Crossing changes and knot invariants

Knot: An embedding $K: S^1 \hookrightarrow \mathbb{R}^3$
or $S^3 = \mathbb{R}^3 \cup \{\infty\}$

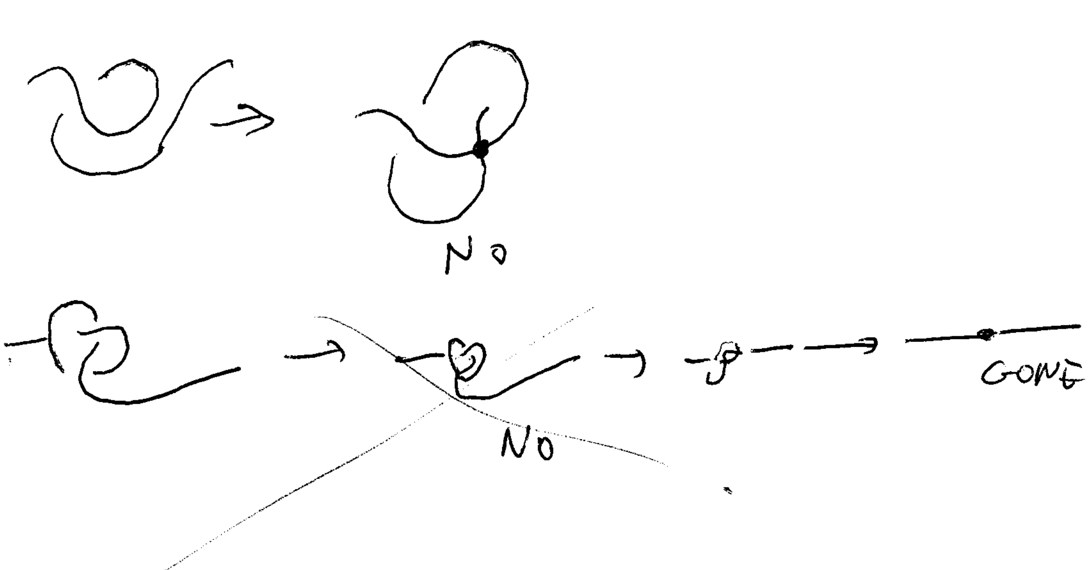
Projection $p: \mathbb{R}^3 \to \mathbb{R}^2$

regular (only ~~transverse~~ transverse double pts)
with an "over-under crossing ~~...~~ indication".

trivial knot.

Knot equivalence: Isotopy.

$K, K': S^1 \to \mathbb{R}^3$ are isotopic iff we can change K to K' by an ambient deformation during which the knot is not allowed to ~~...~~ intersect itself.



he "obvious" questions
- Given two knots $\exists$? an effective process to decide whether they are isotopic.

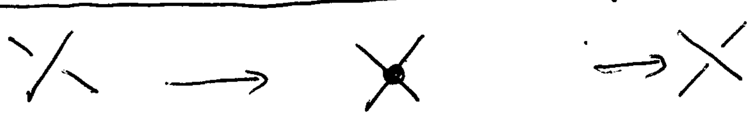
2. What means do we have for distinguishing knots?

Knot invariants: Invariant of K = a quantity $I(K)$ which don't change under knot isotopy,

complete invariant: $I(K) = I(K') \iff K$ isot. to K' ($K = K'$).

- Want complete invariants that can be effectively calculated.

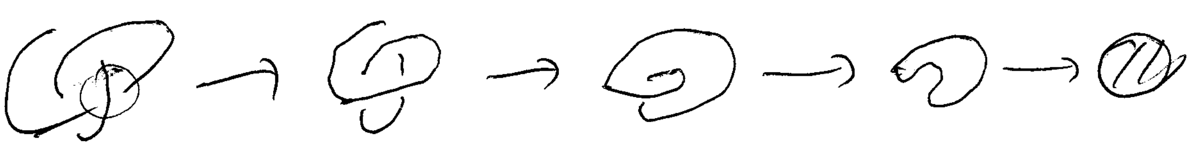
Knot modifications (simplifications)



continuous deformation (generic homotopy).

$\pi_1(\mathbb{R}^3) = \{0\} \Rightarrow$ Any pair of knots is related by a sequence of crossing changes

Any knot can be unknotted by crossing changes



<u>Here, some natural, easy to ask hard to answer questions are.</u>

1) Find ways to determine the minimum number of crossings needed to untie a given knot (calculate the unknotting #: very deep techniques for progress even in the case of knots one sees at the tables! up to 11-crossings. These unknotting numbers were calculated recently by Ozsvath - Szabo).

