surfaces bounded by K_1 in N, S has minimum genus. As usual we isotope S so that $S \cap D$ is an arc α properly embedded on S. As in the proof of Theorem 2.1, S gives rise to Seifert surfaces S_1, S_2 of K_1, K_2, respectively. Now K_1 can be recovered from K_2 by twisting ∂S_2 along α.

Claim: L can be isotoped inside B in the complement of K_1.

Since K_1 is obtained from K_2 by a generalized crossing change supported on L it follows that K_1 lies in B. Since this contradicts our assumption that K_1 is geometrically essential in V, K_2 must be geometrically essential in V. To finish the proof of the lemma, assuming the claim, observe that that if K_1 is not the core C of V, then C is a companion knot of K_1. If K_2 is the core of V, C and K_1 are isotopic in S^3 which by Schubert ([S]) is impossible.

Proof of Claim: Since K_1, K_2 are isotopic in S^3 using Corollary 2.4 of [Ga], as used in the proof of Theorem 2.1, we see that S_1 (resp. S_2) is a minimum genus surface for K_1 (resp. K_2) in S^3. By assumption ∂V is a non-trivial companion torus of K_1. Since the winding number of K_1 in V is zero, the intersection $S_1 \cap \partial V$ (resp. $S_1 \cap \partial V$) is homologically trivial in ∂V. Thus we may replace the components of $S_1 \cap \overline{S^3 \setminus V}$, (resp. $S_2 \cap \overline{S^3 \setminus V}$) with boundary parallel annuli in int(V) to obtain a Seifert surface S_1' (resp. S_2') inside V. It follows, that $S_1 \cap \overline{S^3 \setminus V}$, (resp. $S_2 \cap \overline{S^3 \setminus V}$) is a collection of annuli and S_1' (resp. S_2') is a minimum genus Seifert surface for K_1 (resp. K_2). Now S_2' is a minimum genus Seifert surface for K_2 such that $\alpha \subset S_2'$. By assumption, K_2 lies inside B. Since S_2' is incompressible and V is irreducible, S_2' can be isotoped in B by a sequence of disc trading isotopies in int(V). But this isotopy will also bring α inside B and thus L. □

Next we focus on the case that N' is toroidal and examine the circumstances under which N' contains type II tori. We have the following:

Proposition 4.7. *Let K, K' be knots such that K' is a non-trivial satellite but not composite. Suppose that $K \overset{(m,L_1)}{\longrightarrow} K'$, where L_1 is a crossing circle for K and let the notation be as in Definition 4.1. Then, at least one of the following is true:*

a) L_1 bounds an embedded disc in the complement of K.

b) For every $\emptyset \neq J \subset \{1, \ldots, m\}$*,* $N(s_J)$ *has a Haken system that doesn't contain tori of type* II*.*

c) We have $m \leq 6$*.*

Proof: For $s \in \mathbf{Z}$, let $N(s)$ denote the 3-manifold obtained from N by Dehn filling along T_1 with slope $\frac{1}{s}$. Assume that L_1 doesn't bound an embedded disc in the complement of K and that, for some $\emptyset \neq J_1 \subset \{1, \ldots, m\}$, $N(s_{J_1})$ admits a Haken system that contains tori of type II. We claim that, for every $\emptyset \neq J \subset \{1, \ldots, m\}$, $N(s_J)$ has such a Haken system. To see this, first assume that N doesn't contain essential embedded tori. Then, since $N' = N(s_J)$ and K' is a non-trivial satellite the conclusion follows. Suppose that N contains essential embedded tori. By Lemma 4.6 it follows that an essential torus in N either remains essential in $N(s_J)$, for all $\emptyset \neq J \subset \{1, \ldots, m\}$, or it becomes inessential in all $N(s_J)$ as above. Thus the number of type I tori in a Haken system of $N(s_J)$ is the same for all J as above. Thus, since we assume that $N(s_{J_1})$ has a Haken system containing tori of type II, a Haken system of $N(s_J)$ must contain tori of type II, for every $\emptyset \neq J \subset \{1, \ldots, m\}$. We distinguish two cases:

Case 1: Suppose that $s_1, \ldots, s_m > 0$ or $s_1, \ldots, s_m < 0$. Let $s := \sum_{j=1}^{m} s_j$ and recall that we assumed that N is irreducible. By our discussion above, both of $N(s_1), N(s)$ contain essential embedded tori of type II. By Theorem 1.1 of [Go], we must have

$$\Delta(s, s_1) \leq 5, \tag{4.1}$$

where $\Delta(s, s_1)$ denotes the geometric intersection on T_1 of the slopes represented by $\frac{1}{s_1}$, and $\frac{1}{s}$. Since $\Delta(s, s_1) = |\sum_{j=2}^{m} s_j|$, and $|s_j| \geq 1$, in order for (4.1) to be true we must have $m - 1 \leq 5$ or $m \leq 6$.

