Pergamon

Topology Vol. 37, No. 3, pp. 673–707, 1998
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0040-9383/97 $19.00 + 0.00

PII: S0040-9383(97)00034-7

FINITE TYPE INVARIANTS FOR KNOTS IN 3-MANIFOLDS

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(*Received* 5 *November* 1993; *in revised form* 4 *October* 1995; *final version* 16 *February* 1997)

We use the Jaco–Shalen and Johannson theory of the characteristic submanifold and the Torus theorem (Gabai, Casson-Jungreis) to develop an intrinsic finite type theory for knots in irreducible 3-manifolds. We also establish a relation between finite type knot invariants in 3-manifolds and these in $\mathbb{R}^3$. As an application we obtain the existence of non-trivial finite type invariants for knots in irreducible 3-manifolds. © 1997 Elsevier Science Ltd. All rights reserved

0. INTRODUCTION

The theory of quantum groups gives a systematic way of producing families of polynomial invariants, for knots and links in $\mathbb{R}^3$ or S^3 (see for example [18, 24]). In particular, the Jones polynomial [12] and its generalizations [6, 13], can be obtained that way. All these Jones-type invariants are defined as state models on a knot diagram or as traces of a braid group representation.

On the other hand Vassiliev [25, 26], introduced vast families of numerical knot invariants (*finite type* invariants), by studying the topology of the space of knots in $\mathbb{R}^3$. The computation of these invariants, involves in an essential way the computation of related invariants for special knotted graphs (singular knots). It is known [1–3], that after a suitable change of variable the coefficients of the power series expansions of the Jones-type invariants, are of *finite type*.

In [23], Stanford studied *finite type* invariants for links and graphs in $\mathbb{R}^3$, by only using generalized Redemeister moves and standard PL-techniques available in the 3-space. In [17], Lin was able to free Stanford's work from the use of the special features of $\mathbb{R}^3$, and develop a finite type theory for links in simply connected 3-manifolds. He showed that for these manifolds the theory is equivalent to the one in $\mathbb{R}^3$ (see Theorem 7.2 in [17]). The main ingredient used in Lin's work, is the generic picture of a family of maps from a compact 1-polyhedron in a 3-manifold, parameterized by a 2-disc.

In this paper we develop a *finite type* theory for knots in closed irreducible 3-manifolds. Our approach, intrinsically 3-dimensional, is the one initiated in [17]. However, the presence of a non-trivial fundamental group makes things essentially different and requires the use of more powerful and sophisticated techniques from the topology of 3-manifolds. One of our main tools is the theory of the *characteristic submanifold* as developed by Jaco-Shalen and Johannson [10, 11]. The techniques of this paper and [17], were used in [14] to define a power series invariant for links in rational homology spheres, which generalizes the 2-variable Jones polynomial (HOMFLY).

To describe the main ideas and state the results of this paper we need to establish some notation and terminology.

†This research was partially supported by NSF grants DMS-91-06584 and DMS-96-26140.

Assume that M is an oriented piecewise linear 3-manifold and let $\mathscr{C}$ be the set of conjugacy classes in $\pi_1(M)$. Let $\mathscr{M}=\{\phi:S^1\to M;\ \phi \text{ is a piecewise-linear map}\}$ and let $\mathscr{E}=\{\phi\in\mathscr{M};\ \phi \text{ is a an embedding}\}$. The complement of $\mathscr{E}$, is the *discriminant* of $\mathscr{M}$.

The components of $\mathscr{M}$ are in one-to-one correspondence with the conjugacy classes in $\pi_1(M)$. We will denote by $\mathscr{M}_c$ the component of $\mathscr{M}$ corresponding to $c\in\mathscr{C}$. Finally, let us denote by $\mathscr{K}$ (resp. $\mathscr{K}_c$) the set of isotopy classes of knots in $\mathscr{M}$ (resp. $\mathscr{M}_c$).

Any two knots in some $\mathscr{M}_c$ are related by a sequence of "crossing changes". When we make a crossing change from one knot to another, we produce a singular knot as an intermediate step. By repeating this procedure for singular knots with one double point we produce singular knots with two double points, and so on. So, it is natural to consider immersions $S^1\to M$, whose only singularities are finitely many transverse double points (singular knots).

In this paper we work over a ring $\mathscr{R}$, which is torsion free as an abelian group. A knot invariant $f:\mathscr{K}_c\to\mathscr{R}$, gives rise to a singular knot invariant by repeatedly defining

$$f(K_\times)=f(K_+)-f(K_-). \tag{1}$$

Here K_+ and K_- are the knots obtained by resolving a double point "$\times$", of the singular knot $K_\times$. The functional f is called an invariant of *finite type m*, if its derived singular knot invariant vanishes identically on singular knots with more than m double points, and m is the smallest such integer.

Let us denote by $\mathscr{F}^m$ (resp. $\mathscr{F}_c^m$) the $\mathscr{R}$-module of all *finite type* invariants of type $\leqslant m$, for knots in $\mathscr{K}$ (resp. in $\mathscr{K}_c$). Clearly we have, $\mathscr{F}^m=\bigoplus_{c\in\mathscr{C}}\mathscr{F}_c^m$. We show that the invariants in every $\mathscr{F}_c^m$, are determined by crossing change formulae together with a set of initial data (see Theorem 5.6). The initial data, which consist of the values of the invariants on a special set of singular knots, are given as solutions to a system of linear equations. The equations arise from resolutions of triple points and are given in terms of 4-term relations. The system is finite iff the fundamental group of the manifold is finite, and it is really a generalization of the system arising in [3] from Vassiliev's original work for knots in $\mathbb{R}^3$. Then, we show that every *finite type* knot invariant in $\mathbb{R}^3$ gives rise to an invariant for knots in every irreducible 3-manifold that is not one of the small Euclidean Seifert spaces (Section 3).

THEOREM 0.1. *Let M be a closed, orientable, irreducible 3-manifold, as above. Then $\mathscr{F}_c^m(M)$ contains a submodule isomorphic to $\mathscr{F}^m(\mathbb{R}^3)$, for every m and $c\in\mathscr{C}$, with $c\neq 1$.*

This gives a first existence theorem of non-trivial *finite type* invariants for knots in closed irreducible 3-manifolds.

Let us, now, briefly describe the main ideas of the paper. It turns out that in order to describe the spaces $\mathscr{F}_c^m$ we have to deal with the following problem: Starting with a singular knot invariant f, we want to find necessary and sufficient conditions that f has to satisfy so that it is derived from a knot invariant, via (1). We do so in Sections 3 and 4. This is a question about the "integrability" of the invariant f along a path in $\mathscr{M}_c$. Let Φ be a path in $\mathscr{M}_c$. After perturbation, we may assume that it intersects the *discriminant* in only finitely many points. Moreover, we can assume that when the parameter of the loop passes through such a point the corresponding map changes by a "crossing change". Hence, the maps corresponding to these points are singular knots. The sum of suitably signed values of f on these knots, denoted here by X_Φ, can be thought of as the integral of f along Φ.

In order for f to be derived from a knot invariant, it is necessary and sufficient that X_Φ is independent of Φ, or equivalently that $X_\Phi=0$, for every loop Φ in $\mathscr{M}_c$.

The first thing that we do is to find a set of finite local "integrability conditions" which guarantee that X_Φ only depends on the homotopy class of Φ in $\mathscr{M}_c$. It turns out that these conditions imply the vanishing of the integral X_Φ, along most of the loops Φ. In particular, if the manifold M does not contain any Seifert fibered manifolds over non-orientable surfaces, then these conditions imply the vanishing of the integral X_Φ along any loop Φ. For a null-homotopic loop this assertion is proved by putting the null-homotopy into a nice general position. This is done in Lemmas 3.4 and 3.5. In general we show that the obstruction for integrating a singular knot invariant that satisfies these local integrability conditions, to a knot invariant lies in a certain subgroup of π_1. This obstruction is shown to vanish, directly, in many cases in which $H_1(M)$ is finite (see Proposition 3.10).

To treat the general case, we have to impose a set of "stronger" integrability conditions, on the singular knot invariant f, and employ the theory of the *characteristic submanifold.* First, we need to change our point of view and think of Φ as a map $P \times S^1 \to M$, where P is in general an 1-dimensional compact polyheron. This naturally leads to the study of tori (and annuli) in M and to the use of the theory of the *characteristic submanifold* [10, 11] in order to treat essential Φ's. The idea is to first homotope Φ into the *characteristic submanifold,* which is a Seifert fibered space. This is done by using the "*Enclosing Theorem*" (see Section 2 for the statement). Then, by employing the homotopy classification of essential tori (and annuli) in Seifert fibered spaces [11] we are able to homotope Φ in a "nice position" so that we can see that $X_\Phi = 0$ is implied by the integrability conditions. All this is done in Lemmas 3.11–3.14, 4.6 and 4.7. The case of inessential Φ's is treated in Lemmas 3.8 and 4.5 by a refinement of the arguments used for null-homotopic loops.

Finally, from the *Torus Theorem* [5, 8] we deduce that if M is not Haken, then it is a Seifert manifold, and we use work of Scott [21] to handle non-Haken Seifert manifolds.

The paper is organized as follows: In Section 1 we recall from [17] the generic picture of a family of maps from a compact 1-polyhedron into a 3-manifold, parameterized by a disc. In Section 2 we give the preliminaries from the topology of 3-manifolds that we use in subsequent sections. In Section 3 we treat a special case of the integrability question mentioned above. The main result of this section is Theorem 3.7. In Section 4 we treat the general question of integrability of singular knots invariants (see Theorem 4.1). In Section 5 and Section 6 we describe the structure of the spaces $\mathscr{F}_c^m$ and we prove Theorem 0.1. Finally, we show that the classical Alexander polynomial for knots in a rational homology sphere is equivalent to a sequence of finite type invariants.

1. ALMOST GENERAL POSITION FOR A DISJOINT UNION CIRCLES AND FOR RIGID-VERTEX NULL HOMOTOPIES

In this section we summarize from [17] the results about the generic picture of a family of maps from a compact 1-polyhedron to a 3-manifold, parametrized by a 2-disc.

Let P be an 1-dimensional compact polyhedron. Let M be a 3-manifold and let D^2 be a 2-disc. A map $\Phi : P \times D^2 \to M$ gives rise to a family of maps $\{\phi_x : P \to M;\ x \in D^2\}$, where $\phi_x(*) = \Phi(*, x)$ for $x \in D^2$. Suppose that every ϕ_x is a piecewise-linear map, and let S_ϕ be the closure of the set $\{x \in D^2;\ \phi_x \text{ is not an embedding}\}$. One can see that S_ϕ is a sub-polyhedron of D^2.

Two maps $\phi_1, \phi_2 : P \to M$ are called ambient isotopic if there exists an isotopy $h_t : M \to M$, $t \in [0, 1]$ with $h_0 = id$ and $h_1\phi_1 = \phi_2$.

A double point (resp. triple point) of a map $\phi : P \to M$ is a point $p \in M$ such that $\phi^{-1}(p)$ consists of two (resp. three) points. A double (or triple) point of a piecewise linear map

$\phi : P \to M$ is called transverse if there exist two (or three) 1-simplexes σ_1, σ_2 (or, in addition σ_3) contained in the 1-skeleton of P such that

(1) ϕ is linear and non-degenerate on σ_1 and σ_2 (or, in addition σ_3);

(2) $\phi(\sigma_1) \cap \phi(\sigma_2)$ (or, $\phi(\sigma_1) \cap \phi(\sigma_2) \cap \phi(\sigma_3)$)) is the double (or, triple point) in question;

(3) $\phi(\sigma_1)$ and $\phi(\sigma_2)$ (or, in addition $\phi(\sigma_3)$) intersect transversally in their interiors and $\phi(\sigma_1)$, $\phi(\sigma_2)$, $\phi(\sigma_3)$ do not lie on the same plane.

Let us now introduce some terminology about 1-dimensional polyhedra in 3-manifolds.

Let Q be an 1-dimensional polyhedron. Every point $q \in Q$ has a neighborhood homeomorphic to a bouquet of finitely many arcs such that Q is the common endpoint of these arcs. The number of arcs in the bouquet is called the *valence* of q. A point $q \in Q$ with valence different than 2 is called a *vertex* of Q. A component of the complement of vertices is called an *edge* of Q.

Following [17] we call an 1-dimensional subpolyhedron $Q \subset D^2$ *neat*, if $Q \cap \partial D^2$ consists of finitely many points and each of them is a valence 1 vertex of Q. We call these vertices *boundary vertices* of Q and we call the vertices of Q lying in the interior of D^2 *interior vertices* of Q.

PROPOSITION 1.1 (Lin [17]). *Assume that P is a disjoint union of oriented circles and that M is an oriented 3-manifold. A map $\Phi : S^1 \times D^2 \to M$ can be changed by an arbitrary small perturbation so that S_ϕ is a neat 1-dimensional subpolyhedron of D^2. Moreover, we have*

(1) *If $x, x' \in D^2$ belong to the same component of $D^2 \backslash S_\phi$ or $S_\phi \backslash \{\text{interior vertices}\}$, then ϕ_x and $\phi_{x'}$ are ambient isotopic.*

(2) *The interior vertices of S_ϕ are of valence either four or one.*

(3) *If $x \in S_\phi$ lies on an edge of S_ϕ or is a boundary vertex, then ϕ_x has exactly one transverse double point.*

(4) *If $x \in S_\phi$ is an interior vertex of valence four, then ϕ_x has exactly two transverse double points.*

(5) *If $x \in S_\phi$ is an interior vertex of valence one, then ϕ_x is an embedding ambient isotopic to the nearby embeddings.*

We say that the resulting map in Proposition 1.1 is *in almost general position.* Figure 1 below illustrates $S_\Phi \subset D^2$ for a map Φ in almost general position.

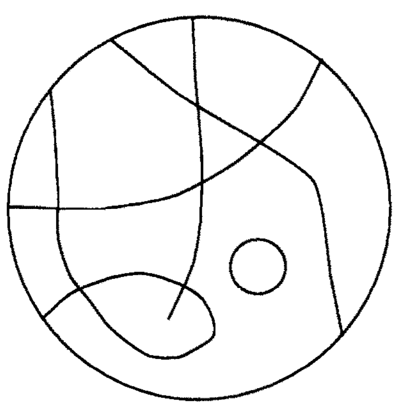

Fig. 1. $S_\Phi \subset D^2$.

Now we assume that P is an 1-dimensional compact polyhedron whose set of vertices of valence $\geqslant 3$, is not empty. Let us denote this set by V. Furthermore, assume that P has no vertices of valence 1, and let M be an oriented 3-manifold.

Definition 1.2. An embedding $\phi: P \to M$ is called a rigid-vertex embedding, if for every vertex $v \in V$, there is a proper 2-disc D in a ball neighborhood $B \subset M$ of $\phi(v)$ is specified such that, the image of a neighborhood of v is contained in D. An isotopy $h_t: M \to M$, $t \in [0,1]$, between two rigid-vertex embeddings ϕ_0 and ϕ_1, is called rigid-vertex isotopy if it carries through the ball–disc pair for every $v \in V$. A homotopy $\phi_t: M \to M$, $t \in [0,1]$, between two rigid embeddings ϕ_0 and ϕ_1, is called rigid-vertex homotopy if there is a neighborhood N of V in P, and an isotopy $h_t: M \to M$, $t \in [0,1]$, such that $\phi_t|N = h_t\phi_0|N$ for all $t \in [0,1]$. Moreover, h_t should carry through the ball-disc pair for every vertex $v \in V$.

LEMMA 1.3 (Lin [17]). *If two rigid-vertex embeddings ϕ_0 and ϕ_1, are homotopic then they are rigid-vertex homotopic.*

If $\{\phi_t\}_{t\in[0,1]}$ is a rigid vertex homotopy with $\phi_0 = \phi_1$ and the ball–disc pairs for vertices of P corresponding to ϕ_0 and ϕ_1 are the same, we have a closed rigid-vertex homotopy. In this case we can assume that there is a neighborhood B of $\phi_0(V) \subset M$ containing $\phi_0(N)$ such that $h_1|B = id$.

Let E be the frame bundle of M, which is a principal SO_3-bundle, and let $\Phi: P \times D^2 \to M$ be a map such that $\Phi|P \times \partial D^2$ is a closed rigid-vertex homotopy. We pull back E via Φ, to get a trivial SO_3-bundle E_1, over $V \times D^2$. The isotopy in the definition of a rigid-vertex homotopy gives rise to a section of E_1 over $V \times \partial D^2$. The obstruction of extending this section on $V \times D^2$ lies in $\bigoplus \pi_1(SO_3) = \bigoplus \mathbb{Z}_2$. If this obstruction is trivial then there is a section of E_1 over $V \times D^2$, extending the one over $V \times \partial D^2$.

Definition 1.4. A map $\Phi: P \times D^2 \to M$ such that $\Phi|P \times \partial D^2$ is a closed rigid-vertex homotopy with vanishing obstruction, is called a rigid-vertex null homotopy.