Case 2: Suppose that not all of $s_1, \ldots, s_m$ have the same sign. Suppose, without loss of generality, that $s_1, \ldots, s_k > 0$ and $s_{k+1}, \ldots, s_m < 0$. Let $s := \sum_{j=1}^{k} s_j$ and $t := \sum_{j=k+1}^{m} s_j$. Since both of $N(s), N(t)$ contain essential embedded tori of type II, by Theorem 1.1 of [Go]

$$\Delta(s, t) \leq 5. \tag{4.2}$$

But $\Delta(t, s) = s - t = \sum_{j=1}^{m} |s_j|$. Thus, in order for (4.2) to be true we must have $m \leq 5$ and the result follows. □

Proposition 4.7 and Lemma 4.5 yield the following corollary:

Corollary 4.8. *Let K, K' be knots and let L_1 be a crossing circle for K. Suppose that the 3-manifold N contains no essential embedded torus and that $K \stackrel{(m,L_1)}{\longrightarrow} K'$. If K' is a non-trivial satellite, then either $m \leq 6$ or L_1 bounds an embedded disc in the complement of K.*

4.4. **Hyperbolic and Seifert fibered manifolds.** In this section we will deal with the case that the manifold N is atoroidal. As already mentioned, by Thurston's uniformization theorem for Haken manifolds ([T1]), N is either hyperbolic or a Seifert fibered manifold.

First we recall some terminology about hyperbolic 3-manifolds. Let N be a hyperbolic 3-manifold with boundary and let T_1 a component of ∂N. In $\mathrm{int}(N)$ there is a cusp, which is homeomorphic to $T_1 \times [1, \infty)$, associated with the torus T_1. The cusp lifts to an infinite set, say $\mathcal{H}$, of disjoint horoballs in the hyperbolic space $\mathbf{H}^3$ which can be expanded so that each horoball in $\mathcal{H}$ has a point of tangency with some other. The image of these horoballs under the projection $\mathbf{H}^3 \longrightarrow \mathrm{int}(N)$, is the *maximal horoball neighborhood* of T_1. The boundary $\mathbf{R^2}$ of each horoball in $\mathcal{H}$ inherits a Euclidean metric from $\mathbf{H}^3$ which in turn induces a Euclidean metric on T_1. A slope $\mathbf{s}$ on T_1 defines a primitive element in $\pi_1(T_1)$ corresponding to a Euclidean translation in $\mathbf{R^2}$. The length of $\mathbf{s}$, denoted by $l(\mathbf{s})$, is given by the length of corresponding translation vector.

Given a slope $\mathbf{s}$ on T_1, let us use $N[\mathbf{s}]$ to denote the manifold obtained from N by Dehn filling along T_1 with slope $\mathbf{s}$. We remind the reader that in the case that the slope $\mathbf{s}$ is represented by $\frac{1}{s}$, for some $s \in \mathbf{Z}$, we use the notation $N(s)$ instead. Next we recall a result of Cooper and Lackenby the proof of which relies on work of Thurston and Gromov. We only state the result in the special case needed here:

Theorem 4.9. (Cooper-Lackenby, [CoLa]) *Let N' be a compact orientable manifold, with $\partial N'$ a collection of tori. Let N be a hyperbolic* *manifold and let* $\mathbf{s}$ *be a slope on a toral component* T_1 *of* ∂N *such that* $N[\mathbf{s}]$ *is homeomorphic to* N'. *Suppose that the length of* $\mathbf{s}$ *on the maximal horoball of* T_1 *in* $\mathrm{int}(N)$ *is at least* $2\pi + \epsilon$, *for some* $\epsilon > 0$. *Then, for any given* N' *and* $\epsilon > 0$, *there is only a finite number of possibilities (up to isometry) for* N *and* $\mathbf{s}$.

Remark 4.10. With the notation of Theorem 4.9, let E denote the set of all slopes $\mathbf{s}$ on T_1, such that $l(\mathbf{s}) \leq 2\pi$. It is a consequence of the Gromov-Thurston "2π " theorem that E is finite. More specifically, the Gromov-Thurston theorem (a proof of which is found in [BHo]) states that if $l(\mathbf{s}) > 2\pi$, then $N[\mathbf{s}]$ admits a negatively curved metric. But in Theorem 11 of [BHo], Bleiler and Hodgson show that there can be at most 48 slopes on T_1 for which $N[\mathbf{s}]$ admits no negatively curved metric. Thus, there can be at most 48 slopes on T_1 with length $\leq 2\pi$.

Using Theorem 4.9 we will prove the following proposition which is a special case of Theorem 4.3 (compare possibility (b) of §4.1):

Proposition 4.11. *Let* K, K' *be knots such that* $K \overset{(m,L_1)}{\longrightarrow} K'$, *where* L_1 *is a crossing circle for* K *and* $m > 0$. *Suppose that* $N := S^3 \setminus \eta(K \cup L_1)$ *is a hyperbolic manifold. Then, there is a constant* $b(K,\ K')$, *that depends only on* K, K', *such that* $m \leq b(K,\ K')$.

Proof: We will apply Theorem 4.9 for the manifolds $N := S^3 \setminus \eta(K \cup L_1)$, $N' := S^3 \setminus \eta(K')$ and the component $T_1 := \partial\eta(L_1)$ of ∂N. Let $s_1, \ldots, s_m$ be integers that satisfy Definition 4.1. That is, for every $\emptyset \neq J \subset \{1, \ldots, m\}$, $N(s_J)$ is homeomorphic to N'. By abusing the notation, for $r \in \mathbf{Z}$ we will use $l(r)$ to denote the length on T_1 of the slope represented by $\frac{1}{r}$. Also, as in the proof of Proposition 4.7, we will use $\Delta(r,t)$ to denote the geometric intersection on T_1 of the slopes represented by $\frac{1}{r}$, $\frac{1}{t}$. Let $A(r,t)$ denote the area of the parallelogram in $\mathbf{R}^2$ spanned by the lifts of these slopes and let $A(T_1)$ denote the area of a fundamental domain of the torus T_1. It is known that $A(T_1) \geq \frac{\sqrt{3}}{2}$ (see, [BHo]) and that $\Delta(r,t)$ is the quotient of $A(r,t)$ by $A(T_1)$. Thus, for every $r, t \in \mathbf{Z}$, we have

$$l(r)l(t) \geq \Delta(r,t)\frac{\sqrt{3}}{2}. \tag{4.3}$$

Let $\lambda > 0$ denote the length of a meridian of T_1; in fact it is known that $\lambda \geq 1$. Assume on the contrary that no constant $b(K,\ K')$ as in the statement of the proposition exists. Then, there exist infinitely many integers s such that $N(s)$ is homeomorphic to N'. Applying (4.3) for $l(s)$ and λ we obtain

$$l(s) \geq |s|\frac{\sqrt{3}}{2\lambda}.$$

Thus, for $|s| \geq \frac{4\pi\lambda+2\lambda}{\sqrt{3}}$ we have $l(s) \geq 2\pi + 1$. But then for $\epsilon = 1$, we have infinitely many integers such that $l(s) \geq 2\pi + \epsilon$ and $N(s)$ is homeomorphic to N'. Since this contradicts Theorem 4.9 the proof of the Proposition is finished. □

Next we turn our attention to the case where $N := S^3 \setminus \eta(K \cup L_1)$ is an atoroidal Seifert fibered space. Since N is embedded in S^3 it is orientable. It is know that an orientable, atoroidal Seifert fibered space with two boundary components is either a *cable space* or a trivial torus bundle $T^2 \times I$. Let us recall how a cable space is formed: Let $V'' \subset V' \subset S^3$ be concentric solid tori. Let J be a simple closed curve on $\partial V''$ having slope $\frac{a}{b}$, for some $a, b \in \mathbf{Z}$ with $|b| \geq 2$. The complement $X := V' \setminus \text{int}(\eta(J))$ is a $\frac{a}{b}$-cable space. Topologically, X is a Seifert fibered space over the annulus with one exceptional fiber of multiplicity $|b|$. We show the following:

Lemma 4.12. *Let K, K' be knots such that $K \stackrel{(m,L_1)}{\longrightarrow} K'$, where L_1 is a crossing circle for K and $m > 0$. Suppose that $N := S^3 \setminus \eta(K \cup L_1)$ is an irreducible, atoroidal Seifert fibered space. Then, there is a constant $b(K,\ K')$ such that $m \leq b(K,\ K')$.*

Proof: As discussed above, N is either a cable space or a torus bundle $T^2 \times I$. Note, however, that in a cable space the cores of the solid tori bounded in S^3 by the two components of ∂N have non-zero linking number. Thus, since the linking number of K and L_1 is zero, N cannot be a cable space. Hence, we only have to consider the case where $N \cong T^2 \times I$. Suppose $T_1 = T^2 \times \{1\}$ and $T_2 := \partial\eta(K) = T^2 \times \{0\}$. By assumption there is a slope $\mathbf{s}$ on T_1 such that the Dehn filling of T_1 along $\mathbf{s}$ produces N'. Now $\mathbf{s}$ corresponds to a simple closed curve on T_2 that must compress in N'. By Dehn's Lemma, K' must be the unknot. It follows that either $g(K) > g(K')$ or K is the unknot. In the later case, we obtain that L_1 bounds a disc disjoint from K contrary to our assumption that N is irreducible. Thus, $g(K) > g(K')$ and the conclusion follows from Theorem 1.3. $\square$.