THEOREM 1.5 (Lin [17]). *Let $\Phi: P \times D^2 \to M$ be a rigid-vertex null homotopy. Then Φ can be changed by an arbitrary small perturbation, so that S_Φ is a neat 1-dimensional subpolyhedron of D^2. Moreover, we have*

(1) *If $x, x' \in D^2$ belong to the same component of $D^2 \backslash S_\Phi$ or $S_\Phi \backslash \{$*interior vertices*$\}$, then ϕ_x and $\phi_{x'}$ are ambient isotopic.*

(2) *If $x \in S_\Phi$ lies on an edge or is a boundary vertex, then ϕ_x has exactly one transverse double point.*

(3) *If $x \in S_\Phi$ is an interior vertex of valence $\geqslant 3$, then either ϕ_x has exactly two transverse double points (in which case the valence is 4), or there exists a point $v \in V$ such that $\phi_x(v) \in \phi_x(int(\sigma))$ for an 1-simplex $\sigma \subset P \backslash V$, and this is the only singularity of ϕ_x.*

(4) *If $x \in S_\Phi$ is an interior vertex of valence 1, then ϕ_x is an embedding isotopic to the nearby embeddings.*

We say that the resulting rigid-vertex null homotopy in Theorem 1.5 is *in almost general position.*

2. PRELIMINARIES FROM THE THEORY OF THE CHARACTERISTIC SUBMANIFOLD

In this section we state several, well known, results about the topology of 3-manifolds, which are used in subsequent sections.

Definition 2.1. A 2-sphere S^2 in a 3-manifold M is compressible in M, if S^2 bounds a 3-cell embedded in M. Otherwise S is called incompressible. A 3-manifold M is called irreducible iff every 2-sphere in M is compressible.

By the Sphere Theorem (see for example [9]), we have that $\pi_2(M)=\{1\}$, if M is irreducible.

Definition 2.2. A surface $F\neq S^2$, properly embedded in a 3-manifold M (or embedded in ∂M), is compressible if there exists a disc $D\subset M$ such that $D\cap F=\partial D$ and ∂D is not homotopically trivial in F. Otherwise F is called incompressible in M. A compact, orientable, irreducible 3-manifold is called a Haken manifold (or a sufficiently large manifold), if it contains a two-sided incompressible surface.

It is well known that every 3-manifold with infinite first homology group is Haken.

Next, we need to recall a few things about *Seifert fibered spaces*. More details may be found in any of [9,10] or [22]. Let (μ,ν) be a pair of relatively prime integers. Let

$$D^2=\{(r,\theta);\ 0\leqslant r\leqslant 1,\ 0\leqslant\theta\leqslant 2\pi\}\subset\mathbb{R}^2$$

be the unit 2-disc defined in polar coordinates. A fibered solid torus of type (μ,ν), is the quotient of the cylinder $D^2\times I$, via the identification $((r,\theta),1)=((r,\theta+2\pi\nu/\mu),0)$. The fibers are the images of the arcs $\{x\}\times I$. If $\mu>1$ the fibered solid torus is said to be *exceptionally fibered* and the core is the *exceptional fiber*. Otherwise the fibered solid torus is *regularly fibered* and each fiber is a *regular fiber*.

Definition 2.3. An orientable 3-manifold S is called a *Seifert fibered space*, if it is a union of pairwise disjoint simple closed curves, called fibers, such that each one has a closed neighborhood, consisting of a union of fibers, which is homeomorphic to a fibered solid torus via a fiber preserving isomorphism.

A fiber of S is called *exceptional* if it has a neighborhood homeomorphic to an *exceptionally fibered* solid torus.

The quotient space obtained from a Seifert fibered manifold S by identifying each fiber to a point is called *the orbit space* and the images of *exceptional fibers* are called *cone points*. A total classification of *Seifert manifolds* up to fiber preserving isomorphism can be found in the beautiful original paper of Seifert [22].

Proposition 2.4. *Assume that S is a Seifert fibered space with orbit space B and fiber projection p. Let $\tilde{B}$ be a d-fold covering of B. For a point $x\in B$, let $\tilde{x}\in\tilde{B}$ be a point over x, and s a point of S which corresponds to x. Consider the set $\tilde{S}=\{(s,\tilde{x});\ \tilde{x}\in\tilde{B}\}$, and define $f:\tilde{S}\to S$ by $f(s,\tilde{x})=s$. Then f is a covering map of degree d, such that if a point $s\in S$ runs along a fiber H, then $(s,\tilde{x})$, for fixed $\tilde{x}$, runs along a curve $\tilde{H}$ which lies one-to-one over H.*

Proof. It follows directly from the definitions. □

Definition 2.5. Let M be a 3-manifold and F a surface, which is not the 2-sphere. A map $\Phi:F\to M$ is called *essential* iff

(i) $\mathrm{Ker}(\{\Phi_*:\pi_1(F)\to\pi_1(M)\})=1$;

(ii) Φ cannot be homotoped to a map $\Phi_1:F\to M$ with $\Phi_1(F)\subset\partial M$.

Definition 2.6. Let S be a Seifert fibered space, with a fixed fibration and let $p: S \to B$ be the fiber projection. Let F be a surface. A map $f: F \to S$ is called *vertical* or *saturated*, with respect to p, if $p^{-1}(pf(F)) = f(F)$ and $f(F)$ contains no exceptional fibers.

A proof of the following classification up to homotopy of singular *essential* tori and annuli in a *Seifert fibered space* can be found in [11].

PROPOSITION 2.7. *Suppose that S is a Seifert fibered space which is a Haken manifold. Let $\Phi: T = S^1 \times S^1 \to S$ be an essential map. Then there exists a Seifert fibration of S, and a homotopy $\Phi_t: T \to S$, $t \in [0,1]$, such that $\Phi_0 = \Phi$ and Φ_1 is vertical with respect to this fibration.*

PROPOSITION 2.8. *Let S be a Seifert fibered space such that $\partial S \neq \emptyset$. Moreover, assume that S is not a solid torus. Let $\Phi: (A, \partial A) \to (S, \partial S)$ be an essential map, where each component of A is an annulus. Then, one of the following is true:*

(1) *There is a Seifert fibration of S, such that Φ can be homotoped, as a map of pairs, to a map Φ_1 which is vertical with respect to this fibration.*

(2) *There exists a fibration of S as an I-bundle over the annulus, torus, Mobius band, or Klein bottle such that A is vertical with respect to this fibration.*

Definition 2.9. Let M be a closed 3-manifold, with or without boundary. A co-dimension zero submanifold $X \subset M$ is called a *characteristic submanifold* if the following hold:

(1) Each component S of X admits a structure as a Seifert fibered space, with fiber projection $p: S \to B$, such that

$$S \cap \partial M = p^{-1}(p(S \cap \partial M))$$

(2) If W is a non-empty codimension zero submanifold of M with $W \subset \overline{(M \backslash X)}$ then $X \cup W$ does not satisfy (1).

(3) If X' is a codimension zero submanifold of M satisfying (1) and (2), then X' can be deformed onto X by a proper isotopy of M.

We state below a version of the "Jaco–Shalen–Johannson decomposition Theorems" which are suitable for our purposes here.

THEOREM 2.10 (Jaco and Shalen [10] and Johannson [11]). *Suppose that M is a Haken 3-manifold, which is either closed or it has incompressible boundary. Then either M contains no essential tori (or annuli) or it contains a non-empty characteristic submanifold.*

In fact as a consequence of the work of Gabai [8] and Casson–Jungreis [5] we have

THE TORUS THEOREM (Casson–Jungreis [5] and Gabai [8]). *Let M be a closed irreducible 3-manifold that admits an essential map $\Phi: T = S^1 \times S^1 \to M$. Then either M contains an embedded torus, and thus is Haken, or it is a Seifert fibered space.*

THE ENCLOSING THEOREM (Jaco and Shalen [10] and Johannson [11]). (a) *Let M be a closed Haken 3-manifold, and let $X \subset M$ be its characteristic submanifold. Let $\Phi: T = S^1 \times S^1 \to M$ be an essential map. Then there exists a homotopy $\Phi_t: T \to M$, $t \in [0,1]$, such that $\Phi_0 = \Phi$ and $\Phi_1(T) \subset X$.*

(b) *Let M be a Haken 3-manifold, with incompressible boundary ∂M. Let A be a 2-manifold each component of which is an annulus, and $f: (A, \partial A) \to (M, \partial M)$ such that $f|\partial A$ is 1–1 and every component of A is non-contractible in M. Let U be a regular* *neighborhood of ∂A in ∂M. Then there exists a Seifert fibered pair $(S, U) \subset (X, \partial X)$, such that f can be homotoped, relative ∂A, to a map $f' : A \to M$ with $f'(A) \subset S$.*

From Propositions 2.7, 2.8 and the *Enclosing Theorem* we get the following proposition, which is going to be very useful in the next sections.

PROPOSITION 2.11 (Johannson [11]). *Let M be a Haken 3-manifold. Suppose that M is closed or boundary incompressible and let $\Phi : T \to M$ be an essential map from the torus or the annulus. Then there exists a homotopy $\Phi_t : T \to M$ such that $\Phi_0 = \Phi$ and $\Phi_1 = \Phi_2 \circ q$, where $q : T \to T$ is a covering map, and $\Phi_2 : T \to M$ is an immersion without triple points.*

Finally, we will need the following theorem of Nielsen (see for example [9, Theorem 13.1]).

THEOREM 2.12. *Suppose F and G are compact, closed surfaces with $\pi_1(F) \neq 1$. Let $f : (F, \partial F) \to (G, \partial G)$ be a map such that $f_* : \pi_1(F) \to \pi_1(G)$ is one-to-one. Then there is a homotopy $f_t : (F, \partial F) \to (G, \partial G)$ with $f_0 = f$ and either*

(i) *$f_1 : F \to G$ is a covering map, or*
(ii) *F is an annulus or Mobius band and $f_1(F) \subset \partial G$.*

If for some component C of ∂F $f|C : C \to f(C)$ is a covering map, then the homotopy can be carried out relatively C.

3. SINGULAR KNOTS AND THEIR INVARIANTS

This section contains the statement and the proof of the first step (see Theorem 3.7 below) that *finite type* knot invariants exist in closed irreducible 3-manifolds. The proof will occupy most of the section.

3.1. Preliminaries

Let M be an oriented 3-manifold and let P be a disjoint union of oriented circles.

Definition 3.1. A singular link of order n is a piecewise-linear map $L : P \to M$ that has exactly n transverse double points. A singular link of order 0 is simply a link. Two singular links L and L' are equivalent if there is an isotopy $h_t : M \to M$, $t \in [0, 1]$ such that $h_0 = id$, $L' = h_1(L)$, and the double points of $h_t(L)$ are transverse for every $t \in [0, 1]$.

Let $p \in M$ be a transverse double point of a singular link L. Then $L^{-1}(p)$ consists of two points $p_1, p_2 \in P$. There are disjoint 1-simplexes, σ_1 and σ_2, on P with $p_i \in int(\sigma_i)$, $i = 1, 2$, such that for a small ball neighborhood B of p in M

$$L \cap B = L(\sigma_1) \cap L(\sigma_2).$$

Moreover, there is a proper 2-disc D in B such that $L(\sigma_1)$, $L(\sigma_2) \subset D$ intersect transversally at p, and the isotopy h_t of Definition 3.1 carries the ball disc pair (B, D) through for all the double points of L.

We can resolve a transverse double point of a singular link of order n in two different ways. Notice, that $L(\sigma_1) \cap L(\sigma_2)$ consists of four points on ∂D. Choose two arcs a_1 and a_2 as shown in Fig. 2.

The orientation of M and that of $L(\sigma_2)$ determine an orientation of $a_1 \cup a_2$. Suppose that it is consistent with the orientation of a_1 and opposite to that of a_2.

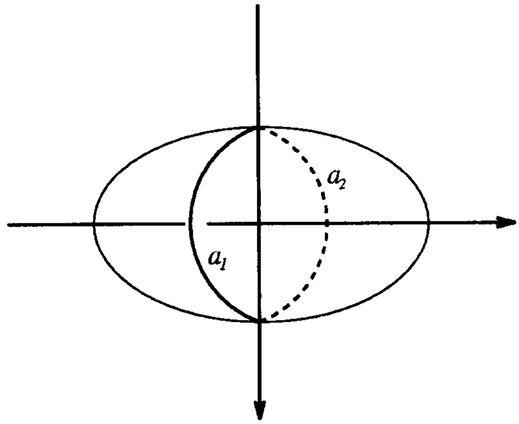

Fig. 2. Resolutions of a transverse double point.

Define

$$L_+ = \overline{L \backslash L(\sigma_2)} \cup a_1$$
$$L_- = \overline{L \backslash L(\sigma_2)} \cup a_2.$$

Clearly L_+, L_- are well defined singular links of order $n-1$. We call L_+ (resp. L_-) the positive (resp. the negative) resolution of L.

Let $\mathscr{K}^n$ (resp. $\mathscr{L}^n$) denote the set of equivalence classes of singular knots (resp. links) of order n in M, and let $\mathscr{R}$ be a ring. A singular link invariant is a map $\mathscr{L}^n \to \mathscr{R}$. In particular, for $n=0$ we have a link invariant. From a link invariant $\mathscr{L} \to \mathscr{R}$ we can always define a singular link invariant $\mathscr{L}^1 \to \mathscr{R}$ as follows:

Let $L_\times \in \mathscr{L}^1$ where $\times$ stands for the only double point. Then L_+, $L_- \in \mathscr{L}^0 = \mathscr{L}$. We can define a singular link invariant $f : \mathscr{L}^1 \to \mathscr{R}$ by

$$f(L_\times) = f(L_+) - f(L_-) \tag{2}$$

As a first step in reversing this procedure we ask the following question: Suppose that we are given a singular link invariant $\mathscr{L}^1 \to \mathscr{R}$. Under what conditions can we find a link invariant $\mathscr{L} \to \mathscr{R}$ so that (2) holds for all $L_\times \in \mathscr{L}^1$? In [1], Bar–Natan thinks of (2) as the definition of the "first partial derivative" of the link with respect to a certain crossing. In this spirit the question above concerns the "integrability" of a singular link invariant (see also discussion in [17]).

Our goal in this section is to answer the above-mentioned question for knots in closed irreducible 3-manifolds, and $\mathscr{R}$ a ring which is torsion free as an abelian group. Before we can state our main result we need some preparation.

We will say that a singular link $L_\times \in \mathscr{L}^1$ is *inadmissible* iff either

(a) the two resolutions L_+ and L_- of $L_\times$ represent isotopic links; or
(b) the double point $\times$ of $L_\times$ belongs on a single component $L_1 \subset L$ and moreover:
  (i) the two *lobes* of L_1 are homotopically essential in $\pi_1(M)$; and
  (ii) the two resolutions L_+ and L_- of $L_\times$, differ by a change of orientation of the component corresponding to L_1.

Definition 3.2. (a) We say that an invariant $f : \mathscr{L}^1 \to \mathscr{R}$ satisfies the *weak* local integrability conditions if and only if we have:

$$f(\infty) = 0 \tag{3}$$

$$f(L_{\times +}) - f(L_{\times -}) = f(L_{+\times}) - f(L_{-\times}). \tag{4}$$

(b) We say that an invariant $f:\mathscr{L}^1 \to \mathscr{R}$ satisfies the *strong* local integrability conditions if and only f satisfies (4) above and we have

$$f(L_{\mathrm{inadm}}) = 0. \tag{3'}$$

Notation. Before we proceed let us explain the notation above. In (3) the kink stands for a singular link $L_\times \in \mathscr{L}^1$ where there is a 2-disc $D \subset M$ such that $L_\times \cap D = \partial D$, and the unique double point of $L_\times$ lies on ∂D. In (4) we start with an arbitrary singular link $L_{\times\times} \in \mathscr{L}^2$. The four singular links in $\mathscr{L}^1$ are obtained by resolving one double point of $L_{\times\times}$ at a time. Finally, in (3′) L_{inadm} stands for any inadmissible singular link in $\mathscr{L}^1$. Clearly, (3′) is a stronger condition than (3).

To continue, let $L:P \to M$ be a link and let $\mathscr{M}^L(P,M)$ denote the space of maps $P \to M$ homotopic to L, equipped with the compact-open topology. Moreover, let $\Phi : P \times S^1 \to M$ be a closed homotopy from L to itself. Then, Φ may be thought of as a loop in $\mathscr{M}^L(P,M)$, based at L.

For a point $t \in S^1$, let ϕ_t denote the link $\Phi(P \times \{t\})$. After perturbation, we can assume that there are only finitely many points $t_1, t_2, \ldots, t_n \in S^1$, ordered cyclicly according to the orientation of S^1, so that $\phi_{t_i} \in \mathscr{L}^1$ and ϕ_s is equivalent to ϕ_t for all $t_i < t,\ s < t_{i+1}$. When t passes through t_i, ϕ_t changes from one resolution of ϕ_{t_i} to another.

Definition 3.3. Let L, Φ and $t_1, t_2, \ldots, t_n \in S^1$ be as above and let $f:\mathscr{L}^1 \to \mathscr{R}$ be an invariant of singular links with one double point. We define the integral of f along Φ to be the following alternating summation:

$$X_\Phi = \sum_{i=1}^{n} \varepsilon_i f(\phi_{t_i})$$

where $\varepsilon_i = \pm 1$ is determined as follows: If $\phi_{t_i+\delta}$, for $\delta > 0$ sufficiently small, is a positive resolution of ϕ_{t_i} then $\varepsilon_i = 1$. Otherwise $\varepsilon_i = -1$.