The following Proposition complements nicely Corollary 4.8. We point out that the proposition is not needed for the proof of the main result. Hence a reader eager to get to the proof of Theorem 4.3 can move to the next section without loss of continuity.

Proposition 4.13. *Let K, K' be non-isotopic hyperbolic knots. Suppose there exists a crossing circle L_1 for K such that $K \stackrel{(m,L_1)}{\longrightarrow} K'$, for some $m \geq 6$. Then, for given K and K', there is only a finite number of possibilities for m and for L_1 up to isotopy in the complement of K.*

Proof: As before, let $N := S^3 \setminus \eta(K \cup L_1)$, $N' := S^3 \setminus \eta(K')$ and let D be a crossing disc for L_1. Since K is not isotopic to K', N is irreducible and ∂-irreducible.

Claim. N is atoroidal.

Proof of Claim. Suppose that N contains an embedded essential torus T and let V denote the solid torus bounded by T in S^3. If L_1 cannot be isotoped to lie in int$V$ then $D \cap T$ contains a component whose interior in D is pierced exactly once by K. This implies that T is parallel to $\partial\eta(K)$ in N; a contradiction. Thus, L_1 can be isotoped to lie inside V. Now let S be a Seifert surface of K that is taut in N. After isotopy, $D \cap S$ is an arc α that is essential on S. By Theorem 2.1, S remains of minimum genus in at least one of $N'' := S^3 \setminus \eta(K)$, N'. Assume S remains of minimum genus in N'; the other case is completely analogous. Since K, K' are hyperbolic T becomes inessential in both of N'', N'. But since K, K' are related by a generalized crossing change, either T becomes boundary parallel in both of N'', N' or it becomes compressible in both of them. First suppose that T is boundary parallel in both of N'', N': Then it follows that the arc α is inessential on S and K is isotopic to K'; a contradiction. Now suppose that T is compressible in both of N'', N': Then, both of K, K' are inessential in V and they can be isotoped to lie in a 3-ball $B \subset$ int$V$. By an argument similar to this in the proof of Lemma 4.6 we can conclude that α, and thus L_1, can be isotoped to lie in B. But this contradicts the assumption that T is essential in N and finishes the proof of the claim.

To continue observe that the argument of the proof of Lemma 4.12 shows that if N is a Seifert fibered space then K' is the unknot. But this is impossible since we assumed that K' is hyperbolic. Thus, by [T1], N is hyperbolic. Let $s_1, \ldots, s_m$ be integers that satisfy Definition 4.1 for K, K'. Thus we have $2^m - 1$ integers s, with $N(s) = N'$. Now [BHo] implies that we can have at most 48 integers so that the corresponding slopes have lengths $\leq 2\pi$ on T_1. Since $m \geq 6$ we have $2^m - 1 > 48$. Thus we have $k_m := 2^m - 49 > 0$ integers s such that $l(s) > 2\pi$ and $N(s) = N'$. By Theorem 4.9, there is only a finite number of possibilities (up to isometry) for N and s. Now the conclusion follows. □

Remark 4.14. Proposition 4.13 implies Theorem 4.3, and thus Theorem 1.2, in the case that K, K' are hyperbolic.

4.5. **The proof of Theorem 4.3.** In this subsection we give the proof of Theorem 4.3. We will need the following theorem which is a special case of a result of McCullough proven in [M].

Theorem 4.15. (McCullough, [M]) *Let M be a compact orientable 3-manifold, and let C be a simple loop in ∂M. Suppose that $h\colon M \to M$ is a homeomorphism whose restriction to ∂M is isotopic to a nontrivial power of a Dehn twist about C. Then, C bounds a disc in M.*

Before we embark on the proof of Theorem 4.3, we recall that for a compact orientable 3-manifold M, $\tau(M)$ denotes the cardinality of a Haken system of tori (see subsection §4.3). In particular, M is atoroidal if and only if $\tau(M) = 0$.

Proof: [Proof of Theorem 4.3] Let K, K' be knots and let L_1 be a crossing circle of K such that $K \stackrel{(m,L_1)}{\longrightarrow} K'$. As before we set $N := S^3 \setminus \eta(K \cup L_1)$ and $N' := S^3 \setminus \eta(K')$. If $g(K) > g(K')$, by Theorem 1.3, we have $m \leq 3g(K) - 1$. Thus, in this case, we can take $b(K, K') :=$

$3g(K)-1$ and Theorem 4.3 holds. Hence, we only have to consider that case that $g(K) \leq g(K')$.