Next, we study various properties of the integral of a singular link invariant along a closed homotopy and we prove some preliminary lemmas which are essential for the proof of Theorem 3.7. The following lemma was proved in [17], but we include the proof here for completeness.

Lemma 3.4. *Let M, P and Φ be as above and let f be a singular link invariant that satisfies the weak local integrability conditions. Moreover, suppose that Φ can be extended to a map $\hat{\Phi}:P \times D^2 \to M$, where D^2 is a 2-disc with $\partial D^2 = \{*\} \times S^1$. Then, the integral of f along Φ vanishes, i.e. $X_\Phi = 0$.*

Proof. We perturb $\hat{\Phi}$ to an almost general position map as in Proposition 1.1. Then, each edge of the set of singularities $S_{\hat{\Phi}}$, corresponds to a singular link of order 1. So by using the invariant f we can assign an element of $\mathscr{R}$ to every edge of $S_{\hat{\Phi}}$. We will reduce the desired conclusion to local integrability conditions around each interior vertex in $S_{\hat{\Phi}}$. More precisely, for every interior vertex of $S_{\hat{\Phi}}$ draw a small circle C around it, so that the number of points in $C \cap S_{\hat{\Phi}}$ is equal to the valence of the vertex. For a picture see Fig. 3. It suffices to show that

$$\sum_{x \in C \cap S_{\hat{\Phi}}} \pm f(\hat{\phi}_x) = 0 \tag{5}$$

for every interior vertex of $S_{\hat{\Phi}}$. Here $\hat{\phi}_x(S^1) = \hat{\Phi}(P \times \{x\})$.

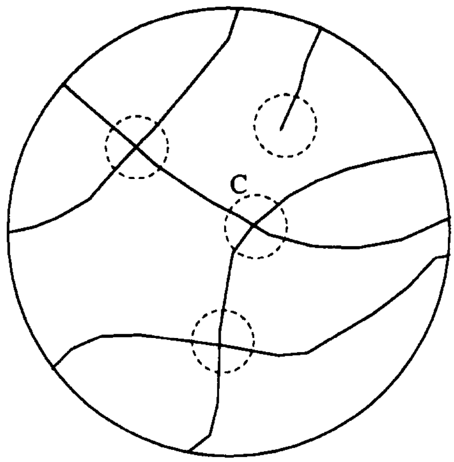

Fig. 3. From local to global integrability conditions.

Case 1: The valence of the interior vertex is one: In this case it is easy to see that for $x \in S_{\hat{\Phi}}$, near that vertex, the unique double point of $\hat{\phi}_x$ is at a kink. So (5) is implied by the local integrability condition (3).

Case 2: The valence of the interior vertex is four: In this case the four points in $C \cap S_{\hat{\Phi}}$ correspond to the four singular links appearing in the local integrability condition (4) and one can easily see that (5) is guaranteed by it. □

LEMMA 3.5. *Assume that M is an orientable 3-manifold with $\pi_2(M) = \{1\}$. Let $f : \mathscr{L}^1 \to \mathscr{R}$ be a singular link invariant satisfying the weak local integrability conditions, and let $\Phi : S^1 \to \mathscr{M}^L(P,M)$ be a loop. Then X_Φ only depends on the free homotopy class of Φ in $\mathscr{M}^L(P,M)$.*

Proof. Let Φ' be another closed homotopy in almost general position such that $\Phi, \Phi' : S^1 \to \mathscr{M}^L(P,M)$ are freely homotopic loops in $\mathscr{M}^L(P,M)$. Then there exists a homotopy $\Phi_t : P \to \mathscr{M}^L(P,M)$ with $t \in [0,1]$, such that $\Phi_0 = \Phi$ and $\Phi_1 = \Phi'$.

Let γ be the path in $\mathscr{M}^L(P,M)$ defined by $\gamma(t) = \Phi_t(L)$. After putting γ in almost general position we have

$$X_{\gamma\Phi'\gamma^{-1}} = X_\gamma + X_{\Phi'} - X_\gamma = X_{\Phi'}.$$

Hence we can assume that both X_Φ and $X_{\Phi'}$ are based at L, and the homotopy Φ_t is taken relatively L. The homotopy Φ_t gives rise to a map $\mathscr{H} : P \times S^1 \times I \to M$. We cut the annulus $S^1 \times I$ into a disc D^2 along a proper arc $\alpha \subset S^1 \times I$. Then, we have

$$X_{\partial D^2} = \pm(X_\Phi - X_{\Phi'} - X_\alpha + X_\alpha).$$

By Lemma 3.4 we obtain $X_{\partial D^2} = 0$, and hence $X_\Phi = X_{\Phi'}$. □

To continue, we first need to introduce some notation. Suppose that P has m components; that is

$$P = \coprod_{i=1}^{i=m} P_i$$

where each P_i is an oriented circle. Let $L : P \to M$ be a link. Pick a basepoint $p_i \in P_i$ and let a_i denote the homotopy class of $L(P_i)$ in $\pi_1(M, L(p_i))$. Finally, we denote by $Z(a_i)$ the centralizer of a_i in $\pi_1(M, L(p_i))$.

LEMMA 3.6. *Assume that M, P, and f are as in the statement of Lemma* 3.5. *Let $L:P\to M$ be a link such that the abelianization of $Z(a_i)$ is finite, for every $i=1,\dots,m$. Then $X_\Phi=0$, for every closed homotopy $\Phi:P\times S^1\to M$ from L to itself.*

Proof. We denote by $\mathscr{M}=\mathscr{M}^L(P,M)$ the space of maps $P\to M$, which are homotopic to L, equipped with the compact-open topology.

Let

$$\pi=\pi_1(\mathscr{M}^L(P,M),\ L).$$

Since $\pi_2(M)=\{1\}$ one can see that π is isomorphic to the direct product of the centralizers $\{Z(a_i)\}_{i=1,\dots,m}$.

By Proposition 3.3 of [17], the assignment $\Phi\to\chi(\Phi)$ is a group homomorphism $\chi:\pi\to\mathscr{R}$.

Since $\mathscr{R}$ is abelian, χ must factor through the abelianization of π which is finite by assumption. Now, since $\mathscr{R}$ is torsion free, we must have $\chi=0$ and thus

$$\chi(\Phi)=X_\Phi=0$$

which is the desired conclusion. □

3.2. The statement of the main result

Our goal in this paragraph is to give the statement of "integrability result" for invariants of singular knots with one double point, in irreducible 3-manifolds and prove it for knots in manifolds which are *atoroidal.*

Recall, that we have denoted by $\mathscr{K}^1$ (resp. $\mathscr{K}$) the set of equivalence classes of singular knots of order 1 (resp. knots) in M. Let $\mathscr{C}$ be the set of all conjugacy classes in $\pi_1(M)$. For every $c\in\mathscr{C}$, let $\mathscr{K}_c$ denote the set of equivalence classes of knots, corresponding to c. For every $c\in\mathscr{C}$, we fix a knot $K_c:S^1\to M$ that represents it. We denote by $\mathscr{TK}=\{K_c$ where $c\in\mathscr{C}\}=$"set of trivial knots". The knots in $\mathscr{TK}$ are going to play for our theory a role similar to the one that the standard unknot plays for the "finite type theory" in S^3. Changing the set $\mathscr{TK}$ amounts to changing our invariants by some constants.

If M is a Seifert fibered manifold, we will say that it is *small*, if it fibers over the 2-sphere and it has at most three exceptional fibers. Let N be the subgroup of $\pi_1(M)$ generated by a regular fiber of M. If M is *small*, then either $\pi_1(M)$ is finite or $\pi_1(S)/N$ is a Euclidean or hyperbolic triangle group.

Now we are ready to state the main theorem of this section.

THEOREM 3.7. *Suppose that M is a closed, oriented, irreducible* 3*-manifold, that it is not a small Euclidean Seifert manifold, and that $\mathscr{R}$ is a ring which is torsion free as an abelian group. Let $f:\mathscr{K}^1\to\mathscr{R}$ be a singular knot invariant.*

(a) *If f satisfies the strong integrability conditions* (3)′ *and* (4), *for every $c\neq 1$, there exists a knot invariant $F:\mathscr{K}_c\to\mathscr{R}$ so that* (2) *holds for all $K_\times\in\mathscr{K}^1$.*

(b) *Moreover if M does not contain any Seifert fibered spaces over non-orientable surfaces, then the following is true: For every $c\neq 1$, there exists a knot invariant $F:\mathscr{K}_c\to\mathscr{R}$ so that* (2) *holds for all $K_\times\in\mathscr{K}^1$ if and only if f satisfies the weak local integrability conditions* (3) *and* (4).

As we will shortly see, in the course of proving Theorem 3.7, we will need to study maps from the torus to M. Let us here introduce some relevant notation. Let $\Phi:T=S^1\times S^1\to M$, which is in general position. Let $l,m:S^1\to T$ be embeddings representing the standard longitute and meridian of T. For an embedding $\lambda:S^1\to T$, let $[\lambda]$ be its homology class in $H_1(T)$. Then $[\lambda]=x[l]+y[m]$ with $x,y\in\mathbb{Z}$. By abusing the notation, we will sometimes write $\lambda = xl + ym$ to denote the simple closed curve $\lambda(S^1) \subset T$. To continue with our notation, let a, $b \subset T$ be oriented simple closed curves such that $[a]$ and $[b]$ give a basis for $H_1(T)$. We choose a basepoint $* \in T$. The map $\Phi : T = S^1 \times S^1 \to M$, gives rise to a family of maps $\{\phi_x : S^1 \to M, x \in b(S^1)\}$, where $\phi_x(S^1) = \Phi(a(S^1) \times \{x\})$. Thus Φ gives rise to a loop $\Phi^{a,b} : b(S^1) \to \mathscr{M}^{K_*}(a(S^1), M)$, where K_* is the knot $\Phi(a(S^1) \times \{*\})$. We will denote by $X_\Phi^{a,b}$ the integral of f along $\Phi^{a,b}$. Notice that $X_\Phi^{l,m}$ is nothing else but the integral X_Φ, of f along Φ.

Proof of Theorem 3.7. One direction of the theorem is clear. That is if a singular knot invariant $f : \mathscr{K}^1 \to \mathscr{R}$ is derived from a knot invariant $F : \mathscr{K} \to \mathscr{R}$, then it satisfies (3) and (4). To see that (3) is satisfied observe that the positive and the negative resolution of the double point in the kink are equivalent. For (4) observe that, using (2), both sides of (2) can be expressed as $F(K_{++}) - F(K_{-+}) - F(K_{+-}) + F(K_{--})$.

We now turn into the proof of the other direction which we will break into several steps. First of all we define F on the elements in $\mathscr{TK}$ by arbitrarily assigning the values $F(K_c)$, for all $K_c \in \mathscr{TK}$.

Let $K \in \mathscr{K}$ be a knot type in M. We use K to also denote a representative $K : S^1 \to M$, of K. Then $K \in \mathscr{K}_c$ for some $c \in \mathscr{C}$ and hence K is homotopic to some $K_c \in \mathscr{TK}$. We choose a homotopy $\phi_t : S^1 \times [0,1] \to M$ such that $\phi_0 = K$ and $\phi_1 = K_c$. After perturbation we can assume that for only finitely many points $0 < t_1 < t_2 < \cdots < t_n < 1$, ϕ_t is not an embedding. Moreover we can assume that ϕ_{t_i}, for $i = 1, 2, \ldots, n$ are singular knots of order 1. For different $t's$ in an interval of $[0, 1] \setminus \{t_1, t_2, \ldots, t_n\}$ the corresponding knots are equivalent. When t passes through t_i, ϕ_t changes from one resolution of ϕ_{t_i} to another.

We define

$$F(K) = F(K_c) + \sum_{i=1}^{n} \varepsilon_i f(\phi_{t_i}).$$

Here $\varepsilon_i = \pm 1$ is determined as follows: If $\phi_{t_i+\delta}$, for $\delta > 0$ sufficiently small, is a positive resolution of ϕ_{t_i} then $\varepsilon_i = 1$. Otherwise $\varepsilon_i = -1$.

To prove that F is well defined we have to show that modulo "the integration constant" $F(K_c)$, the definition of $F(K)$ above is independent of the choice of the homotopy. For this we consider a closed homotopy $\Phi : T = S^1 \times S^1 \to M$. Here the knot direction is l, and the parameter space is m. To prove that F is well defined we need to show that the integral of f along Φ vanishes, i.e.

$$X_\Phi = 0. \tag{6}$$

We will call (6) the "global integrability condition" for Φ.

LEMMA 3.8. *Assume that M and f are as in the statement of Theorem* 3.7 *and $\Phi : T = S^1 \times S^1 \to M$ is a closed homotopy from a knot K to itself. Moreover, assume that K is not homotopically trivial and that $Ker\{\pi_1(T) \to \pi_1(M)\} \neq \{1\}$. Then, $X_\Phi = 0$.*

Proof. We may assume that $\pi_1(M)$ is infinite since otherwise the conclusion is true by Lemma 3.6. Then since M is orientable and irreducible, $\pi_1(M)$ is torsion free (see for example [9, Theorem 9.8 or Corollary 9.9]).

By assumption, there exists a homotopically non-trivial closed curve $\lambda \subset T$, such that $\Phi_*(\lambda) = 1$, where Φ_* is the map induced by Φ, on the fundamental groups. Furthermore, by the discussion above we can assume that λ is a simple closed curve. Hence $\lambda = al + bm$, where $a, b \in \mathbb{Z}$ with $gcd(a, b) = 1$.

By assumption we have that $b \neq 0$. Then consider the $|b|$-fold covering $p : \tilde{T} \to T$ of T, corresponding to the subgroup $\mathbb{Z} \oplus b\mathbb{Z}$ of $\pi_1(T)$. Let $\tilde{l}$ and $\tilde{m}$ be simple closed curves in $\tilde{T}$ lifting l and m respectively, and let $\tilde{\Phi} = \Phi \circ p$. We orient $\tilde{l}$ and $\tilde{m}$ so that p is orientation preserving. Then $p_*(\tilde{m}) = bm$, $p_*(\tilde{l}) = l$, and $p_* \mid \tilde{l} : \tilde{l} \to l$ is a homeomorphism. Let $\tilde{\lambda} \subset \tilde{T}$ be a simple closed curve lifting λ. Then we have $\tilde{\lambda} = a\tilde{l} + \tilde{m}$. We observe that $\tilde{\Phi}_*(\tilde{\lambda}) = 1$ in M.

Notice that $|\langle \tilde{\lambda}, \tilde{l} \rangle| = \langle \tilde{l}, \tilde{m} \rangle = 1$, where $\langle *, * \rangle$ denotes the algebraic intersection number. Therefore we can take the pair $([\tilde{l}], [\tilde{\lambda}])$ as basis for $H_1(\tilde{T})$. We orient $\tilde{l}$ and $\tilde{\lambda}$ suitably, so that the orientation of $\tilde{T}$ induced by them is the same as that induced by $\tilde{l}$ and $\tilde{m}$.

Since $\tilde{\Phi}_*(\tilde{\lambda}) = 1$ we can extend $\tilde{\Phi}$ on a disc D, with $[\partial D] = [\tilde{\lambda}]$. Since M is irreducible we have $\pi_2(M) = \{1\}$, and hence we can extend $\tilde{\Phi}$ on a solid torus $V \cong S^1 \times D$, with the image of S^1 carried by $\tilde{l}$. We continue to denote the extended map by $\tilde{\Phi}$.

By Lemma 3.4 we conclude that $X_{\tilde{\Phi}}^{\tilde{l},\tilde{\lambda}} = 0$. Observe that $X_{\tilde{\Phi}}^{\tilde{l},\tilde{\lambda}} = X_{\tilde{\Phi}}^{\tilde{l},\tilde{m}}$, since a point $x \in \tilde{m}$ such that $\tilde{\phi}_x$ is not an embedding, corresponds to exactly one such point on $\tilde{\lambda}$, and vice versa. Hence, we get that $X_{\tilde{\Phi}}^{\tilde{l},\tilde{m}} = 0$. Finally, since $p_*(\tilde{m}) = bm$ and $p_* \tilde{l} = l$, we have $X_{\tilde{\Phi}}^{\tilde{l},\tilde{m}} = bX_{\Phi}^{l,m}$ and since $\mathscr{R}$ is torsion free, we get $X_\Phi = 0$. This finishes the proof of the lemma. □

Definition 3.9. A compact, oriented, irreducible manifold M, is called atoroidal if every map $f : S^1 \times S^1 \to M$, from the torus to M is inessential.

PROPOSITION 3.10. *Assume that M is a closed, oriented, irreducible 3-manifold which is atoroidal. Then Theorem 3.7 is true for M.*

Proof. Recall that $f : \mathscr{K}^1 \to \mathscr{R}$ is an invariant which satisfies the local integrability conditions (3) and (4). It is enough to show that for every closed homotopy $\Phi : S^1 \times S^1 \to M$ of a knot to itself the global integrability condition (6) is satisfied. If M is atoroidal then the conclusion follows immediately from Lemma 3.8. □

3.3. Closed homotopies of knots and essential tori

Our next goal is to prove "the global integrable condition" (6), in the case that the closed homotopy $\Phi : S^1 \times S^1 \to M$ is an *essential* map. Since *the characteristic submanifold* of M contains up to homotopy all the essential tori, and since X_Φ depends only on the homotopy class of Φ, it is enough to prove (6) for Seifert fibered spaces. We do so in Lemma 3.13 below, but first we need to prove two auxiliary lemmas for essential tori in Seifert spaces.