Next we consider the complexity

$$\rho = \rho(K,\, K',\, L_1) := \tau(N).$$

First, suppose that $\rho = 0$, that is N is atoroidal. Then, N is either hyperbolic or a Seifert fibered manifold ([T1]). In the former case, the conclusion of the theorem follows from Proposition 4.11; in the later case it follows from Lemma 4.12.

Assume now that $\tau(N) > 0$; that is N is toroidal. Suppose, inductively, that for every triple K_1, K_1', L_1', with $\rho(K_1,\, K_1',\, L_1') < r$, there is a constant $d = d(K_1,\, K_1')$ such that: If $K_1 \stackrel{(m,L_1')}{\longrightarrow} K_1'$, then either $m \leq d$ or L_1' bounds an embedded disc in the complement of K_1. Let K, K', L_1 be knots and a crossing circle for K, such that $K \stackrel{(m,L_1)}{\longrightarrow} K'$ and $\rho(K,\, K',\, L_1) = r$. Let $s_1, \ldots, s_m$ be integers satisfying Definition 4.1 for K, K' and L_1. For every $\emptyset \neq J \subset \{1, \ldots, m\}$, let $N(s_J)$ be the 3-manifold obtained from N by Dehn filling of $\partial\eta(L_1)$ with slope $\frac{1}{s_J}$. By assumption, $N' = N(s_J)$. Assume, for a moment, that for some $\emptyset \neq J_1 \subset \{1, \ldots, m\}$, $N(s_{J_1})$ contains essential embedded tori of type II. Then Proposition 4.7 implies that either $m \leq 6$ or L_1 bounds an embedded disc in the complement of K. Hence, in this case, the conclusion of the theorem is true for K, K', L_1, with $b(K,\ K') := 6$. Thus we may assume that, for every $\emptyset \neq J \subset \{1, \ldots, m\}$, $N(s_J)$ doesn't contain essential embedded tori of type II.

We will show the following:

Claim 1: There exist knots K_1, K_1' and a crossing circle L_1' for K_1 such that:

(1) $K_1 \stackrel{(m,L_1')}{\longrightarrow} K_1'$ and $\rho(K_1,\, K_1',\, L_1') < \rho(K,\, K',\, L_1) = r$.

(2) If L_1' bounds an embedded disc in the complement of K_1 then L_1 bounds an embedded disc in the complement of K.

The proof of the theorem assuming Claim 1: By induction, there is $d = d(K_1,\ K_1')$ such that either $m \leq d$ or L_1' bounds a disc in the complement of K_1. Let $\mathcal{K}_m$ denote the set of all pairs of knots K_1, K_1' such that there exists a crossing circle L_1' for K_1 satisfying properties (1) and (2) of Claim 1. Define

$$b = b(K,\, K') := \min \{\, d(K_1,\, K_1') \mid K_1, K_1' \in \mathcal{K}_m \,\}.$$

Clearly b satisfies the conclusion of the statement of the theorem.

Proof of Claim 1: Let T be an essential embedded torus in N. Since T is essential in N, T has to be knotted. Let V denote the solid torus component of $S^3 \setminus T$. Note that K must lie inside V. For, otherwise L_1 must be geometrically essential in V and thus it can't be the unknot. There are various cases to consider according to whether L_1 lies outside or inside V.

Case 1: Suppose that L_1 lies outside V and it cannot be isotoped to lie inside V. Now K is a non-trivial satellite with companion torus T. Let D_1 be a crossing disc bounded by L_1. Notice that if all the components of $D_1 \cap T$ were either homotopically trivial in $D_1 \setminus (D_1 \cap K)$ or parallel to ∂D_1, then we would be able to isotope L_1 inside V contrary to our assumption. Thus $D_1 \cap T$ contains a component that encircles a single point of the intersection $K \cap D_1$. This implies that the winding number of K in V is one. Since T is essential in N we conclude that K is composite, say $K := K_1 \# K_2$, and T is the follow-swallow torus. Moreover, the generalized crossings realized by the surgeries on L_1 occur along a summand of K, say along K_1. By the uniqueness of prime decompositions of knots, it follows that there exists a (not necessarily non-trivial) knot K_1', such that $K' = {K'}_1 \# K_2$ and $K_1 \overset{(m,L_1)}{\longrightarrow} K_1'$. Set $N_1 := S^3 \setminus \eta(K_1 \cup L_1)$ and $N_1' := S^3 \setminus \eta(K_1')$. Clearly, $\tau(N_1) < \tau(N)$. Thus, $\rho(K_1,\, K_1',\, L_1) < \rho(K,\, K',\, L_1)$ and part (1) of the claim has been proven in this case. To see part (2) notice that if L_1 bounds a disc D in the complement of K_1, we may assume $D \cap K = \emptyset$.

Case 2: Suppose that L_1 can be isotoped to lie inside V. Now the link $K \cup L_1$ is a non-trivial satellite with companion torus T. We can find a standardly embedded solid torus $V_1 \subset S^3$, and a 2-component link $(K_1 \cup L_1') \subset V_1$ such that: i) $K_1 \cup L_1'$ is geometrically essential in V_1; ii) L_1' is a crossing disc for K_1; and iii) there is a homeomorphism $f : V_1 \longrightarrow V$ such that $f(K_1) = K$ and $f(L_1') = L_1$ and f preserves the longitudes of V_1 and V. In other words, $K_1 \cup L_1'$ is the model link for the satellite. Let $\mathcal{T}$ be a Haken system for N containing T. We will assume that the torus T is innermost; i.e. the boundary of the component of $N \setminus \mathcal{T}$ that contains T also contains $\partial\eta(K)$. By twisting along L_1 if necessary, we may without loss of generality assume that $\bar{V} := \overline{V \setminus (K \cup L_1)}$ is atoroidal. Then, $\bar{V}_1 := \overline{V_1 \setminus (K_1 \cup L_1')}$ is also atoroidal. For every $\emptyset \neq J \subset \{1, \ldots, m\}$, let $K(s_J)$ denote the knot obtained from K_1 by performing $\frac{1}{s_J}$-surgery on L_1'. By assumption the knots $f(K(s_J))$ are all isotopic to K'.

Subcase 1: There is $\emptyset \neq J_1 \subset \{1, \ldots, m\}$, such that ∂V is compressible in $V \setminus f(K(s_{J_1}))$. By Lemma 4.6, for every $\emptyset \neq J \subset \{1, \ldots, m\}$, ∂V is compressible in $V \setminus f(K(s_J))$. It follows that there is an embedded 3-ball $B \subset \mathrm{int}(V)$ such that: i) $f(K(s_J)) \subset \mathrm{int}(B)$, for every $\emptyset \neq J \subset \{1, \ldots, m\}$; and ii) the isotopy from $f(K(s_{J_1}))$ to $f(K(s_{J_2}))$ can be realized inside B, for every $J_1 \neq J_2$ as above. From this observation it follows that there is a knot $K_1' \subset \mathrm{int}(V_1)$ such that $f(K_1') = K'$ and $K_1 \stackrel{(m, L_1')}{\longrightarrow} K_1'$ in V_1. Let $N_1 := S^3 \setminus \eta(K_1 \cup L_1')$ and $N_1' := S^3 \setminus \eta(K_1')$. Clearly, $\tau(N_1) < \tau(N)$. Hence, $\rho(K_1, K_1', L_1) < \rho(K, K', L_1)$ and the part (1) of Claim 1 has been proven.

We will prove part (2) of Claim 1 for this subcase together with the next subcase.

Subcase 2: For every $\emptyset \neq J \subset \{1, \ldots, m\}$, $f(K(s_J))$ is geometrically essential in V. By Lemma 4.5, the conclusion of the claim is true if K' is composite. Thus, we may assume that K' is a prime knot. In this case, we claim that, for every $\emptyset \neq J \subset \{1, \ldots, m\}$, there is an orientation preserving homeomorphism $\phi : S^3 \longrightarrow S^3$ such that $\phi(V) = V$ and $\phi(f(K(s_{J_1}))) = f(K(s_{J_2}))$. Since we assumed that $N(s_{J_1})$, $N(s_{J_1})$ do not contain essential tori of type II, T remains innermost in the complement of $f(K(s_{J_1}))$, $f(K(s_{J_2}))$. By the uniqueness of the torus decomposition of knot complements [JS] or the uniqueness of satellite structures of knots [S], there is an orientation preserving homeomorphism $\phi : S^3 \longrightarrow S^3$ such that $\phi(V) \cap V = \emptyset$ and $\bar{K} := \phi(f(K(s_{J_1}))) = f(K(s_{J_2}))$ (compare, Lemma 2.3 of [Mo]). Since T is innermost in $\bar{V}$, we have $S^3 \setminus \mathrm{int}(V) \subset \mathrm{int}(\phi(S^3 \setminus \mathrm{int}(V)))$ or $\phi(S^3 \setminus \mathrm{int}(V)) \subset \mathrm{int}(S^3 \setminus \mathrm{int}(V))$. In both cases, by Haken's finiteness theorem, it follows that T and $\phi(T)$ are parallel in the complement of $\bar{K}$. Thus after an ambient isotopy, leaving $\bar{K}$ fixed, we have $\phi(V) = V$. Let $h = f \circ \phi \circ f^{-1} : V_1 \longrightarrow V_1$. Then h preserves the longitude of V_1 up to a sign and $h(K(s_{J_1})) = K(s_{J_2})$. So, in particular, the knots $K(s_{J_1})$ and $K(s_{J_2})$ are isotopic in S^3. Let K_1' denote the knot type in S^3 of $\{K(s_J)\}_{J\subset\{1,\ldots,m\}}$. By our earlier assumptions, $K_1 \stackrel{(m,L_1')}{\longrightarrow} K_1'$. Let $N_1 := S^3 \setminus \eta(K_1 \cup L_1')$ and $N_1' := S^3 \setminus \eta(K_1')$. Clearly, $\tau(N_1) < \tau(N)$. Thus part (1) of Claim 1 has been proven also in this subcase.