Let B be a surface and let $\alpha : S^1 \to B$ be a loop. Choose a subdivision $t_0 < t_1 < \cdots t_{n-1} < t_n = t_0$ of S^1 such that each $\alpha([t_{i-1}, t_i])$ lies in a disc $D_i \subset B$. By translating, in D_1, a neighborhood of $\alpha(t_0)$ to neighborhood of $\alpha(t_1)$ in $D_1 \cap D_2$ and so on, we obtain a homeomorphism h from a neighborhood of $\alpha(t_0)$ to itself. We say that α is *orientation preserving* if such an h is an orientation preserving homeomorphism.

LEMMA 3.11. *Suppose that S is an irreducible*† *Seifert fibered space, with or without boundary, with orbit space B and fiber projection $p : S \to B$. Let $\Phi : T = S^1 \times S^1 \to S$ be vertical with respect to the given fibration. Moreover, suppose that the loop $p(\Phi(T)) \subset B$ is orientation preserving. Then, there exists a trivial fiber bundle $F \cong S^1 \times B'$*

†The only non-irreducible Seifert manifolds are $S^2 \times S^1$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$.

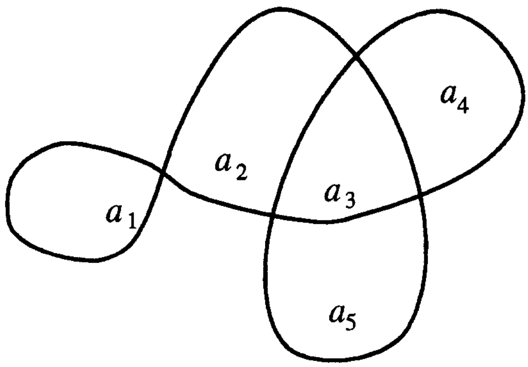

Fig. 4. $\alpha = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n$.

over a surface B', whose boundary ∂F is a collection of disjoint tori $T \cup T_1 \cup \cdots \cup T_k$, and there exists a map $\hat{\Phi}: F \to S$ such that

(i) $\hat{\Phi} \mid T = \Phi$,

(ii) *for every $i = 1, \ldots, k$, $\hat{\Phi} \mid T_i : \to \hat{\Phi}(T_i)$ is the composition of a covering map $T_i : \to \tilde{T}_i$ and an embedding $\tilde{T}_i : \to M$. Here, each $\tilde{T}_i$ is a surface covered by the torus.*

Proof. By Proposition 2.11 we can assume that $\Phi = \Phi_1 \circ q$ where $q: T \to T^1$ is a covering map, and $\Phi_1 : T^1 \to S$ is an immersion without triple points.

Case 1: The degree of the covering q is equal to 1; that is, the map $\Phi: T \to M$ is an immersion without triple points.

Then, the singular set of Φ consists of disjoint parallel, essential simple curves on T. Moreover, the images of these curves, under Φ are regular fibers of the fibration of S. Let us choose one of these simple curves on T, and denote it by H. We also choose a simple closed curve $Q \subset T$ with $\langle H, Q \rangle = 1$, such that $\Phi(Q)$ gives a cross section for the image $\Phi(T)$. By replacing Q with $Q \pm xH$ ($x \in \mathbb{Z}$), we may assume that Q does not wrap around H. Notice that ($[H]$, $[Q]$) give a basis for $H_1(T)$.

By our assumption $\alpha = p(\Phi_1(T)) \subset B$ is an orientation preserving closed curve, whose only singularities are finitely many transverse double points. Hence a regular neighborhood of α in B is a singular annulus.

We distinguish two subcases:

Subcase 1: The orbit space B is an orientable surface. Notice that we have $\partial B \neq \emptyset$ if and only if $\partial S \neq \emptyset$.

By the previous discussion α is a union of simple closed curves, $\alpha = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n$ as shown in Fig. 4. Consider $\{N_i = \alpha_i \times I\}_{i=1,\ldots,n}$ ($I = [-1, 1]$), a collection of regular neighborhoods of the α_i's which are disjoint from the cone points of B. Let $\beta_i = \alpha_i \times \{1\}$, $i = 1, \ldots, n$. Clearly β_i is a simple closed curve on B, and $T_i = p^{-1}(\beta_i)$ is a properly embedded torus in S. Let $B_i = \alpha_i \times [0, 1]$. Then each $F_i = p^{-1}(B_i)$ is a fiber bundle with $\partial F_i = p^{-1}(\alpha_i) \cup p^{-1}(\beta_i)$.

For every $i = 1, \ldots, n$, let k_i be the number of double points of α lying on the boundary of B_i. We choose a collection of simple arcs $\{c_i^j\}_{j=1,\ldots,k_i}$ on B_i such that,

(i) $c_i^j(0)$ is a double point of α,

(ii) $c_i^j((0, 1]) \subset int(B_i)$.

Observe that $A_i^j = p^{-1}(c_i^j) \subset F_i$ is a properly embedded annulus for every $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$.

Let $\{p_k\}_{k=1,\dots,s}$ be the double points of α, $\{p_k^1, p_k^2\}_{k=1,\dots,s}$ be their preimages on $\Phi^{-1}(p^{-1}(\alpha)) = Q$, and let $H_k^j = H \times \{p_k^j\} \subset T$, $k = 1, \dots, s$ and $j = 1, 2$.

Recall our map $\Phi : T \to S$. First we extend Φ along annuli $\{A_k^{12}\}_{k=1}^s$ with $\partial A_k^{12} = H_k^1 \cup H_k^2$, by $\hat{\Phi}(A_k^{12}) = \Phi(H_k^1) = \Phi(H_k^2)$, for every $k = 1, \dots, s$. Let $\bar{F}_i$ be the space obtained from F_i by cutting along the collection of annuli A_i^j, and let A_i^{j1} and A_i^{j2} be the two copies of A_i^j in $\bar{F}_i$. Clearly, Φ extends to a map $\hat{\Phi}$ on $\bar{F} = \bigcup_{i=1}^n \bar{F}_i$. Now, each component of the boundary of $\bar{F}$, $\partial\bar{F}$, is a torus. Let us write $\partial\bar{F} = T \cup T_1 \cup \cdots \cup T_n \cup \mathscr{T}$, where each component in $\mathscr{T}$ is a torus obtained by the union of three annuli, of the form $T_{ijk} = A_i^{j1} \cup A_i^{j2} \cup A_k^{12}$, for some $i = 1, \dots, n$, $j = 1, \dots, k_i$ and $k = 1, \dots, s$.

Since each torus T_{ijk} in $\mathscr{T}$, maps under $\hat{\Phi}$ to an annulus (by our construction we have $\hat{\Phi}(T_{ijk}) = A_i^j$), we see that $\hat{\Phi} \mid T_{ijk}$ is inessential. Thus, and since S is irreducible, we extend $\hat{\Phi}$ on a solid torus V_{ijk} with $\partial V_{ijk} = T_{ijk}$. The fibration of $\bar{F}$ will extend over each V_{ijk} unless one of these solid tori is attached so that the meridian disc is attached to a fiber $\partial\bar{F}$. But this cannot happen since every fiber of $\partial\bar{F}$ is essential in S. Finally, the resulting space, is the desired fiber bundle F. Clearly, we have $\partial F = T \cup T_1 \cup \cdots \cup T_n$, and $\hat{\Phi}(T_i)$ is an embedded torus for every $i = 1, \dots, n$.

Let B' be its orbit space. Notice, that we have constructed both F and B' to be orientable, and since $\partial F \neq \emptyset$, we have that $F \cong S^1 \times B'$. This finishes the proof of Subcase 1.

Subcase 2: B is a non-orientable surface. Let $s : \hat{B} \to B$ be the two-fold orientable covering of B, and let $\hat{S}$ be the Seifert fibered space, with orbifold $\hat{B}$, corresponding to it (see Proposition 2.4). Then $\hat{S}$ is a two-fold covering of S. Let $\tilde{s} : \hat{S} \to S$ be the covering projection. Since $\alpha = p(\Phi(T)) \subset B$ is orientation preserving, we see that $s^{-1}(\alpha)$ has two components α_1 and α_2, such that $s : \alpha_i \to \alpha$ is a homeomorphism. Hence $\tilde{s}^{-1}(\Phi(T))$ has two components T_1 and T_2 such that $\tilde{s} : T_i \to \Phi(T)$ is a homeomorphism. By applying our result of Subcase 1 to the map $\tilde{s}^{-1} \circ \Phi$ we get the desired conclusion.

Case 2: We have that $\Phi = \Phi_1 \circ q$ where $q : T \to T^1$ is a covering map of degree greater than 1. Let H_1 and Q_1 be simple closed curves on T^1 mapping onto a regular fiber of S and a cross section of $\Phi_1(T^1)$, respectively. Let $F_1 \cong S^1 \times B_1$ be a trivial bundle corresponding to the map Φ_1, as constructed in the proof of Case 1.

Suppose H, $Q \subset T$ are curves lifting H_1 and Q_1 respectively, and that $q(H) = aH_1$, $q(Q) = bQ$ for some a, $b \in \mathbb{Z}$. Consider the coverings $q_1 : T \to T^2$ and $q_2 : T^2 \to T^1$ such that,

(i) $q = q_2 \circ q_1$
(ii) If H_2, $Q_2 \subset T^2$ are curves lifting H_1 and Q_1 respectively then $(q_1)_*(H) = H_2$, $(q_1)_*(Q) = bQ_2$ and $(q_1)_*(H_2) = aH_1$, $(q_1)_*(Q_2) = Q_1$.

It is not hard to see that q_2 can be extended to a covering $q_2 : F_2 \to F_1 = q_2(F_2)$, where $F_2 \cong \langle H_2 \rangle \times B_1$, and q_1 to a branched covering $q_1 : F \to F_2 = q_1(F)$, where F a fiber bundle, over a surface B', satisfying all the desired properties. $\square$

LEMMA 3.12. *Assume that S, B, p and $\Phi : T = S^1 \times S^1 \to S$ are as in the statement of Lemma* 3.11. *Assume moreover that $H = S^1 \times \{*\}$ maps onto a cross section of $\Phi(T)$, and $Q = \{*\} \times S^1$ maps onto a regular fiber of S. Moreover, suppose that a neighborhood $N \subset B$ of $p(\Phi(H))$ contains no orientation reversing loops. Then for every $x_1, x_2 \in \{*\} \times S^1$ there exists a homeomorphism $h^{12} : S \to S$ such that,*

(1) *$h^{12} = id$ outside a regular neighborhood N of $\Phi(T)$ in S;*
(2) *$h^{12}(\phi_{x_1}) = \phi_{x_2}$, where $\phi_{x_i} = \Phi(S^1 \times \{x_i\})$, $(i = 1, 2)$;*
(3) *h^{12} is isotopic to the identity map, $id : S \to S$.*

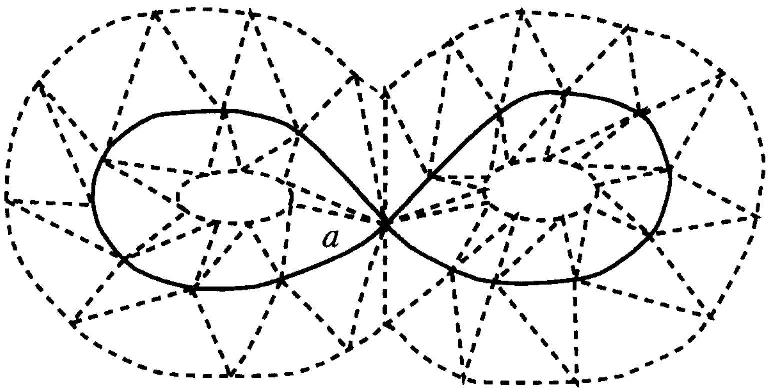

Fig. 5. $N^* \subset B$.

Proof. By our assumption the closed curve $\alpha = p(\Phi(Q)) \subset B$ is orientation preserving and its only singularities are finitely many transverse double points. Recall also that by the definition of a vertical map $\Phi(T) \subset B$ is a union of regular fibers of the fibration of S.

Fix $x_1,\ x_2 \in \{*\} \times S^1$. We will construct $h^{12}: S \to S$ as claimed above.

Consider $\mathscr{D} = \{\Delta_1,\ \Delta_2, \ldots, \Delta_k\}$ a piece-wise linear triangulation of B satisfying the following properties:

1. The double points of the curve α are vertices of $\mathscr{D}$.
2. α is contained in the 1-skeleton of $\mathscr{D}$.
3. The 1-skeleton of $\mathscr{D}$ contains no cone points of B.
4. The interior of every Δ_i, contains at most one cone point of B.
5. A regular neighborhood $N^* \subset B$ of α (triangulated as shown if Fig. 5), contains no cone points of B.

The triangulation $\mathscr{D}$ of B gives rise to a decomposition $\mathscr{V} = \{V_i = p^{-1}(\Delta_i)\}_{i=1,\ldots,k}$, of S into fibered solid tori. For every V_i, $\partial V_i = T_i$ is the union of three fibered annuli each lying above a side of $\Delta_i = p(V_i)$. Let $N = p^{-1}(N^*)$.

The decomposition $\mathscr{V}$ has the following properties:

1. If $V_i \subset N$ then either,

(a) $\partial V_i \cap \phi_{x_j} = V_i \cap \phi_{x_j} = \delta_j$ $(j = 1,2)$, where δ_j is an arc with $\partial\delta_j$ lying on distinct components of one of the annuli consisting $\partial V_i = T_i$, and $p(\delta_1) = p(\delta_2) \subset \alpha$ is an 1-simplex of $\mathscr{D}$ or,

(b) $\partial V_i \cap \phi_{x_j} = V_i \cap \phi_{x_j} = p_j$, $(j = 1,2)$, and $p_j \subset \phi_{x_j}$ is a point with $p(p_1) = p(p_2) \subset \alpha$ a vertex of $\mathscr{D}$.

2. If $V_i \subset \overline{S \backslash N}$ then $\partial V_i \cap \phi_{x_j} = \emptyset$, for $j = 1,2$. Assume that $V_i \subset N$ and let $\partial V_i = T_i = A_i^1 \cup A_i^2 \cup A_i^3$, where $\{A_i^k\}_{k=1,2,3}$, are the three fibered annuli corresponding to the three sides of $\Delta_i = p(V_i)$.

Choose oriented arcs $\gamma_i^{kj} \subset \{A_i^k\}$, where $j = 1,2$ and $i = 1,2,3$, such that

(i) $\partial\gamma_i^{kj}$ lies on distinct components of $\{A_i^k\}$ and $int\,\gamma_i^{kj} \subset int\,\{A_i^k\}$.

(ii) The initial point of $\gamma_i^{(k+1)j}$ is the terminal point of γ_i^{kj} (here k is considered $mod(3)$).

(iii) If V_i is as in the case (1a) above and $\partial V_i \cap \phi_{x_j} = A_i^k \cap \phi_{x_j} = \delta_j$, then $\gamma_i^{kj} = \delta_j$ and terminal point of $\gamma_i^{(k+1)j}$ = initial point of $\gamma_i^{(k+2)j}$.

(iv) If V_i is as in the case (1b) above and $\partial V_i \cap \phi_{x_j} = A_i^k \cap \phi_{x_j} = A_i^{(k+1)} \cap \phi_{x_j} = \{p_j\}$ then we have terminal point of γ_i^{kj} = initial point of $\gamma_i^{(k+1)j}$ and $\gamma_i^{(k+2)1} = \gamma_i^{(k+2)2}$.

(v) Let $c_{ij} = \gamma_i^{1j} \cap \gamma_i^{2j} \cap \gamma_i^{3j}$ $(j = 1,2)$. We have that c_{i1} and c_{i2} are homotopic, simple closed curves on T_i, each intersecting the fibers exactly once.

We are ready, now, to construct map $h^{12} : S \to S$ as claimed in the statement of the lemma. First note that since c_{i1} is homotopic to c_{i2}, and since every $V_i \subset N$ is an ordinary fibered solid torus, there exists $h^{12} | \bigcup_{V_i \subset N} T_i$ such that $h^{12} | \bigcup T_i : T_i \to T_i$ takes c_{i1} onto c_{i2}, translates T_i along the fibers and is isotopic to $id : T_i \to T_i$. We can extend h^{12} to $\bigcup V_i$ and then to the whole space S. The rest of the claims follow easily. □

Remark 3.13. As it was pointed out to me by C. Livingston and P. Kirk, Lemma 3.12 is not true in the case that a neighborhood $N \subset B$ of $p(\Phi(H))$ contains orientation reversing loops. Thus, the weak integrability conditions do not guarantee that a singular knot invariant can be integrated to a knot invariant, in every irreducible 3-manifold. See [15], for examples of invariants that satisfy the weak integrability conditions but cannot be integrated to knot invariants. The reason for that is the following: Let us, for simplicity, assume that $p(\Phi(H))$ contains two orientation reversing sub-loops, say α_1 and α_2, meeting at a double point $k \in p(\Phi(H))$. Let $t = p^{-1}(k)$ denote the fiber of S over k. Then $\Phi^{-1}(t)$ constists of two parallel copies, say Q_1 and Q_2, of the parameter space Q that are identified by opposite orientation, under Φ. One can easily see that each of $\Phi(S^1 \times \{0\})$ and $\Phi(S^1 \times \{\frac{1}{2}\})$ is a singular knot with a double point. Moreover, each of these double points is of the same sign. Then, and by our definition of a resolution of a double point, we can see that both $\Phi(S^1 \times \{0\})$ (resp. $\Phi(S^1 \times \{\frac{1}{2}\})$) are *inadmissible* singular knots.

In general, we can have many pairs of the parameter space Q, that are identified under Φ, with opposite orientations. Each such pair gives rise to two singular knots along Φ. The two resolutions of each of the singular knots appearring along Φ differ by a change of orientation. Then, one can see that all the singular knots along such an Φ are *inadmissible*.

Lemma 3.14. *Assume that S is Seifert fibered space, in which every essential (singular) torus can be homotoped to be vertical with respect to some fibration. Let f and $\mathcal{R}$ be as in the statement of Theorem* 3.7 *and let $\Phi : T = S^1 \times S^1 \to S$ be a closed homotopy from a knot to itself. Assume moreover, that Φ is an essential map and let X_Φ be the integral of f along Φ. Then we have $X_\Phi = 0$.*

Proof. By assumption there exists a homotopy $\Phi_t : T = S^1 \times S^1 \to S$, $t \in [0,1]$, with $\Phi_0 = \Phi$ and a Seifert fibration of S, such that Φ_1 is vertical with respect to this fibration.

By Lemma 3.5 we have $X_{\Phi_1} = X_\Phi$. Therefore it is enough to prove that $X_{\Phi_1} = 0$.

We choose a pair of simple closed curves (H, Q) on T, such that $\Phi_1(H)$ covers a regular fiber of S, and $\Phi_1(Q)$ covers a cross section of the image $\Phi_1(T)$. We orient (H, Q) so that the induced orientation on T is the same with that induced by (l, m). Let B be the orbit space of S and let $\alpha = p(\Phi_1(T)) \subset B$.

Since the orientation of $\Phi_1(l)$ (= knot) does not change along $\Phi(m)$ we have that the orientation of $\Phi_1(H)$ does not change along $\Phi_1(Q)$. Hence the orientation of the fiber does not change along $\alpha = p(\Phi_1(T)) \subset B$. But since S is orientable α has to be an orientation preserving curve on B. Hence Lemma 3.11 applies and we can find a trivial fiber bundle F with fiber H and boundary $\partial F = T \cup T_1 \cup \cdots \cup T_n$, and a map $\hat{\Phi}_1 : F \to S$ such that $\hat{\Phi}_1 | T = \Phi_1$ and $\hat{\Phi}_1 | T_i : T_i \to \hat{\Phi}_1(T_i)$ is a covering map for $i = 1, \ldots, n$.

Let B' be the orbit space of F. Then we have $\partial B' \cap T = B' \cap T = Q$. Let $Q_i = \partial B' \cap T_i = B' \cap T_i \subset T_i$, for $i = 1, \dots, n$.

Suppose that $l = aH + bQ$ and $m = cH + dQ$, where $a, b, c, d \in \mathbb{Z}$ such that $gcd(a,b) = gcd(c,d) = ad - bc = 1$.

Assume that $a \neq 0$. Then consider the covering $\tilde{p}: \tilde{F} \to F$ of F, corresponding to the subgroup $a\mathbb{Z} \times \pi_1(B')$ of $\mathbb{Z} \times \pi_1(B') = \pi_1(F)$. Let $\tilde{T} = \tilde{p}^{-1}(T)$ and let $\tilde{H}$, $\tilde{Q}$, $\tilde{l}$ and $\tilde{m}$ be simple closed curves in $\tilde{T}$ lifting H, Q, l and m respectively. Also let $\tilde{T}_i = \tilde{p}^{-1}(T_i)$ and let $\tilde{H}_i$, $\tilde{Q}_i$, $\tilde{l}_i$ and $\tilde{m}_i$ be simple closed curves in $\tilde{T}_i$ lifting H_i, Q_i, l_i and m_i, for $i = 1, \dots, n$. We orient $\tilde{F}$ suitably so that $\tilde{p}$ is orientation preserving. Then we have $\tilde{l} = \tilde{H} + b\tilde{Q}$ and $\tilde{m} = -c\tilde{H} - ad\tilde{Q}$. Let $\tilde{\Phi}_1 = \Phi_1 \circ \tilde{p}$.

Claim. *We have*

$$X_{\tilde{\Phi}_1}^{\tilde{l},\tilde{Q}} = X_{\tilde{\Phi}_1}^{\tilde{l},\tilde{m}} = aX_{\Phi_1}$$

Proof of claim. Observe that $\tilde{p}_*(\tilde{m}) = am$ and that $\tilde{Q} = c\tilde{l} + \tilde{m}$, for some $c \in \mathbb{Z}$. Then, the first equality follows by observing that every point $x \in \tilde{m}$ such that $\tilde{\Phi}_1(\tilde{l}(S^1) \times \{x\})$ is a singular knot of order 1, gives rise to such a point on $\tilde{Q}$ and *vice versa*. To see the second equality, observe that every point $x \in m$ for which $\Phi_1(l(S^1) \times \{x\})$ is not an embedding, corresponds to $|a|$ points $y \in \tilde{m}$ such that $\tilde{\Phi}_1(\tilde{l}(S^1) \times \{y\})$ is not an embedding.

Notice that the intersection number of $\tilde{l}$ and $\tilde{Q}$ is 1. Hence we may assume, up to a fiber preserving homeomorphism, that $\tilde{F}$ is a trivial bundle over B' with fiber $\tilde{l}$.

To continue with the proof of the lemma, we choose a collection of proper arcs

$$\{\alpha_j\}_{j=1}^{j=m} \subset B'$$

such that (i) B' if cut along the $\{\alpha_i^j\}$'s becomes a disc (this is possible since B' is connected), and

(ii) the endpoints of the $\{\alpha_j\}$'s avoid the points for which $\tilde{\Phi}_1(\tilde{l} \times \{*\})$ is not an embedding. Let Γ denote the space obtained by cutting $\tilde{l} \times B'$ along the collection of annuli

$$\{A_j\}_{j=1}^{j=m}$$

where $A_j = \tilde{l} \times \alpha_j$. Let us denote by $\tilde{\Psi}$ the map induced on Γ, by $\hat{\Phi}_1 \circ \tilde{p}$.

By Lemma 3.4 we have that

$$X_{\tilde{\Psi}} = 0.$$

It is not hard to see that $X_{\tilde{\Psi}} = \pm(X_{\tilde{\Phi}_1}^{\tilde{l},\tilde{Q}} - X_{\tilde{\Phi}_1}^{\tilde{l}_1,\tilde{Q}_1} - \dots - X_{\tilde{\Phi}_1}^{\tilde{l}_m,\tilde{Q}_m})$ and thus we obtain

$$X_{\tilde{\Phi}_1}^{\tilde{l},\tilde{Q}} = X_{\tilde{\Phi}_1}^{\tilde{l}_1,\tilde{Q}_1} + \dots + X_{\tilde{\Phi}_1}^{\tilde{l}_m,\tilde{Q}_m}.$$

Since $\tilde{\Phi}_1|\tilde{T}$ is a covering map we can easily see that we have $X_{\tilde{\Phi}_1}^{\tilde{l}_i,\tilde{Q}_i} = 0$ for every $i = 1, \dots, m$. Hence the right-hand side of the equation above is zero, which implies that $X_{\tilde{\Phi}_1}^{\tilde{l},\tilde{Q}} = 0$. Now from the claim above, and the fact that we are working over a torsion free ring, we obtain that $X_{\Phi_1} = 0$ which finishes the proof of the case $a \neq 0$.

Now, we observe that if $a = 0$, then we have $\Phi_{1*}(\tilde{l}) = b\tilde{Q}$ and by applying Lemma 3.12 and Remark 3.13 we get that $X_{\Phi_1} = 0$. □

By Proposition 2.7, we see that Lemma 3.14 is true for Seifert manifolds which are Haken. In particular, this includes all Seifert manifolds with non-empty boundary.

It is well known that if S is a closed irreducible Seifert manifold, with infinite fundamental group, that is not Haken then the base space is the 2-sphere and it has exactly three exceptional fibers. Moreover, if N is the subgroup of $\pi_1(S)$ generated by a regular fiber of S, then the quotient $\pi_1(S)/N$ is the triangle group $\Delta(p,q,r)$, where p, q and r are the multiplicities of the exceptional fibers. As shown in [21], an essential map $S^1 \times S^1 \to S$, can always be homotoped to a vertical one if $\Delta(p,q,r)$ is a hyperbolic triangle group. Thus, Lemma 3.14 is also true for these manifolds, and the only Seifert spaces that cannot be handled with the techniques introduced so far are those corresponding to Euclidean triangle groups.

Let us call a Seifert manifold small if it has finite fundamental group, or it fibers over the 2-sphere and it has exactly three exceptional fibers.

COROLLARY 3.15. *Theorem 3.7 is true for Haken Seifert fibered spaces, or small Seifert manifolds that either have finite fundamental group or correspond to hyperbolic triangle groups.*

Proof. It follows immediately from Lemmas 3.8, 3.14, and the discussion above. □

3.4. The completion of the proof of Theorem 3.7

Let $\Phi : T = S^1 \times S^1 \to M$ be a closed homotopy from a knot to itself, and let X_Φ be the integral of the invariant f along Φ. We will show that $X_\Phi = 0$, for every Φ as above.

In view of Lemmas 3.6 and 3.8 we may restrict ourselves to essential Φ's.

First suppose that M is a Haken manifold. If Φ is an essential map, then the *characteristic submanifold* S, of M has to be non-empty. Notice that S is a Seifert fibered space which is closed if M itself is a Haken Seifert space, and has non-empty boundary otherwise. In any case S is a Haken Seifert manifold, and thus Lemma 3.14 applies. By the *Enclosing Theorem*, we can homotope Φ to a map $\Phi_1 : T = S^1 \times S^1 \to M$, with $\Phi_1(T) \subset Int S$. Finally, by combining Lemmas 3.5 and 3.14 we obtain $X_\Phi = X_{\Phi_1} = 0$.

Now suppose that M is non-Haken. Then, $H_1(M)$ has to be finite since every 3-manifold with non-zero first Betti number is Haken. (See for example Lemma 6.6 of [9].) By the *Torus Theorem*, if M is not Haken, then it has to be a small Seifert manifold.

By Corollary 3.15 the only case that we need to worry about is the case of small Seifert manifolds corresponding to Euclidean triangle groups. This finishes the proof of 3.7. □

The problem with these manifolds is that there might contain essential immersed tori that cannot be homotoped to immersions without triple points, and thus our arguments in Lemmas 3.11–3.14 do not apply, directly. These manifolds are handled in the rest of this paragraph.

Let us fix an essential map $\Phi : T = S^1 \times S^1 \to M$, and let l and m denote the longidute (knot direction) and meridian (parameter space) of T.

Recall that we denoted by N, the fiber group of M. Let $a \in \pi_1(M)$. If $a \in N$, then the centralizer $Z(a)$, of a is all of $\pi_1(M)$. In general, $Z(a)$ is abelian of order $\leqslant 2$ (see [10]). Although we do not use this fact, let us mention that if $\Phi_*(l) \in N$, then we obtain that $X_\Phi = 0$ from Lemma 3.6.

There are only three Euclidean triangle groups. These are $\Delta(2,4,4)$, $\Delta(3,3,3)$ and $\Delta(2,3,6)$.

(a) The triangle group is $\Delta(2,4,4)$. Then (see [21]) M has a two-fold covering $\tilde{M}$, which is Haken. Clearly, Φ lifts to an essential map $\tilde{\Phi} : T = S^1 \times S^1 \to \tilde{M}$. Then, and for a generic Φ, we may homotope $\tilde{\Phi}$ to a vertical map and apply Lemma 3.11 to obtain a trivial fiber bundle $\tilde{F} \cong S^1 \times B'$ over a surface B', whose boundary $\partial\tilde{F}$ is a collection of disjoint tori

$T \cup T_1 \cup \cdots \cup T_k$, and there exists a map $\hat{\Phi}: \tilde{F} \to \tilde{M}$ such that;

(i) $\hat{\Phi}|T = \tilde{\Phi}$,

(ii) for every $i = 1, \ldots, k$, $\hat{\Phi}|T_i : \to \hat{\Phi}(T_i)$ is the composition of a covering map $T_i : \to T_i$ and an embedding $T_i : \to \tilde{M}$.

Let $p: \tilde{M} \to M$ be the covering map. Now since p is of degree two the images of the embedded boundary tori of $\tilde{F}$ will be immersed tori in M without triple points.

Thus our original map Φ extends on a trivial fiber S^1-bundle, say F, over a surface B, where the boundary of F is a union of disjoint tori with the following property: One of them is the torus T to begin with, and the images of the rest of the components under the extended map Φ are vertical tori. In particular, the integral of our knot invariant around each of these immersed tori is trivial. Then, by an argument similar to the proof of Lemma 3.14, we obtain $X_\Phi = 0$.

(b) The triangle group is $\Delta(2,3,6)$ or $\Delta(3,3,3)$. Then [21] M has a three-fold covering $\tilde{M}$, which is Haken. In fact $\tilde{M}$ fibers over the 2-sphere with four exceptional fibers each of multiplicity four in the first case, and it fibers overs the torus without exceptional fibers in the second case. In both cases the group of covering translations G, has order three. Let us denote by g a generator of G.

For a generic Φ, as in (a) above we may extend the lifting map $\tilde{\Phi}: T = S^1 \times S^1 \to \tilde{M}$, on a trivial fiber bundle $\tilde{F}$, having the properties described in (a). We only need to be concerned with the fact that as we induce new boundary components in creating $\tilde{F}$, we might introduce some (embedded) essential tori whose images under the covering $p: \tilde{M} \to M$ might have triple points, and thus the technique of Lemma 3.14 would not apply to show that the integral of our singular knot invariant, around the newly created loops is trivial.

Notice however (see proof of Lemma 3.11), that the extra tori we use can be taken to be vertical and embedded in $\tilde{M}$. Let us focus on one of these tori, say $T_1 \subset \tilde{M}$, and let gT_1, g^2T_1 be the images of T_1 under the covering translations. Clearly, $T_1 \cap gT_1$ contains a circle that represents a regular fiber of $\tilde{M}$. But then the proof of Lemma 4.4 in [21] applies to show that we can assume that $T_1 \cap gT_1 \cap g^2T_1$ is empty and thus $p(T_1)$ is an immersion without triple points.

However, in both (a) and (b) above, if after homotopying $\tilde{\Phi}$, we are in the situation of Lemma 3.12 or Remark 3.13 or if $\tilde{\Phi}$ is horizontal with respect to the fibration of $\tilde{M}$ then we may have $X_\Phi \neq 0$ (see [15]).

THEOREM 3.16. *Suppose that M is a closed, oriented, irreducible homology 3-sphere and that $\mathscr{R}$ is a ring which is torsion free as an abelian group. Let $f: \mathscr{K}^1 \to \mathscr{R}$ be a singular knot invariant. There exists a knot invariant $F: \mathscr{K} \to \mathscr{R}$ so that (2) holds for all $K_\times \in \mathscr{K}^1$ if and only if f satisfies the weak integrability conditions (3) and (4).*

Proof. It follows from Theorem 3.7 and Lemma 3.6. □

4. SINGULAR KNOTS WITH MORE THAN ONE DOUBLE POINT

Our goal in this section is to answer the following question: Let $f: \mathscr{K}^n \to \mathscr{R}$, $n > 1$ be a singular knot invariant. Under what conditions is there a singular knot invariant $F: \mathscr{K}^{n-1} \to \mathscr{R}$ such that,

$$f(K_\times) = F(K_+) - F(K_-) \tag{7}$$

for every $K_\times \in \mathscr{K}^n$. Here $\times$ denotes one of the double points of a singular knot of order n.

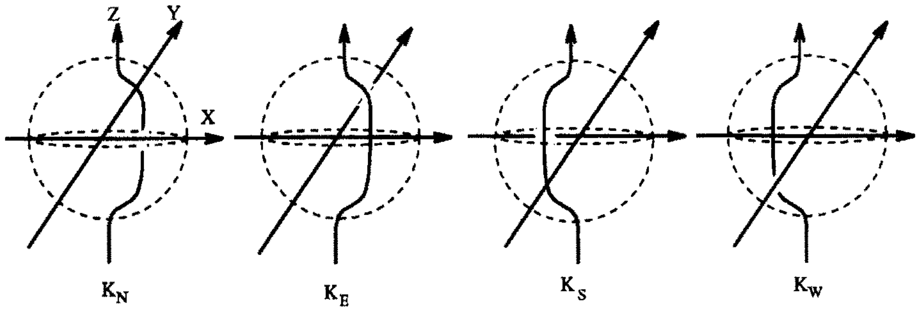

Fig. 6. Resolutions of a transverse triple point.