We now prove part (2) of Claim 1 for both subcases. Note that it is enough to show that if L_1' bounds an embedded disc, say D', in the complement of K_1 in S^3, then it bounds one inside V_1.

Let $D_1' \subset V_1$ be a crossing disc bounded by L_1' and such that $\mathrm{int}(D') \cap \mathrm{int}(D_1') = \emptyset$. Since ∂V_1 is incompressible in $V_1 \setminus K_1$, after a cut and paste argument, we may assume that $E = D_1' \cup (D \cap V_1)$ is a proper annulus whose boundary are longitudes of V_1.

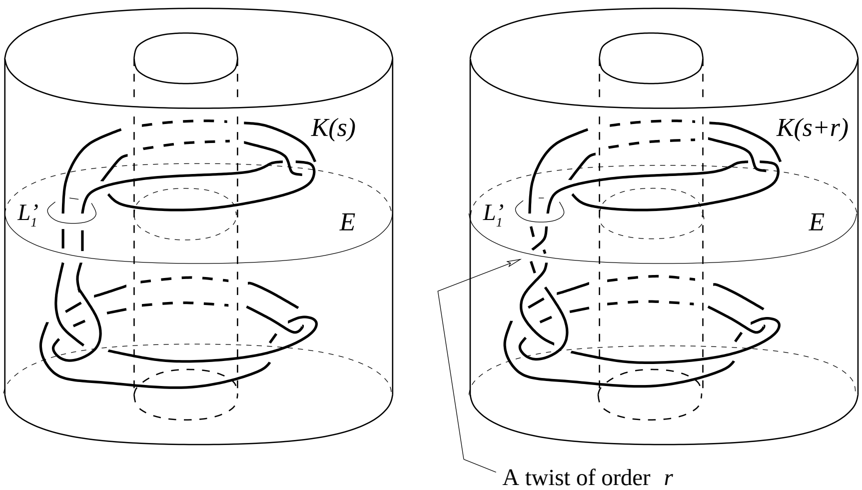

Figure 4. The annulus E contains the crossing circle L_1' and separates V_1 into solid tori V_1' (part above E) and V_1'' (part below E.) In V_1'' the knots $K(s)$, $K(s+r)$ differ by a twist of order r along D_1'.

By assumption, in both subcases, there exist non-zero integers s, r, such that $K(s)$ and $K(s+r)$ are isotopic in S^3. Here, $K(s)$ and $K(s+r)$ denotes the knots obtained from K_1 by a twist along L_1' of order s

and $s+r$ respectively. Let $\hat{h} : S^3 \longrightarrow S^3$ denote the extension of $h : V_1 \longrightarrow V_1$ to S^3. We assume that $\hat{h}$ fixes the core circle C_1 of the complementary solid torus of V_1. Since the 2-sphere $D \cup D_1'$ gives the same (possible trivial) connected sum decomposition of $K_1' = K(s) = K(s+r)$ in S^3, we may assume that $\hat{h}(D) = D$ and $\hat{h}(D_1') = D_1'$ up to an isotopy. During this isotopy of $\hat{h}$, $\hat{h}(C_1)$ and $\hat{h}(V_1)$ remain disjoint. So we may assume that at the end of the isotopy, we still have $\hat{h}(V_1) = V_1$. Thus, we can assume that $h(E) = E$.

The annulus E cuts V_1 into two solid tori V_1' and V_1''. See Figure 4, where the solid torus above E is V_1' and below E is V_1''. We have either $h(V_1') = V_1'$ and $h(V_1'') = V_1''$ or $h(V_1') = V_1''$ and $h(V_1'') = V_1'$. In the case when $h(V_1') = V_1'$ and $h(V_1'') = V_1''$, we may assume that $h|\partial V_1 = \mathrm{id}$ and $h|E = \mathrm{id}$. Thus $K(s+r) \cap V_1' = K(s) \cap V_1'$ and $K(s+r) \cap V_1''$ is equal to $K(s) \cap V_1''$ twisted by a twist of order r along L_1'. Let M denote the 3-manifold obtained from $V_1'' \setminus (V_1'' \cap K(s))$ by attaching a 2-handle to $\partial V_1'' \cap E$ along $K(s) \cap V_1''$. Now $h|\partial M$ can be realized by a Dehn twist of order r along L_1'. By Theorem 4.15, L_1' must bound a disc in M. In order words, L_1' bounds a disc in $V_1 \setminus K(s)$. This implies that L_1' bounds a disc in $V_1 \setminus K_1$