Clearly, for the existence of the invariant F conditions similar to those of Theorem 3.7 are still necessary. However, it turns out that they are not sufficient, in the case that $n>1$.

4.1. A generalization of Theorem 3.7

Let us denote by K_{N}, K_{S}, K_{E} and K_{W} the four singular knots of order $n>1$, which differ only in a small ball in M as shown in Fig. 6. (Our definition here depends on the cyclic ordering of the three arcs X, Y, and Z.)

One easily sees that these four singular knots appear in a series of "crossing changes" from a singular knot $K \in \mathscr{K}^{n-1}$ to itself. Hence in order for (7) to be true, we must have

$$f(K_{\mathrm{N}}) - f(K_{\mathrm{S}}) + f(K_{\mathrm{E}}) - f(K_{\mathrm{W}}) = 0.$$

In the next section we are going to see that K_{N}, K_{S}, K_{E} and K_{W}, arise as resolutions of a triple point.

We will say that a singular knot K, of order n is *inadmissible* if it contains a double point $p \in K$ such that either

(a) the two resolutions K_+ and K_-, of K with respect to p, are isotopic singular knots of order $n-1$; or

(b) the two *lobes* of K with respect to p are non-trivial in $\pi_1(M)$, and the two resolutions K_+ and K_- differ by a change of orientation.

THEOREM 4.1. *Assume M is a closed, oriented, irreducible,* 3*-manifold as in* 3.7, *and $\mathscr{R}$ is a ring which is torsion free as an abelian group. Let $f : \mathscr{K}^n \to \mathscr{R}$, $n>1$ be a singular knot invariant.*

(a) *For every $c \neq 1$, there is a singular knot invariant $F : \mathscr{K}_c^{n-1} \to \mathscr{R}$ such that* (7) *is true if*

$$f(K_{\mathrm{inadm}}) = 0 \tag{8}$$

$$f(L_{\times +}) - f(L_{\times -}) = f(L_{+\times}) - f(L_{-\times}) \tag{9}$$

$$f(K_{\mathrm{N}}) - f(K_{\mathrm{S}}) + f(K_{\mathrm{E}}) - f(K_{\mathrm{W}}) = 0. \tag{10}$$

(b) *Moreover, if M is as in Theorem* 3.7(b) *then there is a singular knot invariant $F : \mathscr{K}_c^{n-1} \to \mathscr{R}$ such that* (7) *is true if and only if f satisfies* (9), (10) *and*

$$f(\infty) = 0. \tag{8$'$}$$

As in Theorem 3.7 we call (8) (resp. (8)$'$), (9) and (10) the "*strong* (*resp. weak*) *integrability conditions*".

The main ingredients used in our proof here, are the homotopy classification of essential annuli in Haken manifolds (Proposition 2.11), and the generic picture for rigid-vertex null homotopies described in Theorem 1.5.

We need some preliminaries before we are ready to proceed with the proof of Theorem 4.1.

Suppose that $K: S^1 \to M$ is a singular knot with m double points. Let P_K be the 1-dimensional compact polyhedron obtained from K as follows: For every double point of K we identify its two preimages on S^1. We will call P_K the *configuration* of K. We say that two singular knots $K, K' \in \mathscr{K}^m$ respect the same configuration, iff there is a homeomorphism $\psi: P_K \to P_{K'}$, which lifts to an orientation preserving homeomorphism $\tilde{\psi}: S^1 \to S^1$. By definition all double points of K and K' are transverse. The homeomorphism ψ induces a one-to-one correspondence between the double points of K and those of K'. After isotopy, we may assume that the double points of K match those of K' according to the above mentioned one-to-one correspondence. Furthermore, we can assume that for each double point there is a ball neighborhood $B \subset M$ with $K \cap B = K' \cap B$ consisting of two line segments that intersect transversally at the given double point. By taking the B's sufficiently small, we can assume that they are disjoint. Let $\{C_i\}$ (resp. $\{C_i'\}$) be the set of components of $K \cap \overline{M \setminus \cup B}$ (resp. $K \cap \overline{M \setminus \cup B}$). Now, ψ induces an one-to-one correspondence between the components of $K \cap \overline{M \setminus \cup B}$ and those of $K' \cap \overline{M \setminus \cup B}$. Suppose that the component C_i of $K \cap \overline{M \setminus \cup B}$, corresponds to C_i' of $K' \cap \overline{M \setminus \cup B}$.

Definition 4.2. Let K and K' be two singular knots in some $\mathscr{K}_c^m$ that respect the same configuration, P_K. With the notation being as above, we will say that K and K' are similar iff, for every i, C_i and C_i' are homotopic in $\overline{M \setminus \cup B}$, relatively to the boundary.

LEMMA 4.3. *Let $K, K' \in \mathscr{K}^{n-1}$ be two similar singular knots, and let P_K be their common configuration. Then there is a rigid-vertex homotopy $\phi_t: P_K \to M$, $t \in [0,1]$, with $\phi_0 = K$, $\phi_1 = K'$ and there are finitely many points $0 < t_1 < t_2 < \cdots < t_s < 1$ such that*

(a) *each $\phi_{t_i} \in \mathscr{K}^n$, $i = 1, \ldots, s$,*

(b) *for every $t_i < t$, $s < t_{i+1}$, ϕ_t and ϕ_s are equivalent singular knots in $\mathscr{K}^{n-1}$; and*

(c) *when t passes through t_i, ϕ_t changes from one resolution of ϕ_{t_i} to another.*

Proof. It follows immediately by putting the homotopies in the definition of similarity into general position. □

Remark 4.4. Notice, that every singular knot $K \in \mathscr{K}^n$ is similar to itself. Let $\phi_t: P_K \to M$ $t \in S^1$, be a rigid-vertex homotopy as in Lemma 4.3 with $\phi_0 = K = \phi_1$. In general, $\phi_0(P_K)$ and $\phi_1(P_K)$ will differ by a permutation of the vertices, of order, say, k. Consider $q: S \cong S^1 \to S^1$ the k-fold covering of the parameter space of the homotopy above. Define a new homotopy $\phi_s': P_K \to M$ $s \in S$, by

$$\phi_s'(P_K) = \phi_{q(s)}(P_K).$$

Clearly, the homotopy $\{\phi_s'\}$ is a rigid-vertex homotopy, as well. Then, it is not hard to see that for every vertex of v of P_K, there is a neighborhood $N \subset P_K$ of v, and there exists a proper 2-disc D, in a ball neighborhood B of $\phi_0'(v)$, such that $\phi_0'(N) \subset D$ and $\phi_1'(N) \subset D$. Then, and possibly after an isotopy taking place in B, we may assume that $\phi_1'|B = \phi_0'|B$. Thus, $\{\phi_s'\}$ gives rise to a map $\Phi': P_K \times S \to M$.

Proof of Theorem 4.1. We need only to prove the sufficiency of *the integrability conditions*.

For every $c \in \mathscr{C}$ we choose a set of representatives of similarity classes in $\mathscr{K}_c^{n-1}$, and assign the values of the singular knot invariant $F: \mathscr{K}^{n-1} \to \mathscr{R}$ on this set, arbitrarily.

Now let $K \in \mathscr{K}^{n-1}$ and let K' be the representative chosen from the similarity class of K. Let ϕ_t, $t \in [0,1]$ be the homotopy from K to K' given in Lemma 4.3.

We define

$$F(K) = F(K') + \sum_{i=1}^{s} \varepsilon_i f(\phi_{t_i}) \tag{11}$$

where $\varepsilon_i = \pm 1$ are determined as in the proof of Theorem 3.7.

To prove that F is well defined we have to prove that

$$\sum_{i=1}^{s} \varepsilon_i f(\phi_{t_i}) = 0 \tag{12}$$

for any rigid-vertex closed homotopy, $\{\phi_t\}_{t \in S^1}$, from K to itself, that satisfies the requirements of Lemma 4.3.

Now observe that the quantity on the left-hand side of (12) is multiplied by an integer, if we replace $\{\phi_t\}_{t \in S^1}$ by $\{\phi'_s\}_{s \in S}$ of Remark 4.4. Thus, and since $\mathscr{R}$ is torsion free, we may assume that $\{\phi_t\}_{t \in S^1}$ gives rise to a map

$$\Phi: P_K \times S^1 \to M.$$

Let X_Φ denote the quantity in the left-hand side of (12). We have to show that

$$X_\Phi = 0. \tag{13}$$

Let $r = n-1$. Choose a basepoint on P_K and let $V = \{v_1, \ldots, v_r\}$ be the set of vertices of P_K in the order we encounter them as we travel along P_K, following the orientation induced by that of the S^1. Also let $\{e_1, \ldots, e_{2r}\}$ be an ordering of the edges of P_K guided by the above ordering of the vertices.

The proof of (13) occupies the rest of this section.

Lemma 4.5. *Let M and Φ be as above. Moreover, assume that X_Φ is the integral of an invariant of singular knots, that satisfies the weak local integrability conditions. Then we have*:

(a) *If $\gamma_i = \Phi(\{v_i\} \times S^1)$ represents a torsion element in $\pi_1(M)$, then $X_\Phi = 0$.*

(b) *In general, if $Ker\{\pi_1(P_K \times S^1) \to \pi_1(M)\} \neq \{1\}$, and $\Phi(P_K)$ is not homotopically trivial then, $X_\Phi = 0$.*

Proof (a). By our assumption there exists an integer m such that $m\gamma_i$ is homotopically trivial in M. Without loss of generality we may assume that $m=1$. For, otherwise we pass to the covering $id \times p: P_K \times S^1 \to P_K \times S^1$, where $p: S^1 \to S^1$ is the m-fold cover.

Since, γ_i is homotopically trivial, we can extend Φ on a disc $\{v_i\} \times D^2$ with $\partial D^2 = S^1$, for every $i = 1, \ldots, r$. Let $S_1^2 = (\{v_1\} \times S^1) \cup (\langle \overline{e_1} \rangle \times S^1) \cup (\{v_2\} \times S^1)$. Clearly S_1^2 is a 2-sphere. Since M is irreducible $\Phi | S_1^2$ extends on a 3-cell, $B_1 \cong \langle \overline{e_1} \rangle \times D^2$. By repeating this procedure till we exhaust all the edges of P_K we can extend Φ on $P_K \times D^2$.

Now notice that the obstruction to extend $\Phi | P_K \times S^1$ to a rigid-vertex null homotopy is annihilated by 2. Since $\mathscr{R}$ is torsion free we may assume that this obstruction is trivial, and hence $\Phi: P_K \times D^2 \to M$ is a rigid-vertex null homotopy. We use Theorem 1.5 to put Φ into almost general position.

Now we can reduce the global integrability condition (13) to local integrability conditions around the interior vertices of S_Φ. Let x be an interior vertex of S_Φ. First we notice that

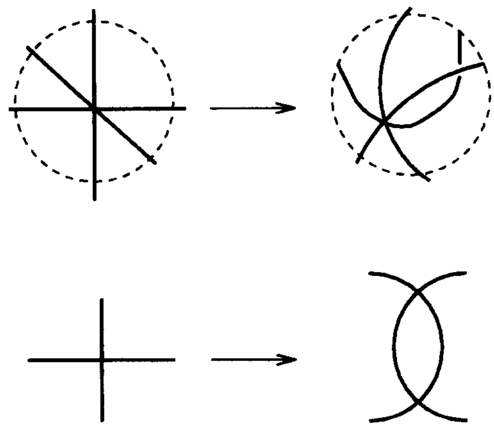

Fig. 7. Modification of a non-transverse triple point.

since all vertices of P_K have valence 4 it follows that x has valence 1 or 4. We have the following three cases:

Case 1: The vertex x is of valence 1.

Case 2: The vertex x is of valence 4 and ϕ_x has exactly two transverse double points.

Case 3: x is of valence 4 and there is an 1-simplex $\sigma \subset P_K$, and a vertex $p_0 \in P_K$ such that $\phi_x(p_0) \in \phi_x(Int(\sigma))$ and this is the only singularity of ϕ_x.

Cases 1 and 2 correspond to our integrability conditions (8′), (9). We now explain how case 3 corresponds to condition (10) of our theorem.

We choose a small enough neighborhood N^* of x in D^2 so that we can assume that $\phi_x(\sigma \times N^*) \subset M$ is a ball and $\phi_y(p) = \phi_x(p)$, for every $x \in N^*$ and $p \in \overline{P_K \backslash \sigma}$. Furthermore, $\phi_x(P_K) \cap \phi_x(\sigma \times N^*)$ consists of three line segments intersecting at $\phi_x(p_0)$ and one of them is $\phi_x(\sigma)$. The triple point of ϕ_x might not be a transverse double point; i.e the three line segments above may lie on the same plane. Then we perturb Φ as shown in Fig. 7.

Under this perturbation S_Φ changes as shown in the bottom of Fig. 7. Now a moment's thought will convince the reader that the newly created vertex has to be as in Case 2 above, and that the vertex x to begin with, corresponds to the local integrable condition (10).

(b). The statement (b) of the Lemma is reduced to (a) by a modification of the arguments in Lemma 3.8 along the lines of Lemmas 3.3.3 and 3.3.4 of [14]. □

LEMMA 4.6. *Let $\Phi, \Phi' : P_K \times S^1 \to M$ be two rigid-vertex homotopies in general position and let $\Phi_t : P_K \times S^1 \to M$ $t \in [0,1]$, be a homotopy such that $\Phi_0 = \Phi$ and $\Phi_1 = \Phi'$. Then we have $X_\Phi = X_{\Phi'}$.*

Proof. Choose a basepoint $y \in S^1$, and let $L = \Phi(P_K \times \{y\})$. Let us denote by $\Sigma^L = \Sigma^L(P_K, M)$, the space of maps $P_K \to M$, which are homotopic to L, equipped with the compact-open topology. Then, Φ and Φ' may be viewed as freely homotopic loops in Σ^L.

Let γ be the path in Σ^L defined by $\gamma(t) = \Phi_t(L)$. After putting γ in almost general position we have

$$X_{\gamma\Phi'\gamma^{-1}} = X_\gamma + X_{\Phi'} - X_\gamma = X_{\Phi'}.$$

Hence, we can assume that both Φ and Φ' are based at L and the homotopy Φ_t is taken relatively L. The homotopy Φ_t gives rise to a map $\mathscr{H} : P \times S^1 \times I \to M$. We cut the annulus $S^1 \times I$ into a disc D^2 along a proper arc $\alpha \subset S^1 \times I$. We have

$$X_{\partial D^2} = \pm(X_\Phi - X_{\Phi'} - X_\alpha + X_\alpha).$$

By Lemma 4.5 we obtain $X_{\partial D^2} = 0$, and hence $X_\Phi = X_{\Phi'}$. □

4.2. Homotopies of singular knots and essential annuli

We will need the following Lemma, the proof of which is identical to that of Lemma 3.12.

LEMMA 4.7. *Assume that S is a Seifert fibered space with non-empty boundary. Let B be the orbit space of S and $p:S\to B$ be the fiber projection. Let A be a 2-manifold each component of which is an annulus and let $G:A\to S$ be a map which is vertical with respect to the fibration of S. Moreover, suppose that the following is true: For every double point k of $p(G(A))$, the two components of $G^{-1}(p^{-1}(k))$ are identified with the same orientation. Let $A=(\coprod I)\times S^1$ and let $g_x=G((\coprod I)\times\{x\})$. Then for every $x_1,x_2\in\{*\}\times S^1$ there exists a homeomorphism $h^{12}:S\to S$ such that,*

(1) *$h^{12}=id$ outside a regular neighborhood N of the interior of $G(A)$ in S;*
(2) *$h^{12}=id$ on $\partial S\backslash\partial N$;*
(3) *$h^{12}(g_{x_1})=g_{x_2}$;*
(4) *h^{12} is isotopic, relatively $\partial S\backslash\partial N$, to the identity map $id:S\to S$.*

We will say that a closed rigid-vertex homotopy Φ, from a singular knot K to itself is *simple* iff there is a neighborhood $N\subset P_K$, of the set of vertices, V, of P_K such that

$$\Phi(N\times S^1)\cap\Phi(X\times S^1)=\emptyset$$

where X is the interior of $P_K\backslash N$. We need the following lemma.

LEMMA 4.8. *Let K be a singular knot with configuration P_K and let V be the set of vertices of P_K. Let Φ be a closed rigid-vertex homotopy from K to itself, such that*

$$\operatorname{Ker}\{\pi_1(P_K\times S^1)\to\pi_1(M)\}=\{1\}.$$

Recall the notation $\phi_t=\Phi(P_K\times\{t\})$. Then, there exist $s_1,\dots,s_k\in[0,1]$, such that if we let $\Phi^1,\Phi^2,\dots,\Phi^{k+1}$ to denote the restriction of Φ on $[0,s_1],[s_1,s_2],\dots,[s_k,1]$, respectively, the following is true: There exist homotopies

$$\Psi_1,\dots,\Psi_k:P_K\times[0,\ 1]\to M$$

such that (i) *We have $\Psi_i(P_K\times\{0\})=\phi_0=K$ and $\Psi_i(P_K\times\{1\})=\phi_{s_i}$, for all $i=1,\dots,k$; and* (ii) *Each of the closed homotopies*

$$\tilde\Phi_1=\Phi^1\circ\Psi_1,\quad \tilde\Phi_2=\Psi_1^{-1}\circ\Phi^2\circ\Psi_2,\dots,\quad \tilde\Phi_{k+1}=\Psi_k^{-1}\circ\Phi^{k+1}$$

is homotopic to a simple homotopy.

In particular, we have that

$$X_\Phi=X_{\tilde\Phi_1}+\dots+X_{\tilde\Phi_{k+1}}.$$

Proof. By general position and our assumption above, we may suppose that the intersections $\Phi(N\times S^1)\cap\Phi(X\times S^1)$ consist of finitely many embedded discs each intersecting transversally one of the embedded curves $\Phi(V\times S^1)$. First choose $s_1\in[0,1]$, so that the open homotopy Φ^1 is simple, and let $\Psi_1:P_K\times[0\ 1]\to M$, with $\Psi_1(P_K\times\{0\})=\phi_0=K$ and $\Psi_1(P_K\times\{1\})=\phi_{s_1}$ be an open homotopy, so that the interiors of Φ^1 and Ψ_1 (viewed as paths in the $\mathscr{M}$) do not intersect. By pushing the intersections of $\Phi^1(N\times[0,s_1])\cap\Psi_1(X\times[0,1])$ and $\Psi_1(N\times[0,1])\cap\Phi^1(X\times[0,s_1])$ into $\Phi(N\times[s_1,1])$, we may assume that $\tilde\Phi_1=\Phi^1\circ\Psi^1$ is simple. Then we choose $s_1<s_2<1$, so that the open homotopy Φ^2 is simple, and $\Psi_2:P_K\times[0\ 1]\to M$, with $\Psi_2(P_K\times\{0\})=\phi_{s_1}$ and $\Psi_1(P_K\times\{1\})=\phi_{s_2}$ and such that

$\Psi_2(X\times[0,1]))\cap\Phi(N\times S^1)=\Psi_2(X\times[0,1])\cap\Phi^1(N\times[0,s_1])$. By rearranging the singularities further, we may assume that $\tilde{\Phi}_2=\Psi_1^{-1}\circ\Phi^2\circ\Psi_2$ is simple. The reader, can see that we can proceed this way till we reach all requirements claimed in the statement above. □

The completion of the proof of Theorem 4.1. By Lemma 4.5 we can assume that the $\mathrm{Ker}\{\pi_1(P_K\times S^1)\to\pi_1(M)\}=\{1\}$. Thus, in particular the curves $\gamma_i=\Phi(\{v_i\}\times S^1)$ are essential.

Since Φ is a rigid-vertex homotopy, there is a neighborhood $N=\bigcup_{i=1}^r N_i$ of V in P_K, and an isotopy $h_t:P_K\to M$, $t\in[0,1]$, such that $\phi_t|N=h_t\phi_0|N$ for all $t\in[0,1]$, where $\phi_t=\Phi(P_K\times\{t\})$. Let X be the interior of $P_K\backslash N$. By our definition of similarity, Lemmas 4.6 and 4.8 we may suppose that $\Phi(N)\cap\Phi(X\times S^1)=\emptyset$.

Now, let $U_i\subset M$ be a regular neighborhood of the simple closed curve $\gamma_i=\Phi(\{v_i\}\times S^1)$ such that;

(a) Every U_i is a solid torus whose meridinal disc $D_i\times\{*\}$ contains the two arcs of $\Phi(N_i)\times\{*\}$ transversally intersecting at $\Phi(v_i)\times\{*\}$.

(b) $\Phi(N_i\times S^1)\subset U_i$ and $\Phi(\partial N_i\times S^1)\subset\partial U_i$.

(c) $\Phi(P_K\times S^1)\cap(\bigcup U_i)=\Phi(N\times S^1)\cap(\bigcup U_i)$.

Without loss of generality we may assume that ∂U_i intersects the singular knot $\Phi(P_K\times\{*\})$ at precisely four points, say $\{p_i^j\}_{j=1,\dots,4}$. Let v_i^j $(j=1,\dots,4)$ be the preimage of p_i^j on ∂N_i. Clearly, $P_K\backslash intN$ is a union of $2r$ arcs whose set of boundary points is $\{p_i^j\}$, $j=1,\dots,4$, $i=1,\dots,r$. Let us call these arcs $\{\alpha_1,\dots,\alpha_{2r}\}$ and let $A_s=\alpha_s\times S^1$, $s=1,\dots,2r$. By our assumptions we have that $\Phi(A_s\cap(\bigcup U_i))=\Phi(\partial A_s\cap(\bigcup\partial U_i))$ for every $s=1,\dots,2r$.

Let $\bar{M}=M\backslash\bigcup U_i$. Since M is irreducible and each γ_i is an essential curve, $\bar{M}$ has to be irreducible. Moreover, $\bar{M}$ is Haken. Since $\partial\bar{M}$ consists of tori, $\bar{M}$ either is a solid torus or its boundary is incompressible. If $\bar{M}$ is a solid torus, then the number of double points of P_K has to be equal to 1. But then M is obtained by glueing together two solid tori and thus $\pi_1(M)$ has to be finite. In this case the conclusion $X_\Phi=0$, follows by Lemma 4.5. So we may assume that the boundary of $\partial\bar{M}$ is incompressible.

Let $A=\bigcup_{i=1}^{2r}A_i$ and let $\Psi=\Phi|A$.

Then $\Psi:(A,\partial A)\to(\bar{M},\partial\bar{M})$ satisfies the hypothesis of (b) of the *Enclosing Theorem* and hence we may find a Seifert fibered pair $(S,U)\subset(\bar{M},\partial\bar{M})$ such that Ψ can be homotoped, relative ∂A, to a map $\Psi':A\to\bar{M}$ with $\Psi'(A)\subset S$.

First, suppose that $\Psi'|A_i$ is inessential on some component A_i, of A. Then, $\Psi'(A_i)$ can be homotoped on the boundary of S, relatively $\partial(A_i)$. Notice that since all our homotopies have been carried out relatively to ∂A, we may conclude that our map $\Phi:P_K\times S^1\to M$ is homotoped to a map $\Phi_1:P_K\times S^1\to M$ such that $\Phi_1(A_i)$ lies on an embedded torus T in M. Observe that the images, under Φ_1, of the two components of ∂A_i are disjoint, and hence they decompose T into two annuli C and B. Then $\Phi_1(A_i)$ has to lie on one of them, say B. By Nielsen's theorem, and since the components of $\Phi_1(\partial A_s)$ are disjoint, we may homotope $\Phi_1|A_i:A_i\to B$, to a covering map, relatively $\partial(A_i)$. Let us continue to denote the resulting map by Φ_1. In view of Lemma 4.6 it is enough to prove that $X_{\Phi_1}=0$. But notice that since $\Phi_1|A_i:A_i\to B$ is a covering and since $\Phi_1(A\backslash A_i)$ lies away from the torus T, the only contributions to X_{Φ_1} will come from double points between some $\Phi_1(\alpha_k\times\{*\})$ and $\Phi_1(\alpha_l\times\{*\})$, with $k,l\neq i$. Of course k may be equal to l.

In view of the above discussion, we may assume that $\Psi':(A,\partial A)\to(S,\partial S)$ is an essential map. Moreover, we can assume that S is not a solid torus since every non-contractible annulus in a solid torus, can be deformed into the boundary. Then by Proposition 2.8 we may assume that, either

Case 1: there is a homotopy of pairs $\Psi'_t:(A,\partial A)\to(S,\partial S)$, where $t\in[0,1]$, such that (i) $\Psi'_0=\Psi'$ and (ii) Ψ'_1 is vertical with respect to the fibration of S, or

Case 2: there exists a fibration of S as an I-bundle over the annulus, torus, Mobius band, or Klein bottle such that A is vertical with respect to this fibration.

First we suppose that we are in Case 1. Since the above-mentioned homotopy is actually a homotopy of pairs, and since homotopic curves on a torus are isotopic, we may assume that H is taking place relatively ∂A. Thus, our map $\Phi:P_K\times S^1\to$ has been homotoped in M to a map $\Phi_1:P_K\times S^1\to$ with the property that $\Phi_1|A=\Psi'_1|A$ is vertical in S.

Let B be the orbit space of S and let $p:S\to B$ be the fiber projection. Then, by the homotopy classification of essential annuli, $p(\Psi'_1(A))$ is a union of $2r$ arcs $\{\beta_1,\ldots,\beta_{2r}\}$ such that;

(i) $\partial\beta_i\subset\partial B$, $i=1,\ldots,2r$;

(ii) the only singularities of β_i are finitely many transverse double points;

(iii) two arcs β_i and β_j intersect at finitely many transverse double points;

(iv) the union $\beta_1\cup\cdots\cup\beta_{2r}$ is disjoint from the cone points of B.

Hence, the core H_i of every annulus A_i maps, under Ψ'_1, onto a regular fiber of S and there exists a simple essential arc $Q_i\subset A_i$, intersecting H_i once, which maps onto a cross section of $\Psi'_1(A_i)$. Without loss of generality, we may assume that $Q_i=\alpha_i$. First suppose that the requirements of Lemma 4.7 are satisfied. Then for every x_1 and x_2 on the parameter curve of Ψ' there is a homeomorphism $h^{12}:S\to S$, with $h^{12}(\psi'_{x_1})=\psi'_{x_2}$, where $\psi'_{x_i}=\Psi'(\bigcup\alpha_j\times\{x_i\})$ $(i=1,2)$.

Now h^{12} is the identity on the boundary of S, except, possibly, on a collection of disjoint embedded annuli on which it is a translation along the fibers of S. Now it is not hard to see that h^{12} may be extended to a homomorphism $h^{12}:M\to M$, which is isotopic to $id:M\to M$ and such that $h^{12}(K^1)=h^{12}(K^2)$, where $K^i=\Phi_1(P_K\times\{x_i\})$. Thus we obtain $X_{\Phi_1}=0$.

Now suppose that Lemma 4.7 does not apply. Then, by an argument similar to that in Remark 3.13 we will show that there is a closed homotopy $\Phi_1:P_K\times S^1\to M$ such that (i) $X_\Phi=X_{\Phi_1}$ and (ii) Φ_1 contains finitely many singular knots of order $n+1$, each of which is inadmissible. Thus, the result will follow from our integrability condition (8).

Now if we are in the situation of Case 2, and by using the fact that $\Psi(\partial A)$ is an embedding, and an argument similar to that in the proof of Proposition 5.13 of [11], one can see that $X_\Phi=0$. □

As in the case of Theorem 3.7 we have a stronger version of Theorem 4.1, if we restrict ourselves to homology spheres.

THEOREM 4.9. *Assume M is a closed, oriented, irreducible, homology 3-sphere and $\mathscr{R}$ is a ring which is torsion free as an abelian group. Let $f:\mathscr{K}^n\to\mathscr{R}$, $n>1$ be a singular knot invariant. There is a singular knot invariant $F:\mathscr{K}_c^{n-1}\to\mathscr{R}$ such that* (7) *is true if and only if f satisfies the weak integrability conditions* (8′), (9) *and* (10).

5. FINITE TYPE INVARIANTS FOR KNOTS IN 3-MANIFOLDS

As we have already mentioned in Section 3, from a knot invariant $f:\mathscr{K}\to\mathscr{R}$ we can derive a singular knot invariant $f:\mathscr{K}^{(1)}\to\mathscr{R}$ by

$$f(\times)=f(K_+)-f(K_-).$$

By iterating this procedure, we can derive a singular knot invariant $f:\mathscr{K}^n\to\mathscr{R}$, for every n. More precisely, let $K\in\mathscr{K}^n$. By considering all positive and negative resolutions of K we get 2^n knots, which we denote by $K_1, \ldots, K_{2^n}$. Then the nth derived singular knot invariant is defined by

$$f(K) = \sum_{i=1}^{2^n} \varepsilon_i f(K_i)$$

where $\varepsilon_i = 1$ if we have made an even number of negative resolutions. Otherwise $\varepsilon_i = -1$.

Definition 5.1. A knot invariant $f : \mathscr{K} \to \mathscr{R}$ is called of finite type m, iff its derived singular knot invariant vanishes on singular knots with more than m double points, and m is the smallest such integer.

Let us denote by $\mathscr{F}^m$ (respectively, $\mathscr{F}_c^m$) the $\mathscr{R}$-module of invariants of type $\leqslant m$, for knots in $\mathscr{K}$ (respectively, $\mathscr{K}_c$). Clearly we have $\mathscr{F}^m = \bigoplus_{c \in \mathscr{C}} \mathscr{F}_c^m$.

LEMMA 5.2. *Let $f \in \mathscr{F}^m$ be a knot invariant. Then for every $K \in \mathscr{K}_c^m$, $f(K)$ depends only on the similarity class of K.*

Proof. Suppose $f \in \mathscr{F}^m$ and let K, $K' \in \mathscr{K}_c^m$ be similar. Then by Lemma 4.3 K can be changed to K' by a sequence of "crossing changes". Hence,

$$f(K) = f(K') + \sum_{i=1}^{s} \varepsilon_i f(K_i)$$

where $K_i \in \mathscr{K}^{m+1}$, $i = 1, \ldots, s$. Since the type of f is $\leqslant m$, we must have $f(K_i) = 0$ and hence $f(K) = f(K')$. □

Definition 5.3. (a) A similarity class in $\mathscr{K}_c^n$ is called inadmissible if every singular knot $K : S^1 \to M$ in this similarity class is inadmissible.

(b) A similarity class in $\mathscr{K}_c^n$ is called strongly inadmissible if for every singular knot $K : S^1 \to M$ in this similarity class we have

(1) there is an interval $I \subset S^1$ such that ∂I is the preimage of a double point of K, and I contains no preimages of other double points; and

(2) $K(I)$ is homotopically trivial in M.

Otherwise the similarity class is called admissible.

For every $c \in \mathscr{C}$, we choose a set of representatives of the similarity cases in $\mathscr{K}_c^j$, denoted by Ω_c^j $(j = 1, \ldots, m)$. This choice should be subject to the following restriction: If $K : S^1 \to M$ is the chosen representative of a strongly inadmissible similarity class, then there is a disc $D \subset M$, with $D \cap K(S^1) = D \cap K(I)$, where I is as in Definition 5.3. We will show that every $\mathscr{F}_c^m$ is determined by a system homogeneous linear equations. The unknowns of the system will be the values of the invariants on $\{\Omega_c^j\}_{j=1,\ldots,m}$, and the equations arise from resolutions of triple points. Hence, the existence of non-trivial *finite type* invariants of type $\leqslant m$ will be reduced to the existence of non-trivial solutions for this system.

In order to explicitly describe the above-mentioned system, we need to introduce and study immersions $S^1 \to M$ that have a transverse triple point. To begin, let us denote by $\mathscr{K}^{(j,1)}$ the set of ambient isotopy classes of piecewise-linear maps $K^1 : S^1 \to M$, having exactly j transverse double points and one transverse triple point. The isotopy should preserve the transversality of the double points and that of the triple point. We will also use K^1 to denote the isotopy class of $K^1 : S^1 \to M$.

Let $K^1 \in \mathscr{K}^{(j,1)}$ and let $p \in M$ be the triple point of K^1. Let $B \subset M$ be a ball neighborhood of p in M such that $B \cap K^1$ consists of three linearly independent oriented line segments X, Y and Z. Then it is easy to see that there are six different ways to resolve p into two double points. More precisely, if we fix a cyclic ordering of $\{X, Y, Z\}$ then there are four different resolutions of p giving rise to knots K^1_{N}, K^1_{S}, K^1_{E}, $K^1_{\mathrm{W}} \in \mathscr{K}^{j+2}$, which differ only in a ball as shown in Fig. 6. We denote by $\mathscr{K}_c^{(j,1)}$ the set of ambient isotopy classes of immersions, with j transverse double points and a triple point, whose all resolutions in the above sense represent singular knots in $\mathscr{K}_c^{j+2}$.

Suppose that $K^1 \in \mathscr{K}^{(j,1)}$. Let $\{p_1, \ldots, p_j\}$ be the double points of K^1, and let p be its triple point. We construct a 1-dimensional compact polyhedron P_{K^1}, by identifying points on S^1 whose image under K^1 is the same. The polyhedron P_{K^1}, is called the *configuration* of K^1. We say that two immersions $K^1_0, K^1_1 \in \mathscr{K}^{(j,1)}$ *respect the same configuration*, iff there is a homeomorphism $\chi : P_{K^1_0} \to P_{K^1_1}$, which lifts to an orientation preserving homeomorphism $\tilde{\chi} : S^1 \to S^1$. The homeomorphism χ induces an one-to-one correspondence between the double (triple) points of K^1_0 and these of K^1_1. After isotopy, we may assume that the double points (resp. triple point) of K^1_0 match the double points (resp. triple point) of K^1_1, according to the above mentioned one-to-one correspondence.