In the case when $h(V_1') = V_1''$ and $h(V_1'') = V_1'$, we may assume that $h|\partial V_1$ and $h|E$ are rotations of $180°$ with an axis on E passing through the intersection points of D_1' with $K(s)$ and $K(s+r)$. Thus $K(s+r) \cap V_1'$ and $K(s) \cap V_1''$ differ by a rotation, and $K(s+r) \cap V_1''$ is equal to $K(s) \cap V_1'$ twisted by a twist of order r along L_1' followed by a rotation. Now we consider the 3-manifold N obtained from $V_1' \setminus (V_1' \cap K(s))$ by attaching a 2-handle to $\partial V_1' \cap E$ along $K(s) \cap V_1'$. As above we conclude that a Dehn twist of order r along L_1' extends to N and we complete the argument by applying Theorem 4.15. □

5. Applications and examples

5.1. **Applications to nugatory crossings.** Recall that a crossing of a knot K with crossing disc D is called *nugatory* if ∂D bounds a disc disjoint from K. This disc and D bound a 2-sphere that decomposes K into a connected sum, where some of the summands may be trivial.

Clearly, changing a nugatory crossing doesn't change the isotopy class of a knot. An outstanding open question is whether the converse is true (see Problem 1.58 of Kirby's Problem List ([GT]):

Question 5.1. (Problem 1.58, [GT]) *If a crossing change in a knot K yields a knot isotopic to K is the crossing nugatory?*

The answer is know to be *yes* in the case when K is the unknot ([ScT]) and when K is a 2-bridge knot ([To]). In [To], I. Torisu conjectures that the answer is always *yes.* Our results in Section five yield the following corollary that shows that an *essential* crossing circle of a knot K can admit at most finitely many twists that do not change the isotopy type of K:

Corollary 5.2. *For a crossing of a knot K, with crossing disc D, let $K(r)$ denote the knot obtained by a twist of order r along D. The crossing is nugatory if and only if $K(r)$ is isotopic to K for all $r \in \mathbf{Z}$.*

Proof: One direction of the corollary is clear. To obtain the other direction apply Theorem 4.3 for $K = K'$. □

In the view of Corollary 5.2, Question 5.1 is reduced to the following: With the same setting as in Corollary 5.2, let $K_+ := K$ and $K_- := K(1)$. If K_- is isotopic to K_+ is it true that $K(r)$ is isotopic to K, for all $r \in \mathbf{Z}$?

5.2. **Examples.** In this subsection, we outline some methods that for every $n > 0$ construct knots K, K' with $K \xrightarrow{n} K'$. It is known that given $n \in \mathbf{N}$ there exists a plethora of knots that are n-adjacent to the unknot. In fact, [AK] provides a method for constructing all such knots. It is easy to see that given knots K, K' such that K_1 is n-adjacent to the unknot, the connected sum $K := K_1 \# K'$ is n-adjacent to K'. Clearly, if K_1 is non-trivial then $g(K) > g(K')$. To construct examples K, K' in which K is not composite, at least in an obvious way, one can proceed as follows: For $n > 0$ let K_1 be a knot that is n-adjacent to the unknot and let $V_1 \subset S^3$ be a standard solid torus. We can embed K_1 in V_1 so that i) it has non-zero winding number; and ii) it is n-adjacent to the core of V_1 inside V_1. Note that there might be many different ways of doing so. Now let $f : V_1 \longrightarrow S^3$ be any embedding that knots V_1. Set $V := f(V_1)$, $K := f(K_1)$ and let K' denote the core of V. By construction, $K \xrightarrow{n} K'$. Since K_1 has non-zero winding number in V_1 we have $g(K) > g(K')$ (see, for example, [BZ]).

We will say that two ordered pairs of knots $(K_1,\ K_1')$, $(K_2,\ K_2')$ are isotopic iff K_1 is isotopic to K_2 and K_1' is isotopic to K_2'. From our discussion above we obtain the following:

Proposition 5.3. *For every $n \in \mathbf{N}$ there exist infinitely many non-isotopic pairs of knots $(K,\ K')$ such that $K \xrightarrow{n} K'$ and $g(K) > g(K')$.*

Remark 5.4. We should point out that, at this point, we don't know of any examples of knots $(K,\ K')$ such that $K \xrightarrow{n} K'$ and $g(K) < g(K')$. In fact the results of [K1], and further examples constructed by Torisu [To], prompt the following question: Is it true that if $K \xrightarrow{n} K'$ for some $n > 1$, then either $g(K) > g(K')$ or K is isotopic to K'?

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