Furthermore, we can assume that for each double point (resp. the triple point) there is a ball neighborhood $B \subset M$ with $K^1_0 \cap B = K^1_1 \cap B$ consisting of two (resp. three) line segments that intersect transversally at the given double (resp. triple) point. By taking these balls sufficiently small, we can assume that they are disjoint. Let $\{C_i\}$ (resp. $\{C'_i\}$) be the set of components of $K^1_0 \cap \overline{M \setminus \cup B}$ (resp. $K^1_1 \cap \overline{M \setminus \cup B}$). Now χ induces an one-to-one correspondence between the components of $K^1_0 \cap \overline{M \setminus \cup B}$ and those of $K^1_1 \cap \overline{M \setminus \cup B}$. Suppose that the component C_i of $K^1_0 \cap \overline{M \setminus \cup B}$ corresponds to C'_i of $K^1_0 \cap \overline{M \setminus \cup B}$.

Definition 5.4. Two immersions K^1_0, $K^1_1 \in \mathscr{K}^{(j,1)}$ are called *similar* iff;

(1) K^1_0 and K^1_1 belong in the same $\mathscr{K}_c^{(j,1)}$, for some $c \in \mathscr{C}$;

(2) they respect the same configuration,

(3) with the notation being as above, for every i, $\{C_i\}$ and $\{C'_i\}$ are homotopic in $\overline{M \setminus \cup B}$, relative to the boundary.

Lemma 5.5. *Let K^1_0, $K^1_1 \in \mathscr{K}^{(j,1)}$ be two* similar *immersions and let P be their common configuration. Then there is a rigid-vertex homotopy $\phi_t : P \to M$ $t \in [0, 1]$, with $\phi_0 = K^1_0$, $\phi_1 = K^1_1$ and there are finitely many points $0 < t_1 < t_2 < \cdots < t_s < 1$ such that*

(a) $\phi_{t_i} \in \mathscr{K}^{(j+1,1)}$, $i = 1, \ldots, s$;

(b) *for every $t_i < t, s < t_{i+1}, \phi_t$ and ϕ_s are equivalent;*

(c) *when t passes through t_i, ϕ_t changes from one resolution of ϕ_{t_i} to another.*

Proof. It follows immediately from the definition of similarity. □

Now let us fix a conjugacy class $c \in \mathscr{C}$ and let $f \in \mathscr{F}_c^m$. Suppose that if $K \in \Omega_c^j$ is a representative that respects an inadmissible similarity class then we have

$$f(K) = 0. \tag{14}$$

If the similarity class in question is strongly inadmissible then (14) is always true. Let $K^1 \in \Omega_c^{(j,1)}$, $j = 1, \ldots, m-2$, and let K^1_{N}, K^1_{S}, K^1_{E}, $K^1_{\mathrm{W}} \in \mathscr{K}_c^{j+2}$ be the resolutions of K^1 as above. Then since K^1_{N}, K^1_{S}, $K^1_{\mathrm{E}}, K^1_{\mathrm{W}}$ are results of a sequence of "crossing changes" of a singular knot in $\mathscr{K}_c^{j+1}$ to itself (see Section 4) we must have

$$f(K^1_{\mathrm{N}}) - f(K^1_{\mathrm{S}}) + f(K^1_{\mathrm{E}}) - f(K^1_{\mathrm{W}}) = 0. \tag{15}$$

Let us denote by $\hat{K}^1_{\mathrm{N}}$, $\hat{K}^1_{\mathrm{S}}$, $\hat{K}^1_{\mathrm{E}}$, $\hat{K}^1_{\mathrm{W}}$ the representative in $\Omega_c^{(j+2)}$ of the similarity class of K^1_{N}, K^1_{S}, K^1_{E}, K^1_{W}, respectively. Now since $f|\mathscr{K}^c_m$ depends only on the similarity class and not on the particular representative we must have

$$f(\hat{K}^1_{\mathrm{N}}) - f(\hat{K}^1_{\mathrm{S}}) + f(\hat{K}^1_{\mathrm{E}}) - f(\hat{K}^1_{\mathrm{W}}) = 0 \tag{16}$$

for every $K^1 \in \Omega_c^{(m-2,\ 1)}$. However, if we take $K^1 \in \Omega_c^j$, $j = 0, \ldots, m-3$, then the resolutions K^1_{N}, K^1_{S}, K^1_{E}, K^1_{W} may not agree with their representatives, in Ω_c^{j+2}, of the similarity class they respect. Nevertheless, we have agreement up to "crossing changes" and we can express any of the differences $f(K^1_{\mathrm{N}}) - f(\hat{K}^1_{\mathrm{N}})$, $f(K^1_{\mathrm{S}}) - f(\hat{K}^1_{\mathrm{S}})$, $f(K^1_{\mathrm{E}}) - f(\hat{K}^1_{\mathrm{E}})$ and $f(K^1_{\mathrm{W}}) - f(\hat{K}^1_{\mathrm{W}})$ as a linear combination of values of f on finitely many elements in $\Omega_c^{j+3}, \ldots, \Omega_c^m$. Thus we get,

$$f(\hat{K}^1_{\mathrm{N}}) - f(\hat{K}^1_{\mathrm{S}}) + f(\hat{K}^1_{\mathrm{E}}) - f(\hat{K}^1_{\mathrm{W}}) = f.l.c\{f(\Omega_c^{(i)}),\ i = 1, \ldots, m\} \tag{17}$$

where $f.l.c$ stands for "finite linear combination".

If we view $\{f(\Omega_c^j)\}_{j=1,\ldots,m}$ as unknowns then (14), (16), (17) give us the system of linear equations promised earlier. We denote this system by S_c. Moreover, let S_c' be the system obtained from S_c by removing all the equations of type (14) for similarity classes in $\{(\Omega_c^j)\}_{j=1,\ldots,m}$ that are not *strongly inadmissible*. We have,

THEOREM 5.6. *Assume that M is a closed, oriented, irreducible 3-manifold, as in Theorem 3.7, and let $c \in \mathscr{C}$.*

(a) *If M is as in 3.7(b), then any invariant $f \in \mathscr{F}_c^m$ $(c \neq 1)$ is completely determined by $f(K_c)$ and its values on Ω_c^j, $j = 1, \ldots, m$ subject to (14), (16) and (17). That is, there exist non-trivial finite type invariants of type $\leqslant m$ if and only if there exist non-trivial solutions for the system S_c'. Moreover, if M is a homology 3-sphere then the conclusion is true for $c = 1$ also.*

(b) *In general, a solution to the system S_c, gives rise to an invariant $f \in \mathscr{F}_c^m$ for $c \neq 1$.*

Proof. We have to show that every solution of the system S_c, gives rise to an invariant $f \in \mathscr{F}_c^m$. By Lemma 5.2, a solution of S_c defines an invariant of singular knots of order m. Let us denote this invariant by f. By (14) and (16) we see that f satisfies the local integrability conditions of Theorem 4.1 and thus it may be integrated to an invariant of singular knots with $m-1$ double points. Inductively, one can show that the integrability conditions (8), (9), and (10) (resp. (8′), (9)) of Theorems 4.1 and 3.7 are always guaranteed by the equations (14), (16) and (17), and thus f can be integrated to a knot invariant. The details are similar to these of the proof of Theorem 7.7 in [17] and are left to the reader. Now suppose that M is a homology sphere. Then the conclusion follows from Theorems 3.16 and 4.9. □

6. EXISTENCE OF NON-TRIVIAL FINITE TYPE INVARIANTS AND A CONNECTION WITH THE CONWAY POTENTIAL FUNCTION

In this section we prove that $\mathscr{F}^m(M)$ is non-trivial for every type m, and every closed irreducible manifold M as in Theorem 3.7. See Corollary 6.4. Finally, we show that the classical Alexander polynomial of a knot in a rational homology sphere is equivalent to a sequence of *finite type* invariants.

Let $c \in \mathscr{C}$, and let Ω_c^j (resp. $\Omega_c^{(j-2,\ 1)}$), be sets of representatives of similarity classes in $\mathscr{K}_c$ (resp. in $\mathscr{K}_c^{(j-2,1)}$), for $j = 1, \ldots, m$. We denote by S_c the system of equations (14), (16) and (17). Recall that S_c determines the values of finite type invariants of type $\leqslant m$ on $\{\Omega_c^j\}_{j=1,\ldots,m}$.

Before we proceed we need the following lemma, whose proof follows directly from the definitions.

LEMMA 6.1. *Let M be a 3-manifold and $\mathscr{C}$ be the set of conjugacy classes in $\pi_1(M)$. Choose $c \in \mathscr{C}$. For every $j = 1, \ldots, m$, let A_c^j and $A_c^{(j,1)}$ be a subset of Ω_c^j and $\Omega_c^{(j-2,1)}$, respectively. Let $A_c = \bigcup A_c^j$ and let $A_c^1 = \bigcup A_c^{(j,1)}$, and suppose that the following property is satisfied: If K^1 is an element in A_c^1 (resp. in the complement $\bigcup \Omega_c^{(i,1)} \setminus A_c^1$) then eqs. (14)–(17) obtained by resolving the triple point of K^1 involve only elements in A_c (resp. in the complement $\bigcup \Omega_c^{(i,1)} \setminus A_c$).*

Then, S_c breaks into two independent subsystems S_c^1 and S_c^2 and hence $\mathscr{F}_c^m \cong F_c^1 \oplus F_c^2$ where F_c^i is the solution space of S_c^i, for $i = 1, 2$.

It is well known (see for example [20]) that every closed, orientable 3-manifold, M, can be obtained from S^3 by surgery along a link $L^0 \in S^3$. Then we can find a link $L \in M$ such that, $M \setminus L$ and $S^3 \setminus L^0$ are homeomorphic.

THEOREM 6.2. *Let M be a closed, orientable, irreducible 3-manifold as in Theorem 3.7. Then, $\mathscr{F}_c^m(M)$ contains a subspace isomorphic to $\mathscr{F}^m(S^3)$ for every m, and every $c \in \mathscr{C}$, with $c \neq 1$. Moreover, if M is a rational homology 3-sphere the conclusion is true for $c = 1$ as well.*

Proof. Let us fix links L and L^0 as in the discussion above and let us fix a homeomorphism $h: M \setminus L \to S^3 \setminus L^0$.

Let $c \in \mathscr{C}$, and $K_c \subset M \setminus L$ be a knot representing c. We take a 3-ball $B^3 \subset M \setminus L$ which intersects K_c in a small unknotted arc.

In B^3, we choose a set A^j, of representatives of similarity classes of $\mathscr{K}^j(S^3)$, and a set $A^{(j-2,1)}$ of representatives of similarity classes of $\mathscr{K}^{(j,1)}(S^3)$. Here, $j = 0, \ldots, m$. Let $A = \bigcup A^j$ and let $A^1 = \bigcup A^{(j-2,1)}$.

For an immersion K in some A^j (resp. $A^{(j-2,1)}$) we can construct various immersions in $\mathscr{K}_c^j$ (resp. in $\mathscr{K}_c^{(j-2,1)}$) by forming a "connected sum", in B^3, of K and K_c. Let A_c (resp. A_c^1) denote the set of all immersions in $\mathscr{K}_c^j$ (resp. in $\mathscr{K}_c^{(j-2,1)}$) that can be obtained that way. By deleting any repetitions, we may assume that no two immersions in A_c or A_c^1 belong in the same similarity class in M. Observe, that all the singular knots obtained by resolving the triple point of an immersion in A_c^1 belong to similarity classes represented in A_c.

Now, we complete A_c (resp. A_c^1) to a set Ω_c, (resp. Ω_c^1) of representatives of similarity classes for $\bigcup \mathscr{K}_c^j$ (resp. $\bigcup \mathscr{K}_c^{(j-2,1)}$).

We claim that all the singular knots obtained by resolving the triple point of an immersion in $\Omega_c^1 \setminus A_c^1$ belong to similarity classes represented in $\Omega_c \setminus A_c$. To see this let us pick an immersion K^1 in $\Omega_c^1 \setminus A_c^1$. First suppose that K^1 contains at least two arcs, connecting two double points or a double point and the triple point, which cannot be homotoped to lie in a 3-ball. Then, each singular knot obtained by resolving the triple point K^1 has the same property and thus it cannot be similar to an immersion in A_c. If K^1 contains exactly one arc that cannot be homotoped to lie in a 3-ball, then it is easy to see that has to be similar to one of the immersions in A_c^1.

By Lemma 6.1 we obtain that S_c breaks into two independent subsystems S_c^1 and S_c^2, where S_c^1 involves only elements in A_c. We will show that every invariant of type $f \in \mathscr{F}^m(S^3)$ gives rise to a solution of S_c^1 and thus to an invariant $\hat{f}$, of type m for knots in $\mathscr{K}_c(M)$.

We can assume that $c \neq 1$. For an immersion $K \in A_c$ we define $\hat{f}(K) = f(h(K))$. Let $K^1 \in A_c^1$, and let K_{N}, K_{S}, K_{E} and K_{W} be the singular knots obtained by resolving the triple point of K^1. Notice that by our choices above, the homotopies taking each of these resolutions to the representative of their similarity class in A_c, can be carried out entirely in B^3. Then, the fact that this choice of values satisfies the equations of type (14), (16) and (17) follows from the fact that f is an invariant of type m for knots in B^3.

If M is a rational homology sphere and $c=1$, then we may take $A_c=A$ and there is nothing to prove. □

Let us, now, suppose that $\mathscr{R}=\mathbb{R}$. Then it is known that $\mathscr{F}^m(S^3)\neq\{0\}$ for every m. More precisely,

THEOREM 6.3 (Birman [2], Birman and Lin [3], Bar-Natan [1] and Lin [16]). *To every irreducible representation of a semisimple Lie algebra corresponds a solution to the system describing $\mathscr{F}^m(S^3)$ for every m, and hence a knot invariant of type m.*

Combining this with Theorem 6.2 above we get,

COROLLARY 6.4. *Let M be a closed, compact, orientable, irreducible 3-manifold as in Theorem 3.7. Then, to every invariant of finite type for knots in S^3 corresponds an invariant of finite type for knots in M. In particular every irreducible representation of a semisimple Lie algebra gives rise to a finite type invariant of every type in M.*

For the rest of the section we assume that M is a *rational homology sphere.*

Let $K\in M$ be a knot and let $\Delta_K(t)$ be the symmetrized version of its Alexander polynomial. For details see [27]. In [4] it is proved that there is a well-defined Conway potential function $\nabla_K(t)$, (see [6]), satisfying

$$\nabla_K(t)=\frac{\Delta_K(t^2)}{t^d-t^{-d}} \tag{18}$$

$$\nabla_{K_+}(t)-\nabla_{K_-}(t)=(t-t^{-1})\nabla_{K_0}(t) \tag{19}$$

where d is defined as follows: Let $G=H_1(M\setminus K)$ and let T be the torsion subgroup of G. We define d to be the quotient $|T|/|H_1(M)|$. We have,

PROPOSITION 6.5. *The coefficients of the power series obtained by $\nabla_K(t)$ if we substitute $t=\mathrm{e}^x$ are finite type invariants.*

Proof. Guided by (19) we extend

$$\nabla_{K_\times}(t)=\nabla_{K_+}(t)-\nabla_{K_-}(t).$$

Let us denote by $P_K(x)$ the power series obtained from $\nabla_K(t)$ by substituting $t=\mathrm{e}^x$. Then using (19) we see that x divides $P_{K_\times}(x)$. Hence, if K has more than j points then $P_K(x)$ is divisible by at least x^{j+1}, and hence the mth coefficient of the power series is an invariant of type m. □

7. CONCLUDING REMARKS

1. In this paper we have restricted ourselves to knots in closed irreducible 3-manifolds. However, all of our arguments generalize immediately to compact, irreducible 3-manifolds with incompressible boundary. With some extra work, we can also generalize our results here for many classes of compact irreducible manifolds whose boundary is not incompressible.

These results, as well as the question of the functoriality behavior of *finite type* theories will be addressed in a forthcoming paper.

2. In [14], we prove Theorem 3.7 for links in a large class of rational homology 3-spheres. As an application we obtained a formal power series link invariant, which generalizes the 2-variable Jones polynomial (HOMFLY).

3. It is not hard to see that the submodule inclusions of Theorem 6.2 are in general proper. For example, for the real projective space $\mathbb{R}P^3$ the dimension of the space of type 2 invariants, corresponding to the trivial conjugancy class of $\pi_1(\mathbb{R}P^3)$ is equal to three. On the other hand, there is only one invariant of type 2 for knots in S^3.

For the moment, it is not clear to us what is the relation of the invariants discussed in this paper or in [14], to the Witten–Reshetikhin–Turaev link invariants [19, 27].

Acknowledgements— results contained in this paper formed part of my Ph.D. thesis (Columbia University 1995). I would like to thank my advisor, Joan Birman, for her insightful guidance and support and Xiao-Song Lin for many conversations, which were crucial at the ealry stages of this work. Special thanks are due to John Hempel, Feng Luo and Peter Scott for helpful correspondence and discussions about positions of tori in 3-manifolds. I am grateful to Thang Le and to the referee, who gave an earlier version of this paper a very careful reading and pointed out some mistakes. I am indebted to Paul Kirk and Chuck Livingston for their interest in my work and for their comments and suggestions that improved this paper in many ways. I thank, Dror Bar-Natan, Pierre Deligne, Peter Landweber, Bill Menasco, George Pappas and Vladimir Turaev for useful comments and discussions. Finally, I would like to acknowledge with thanks the hospitality of the Institute for Advanced Study, where this paper was written.

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