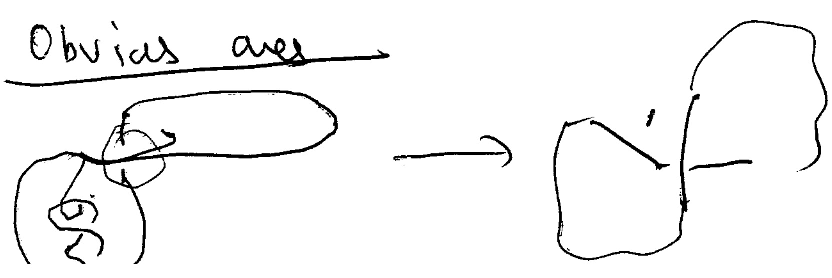
2) Determine (locate) which of crossings ~~changes~~ "simplify" knots (with respect to a complexity function).

3) Determine which crossings leave the isotopy class of a knot unchanged (nugatory).

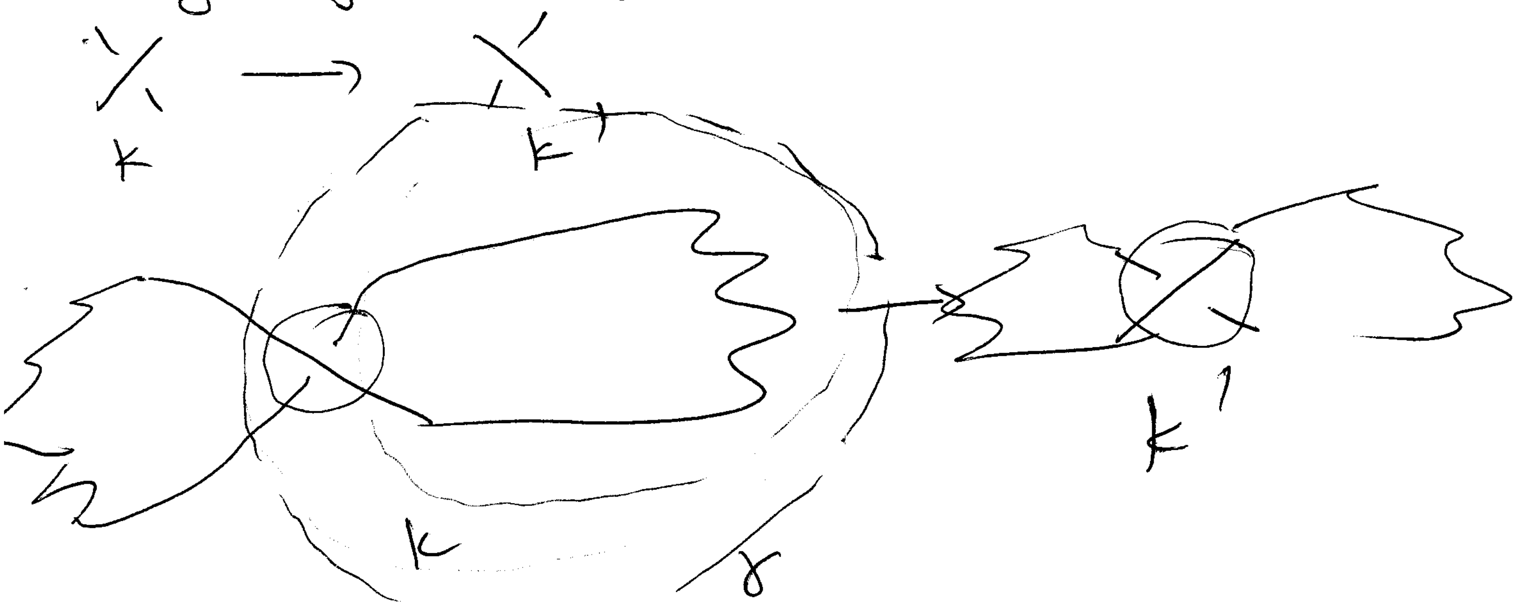
I'll discuss 3: Unknown for even a single crossing change, in general.

<u>A single crossing change</u>: How does a crossing change that will not change the isotopy class of the knot look like.

<u>Obvious ones</u>



A nugatory crossing change



$\exists$ a s.c.c $\gamma \subset \mathbb{R}^2$ containing the crossing and intersecting K, K' exactly twice.

IF both sides of γ contain "knotted" parts K is called composite.

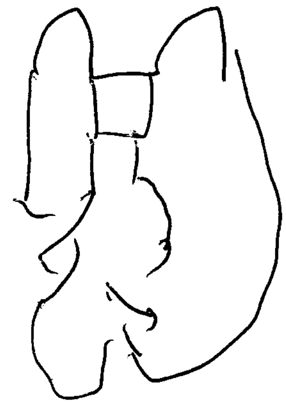
Prime $\Rightarrow$ No such γ can be found for any of the projections of K.

<u>Question</u> (From Kirby's list).

$K \longrightarrow K'$: $K = K' \overset{?}{\Longrightarrow}$ the crossing change is nugatory?

A deep result of D. Gabai ('88) $\Longrightarrow$ yes if K, K' are the trivial knot.
Scharlemann-Thompson (1989)

2. 1. Torisu (1997?) : Yes if K is a 2-bridge knot ~~$\emptyset$~~



$\boxed{\text{Ingredient}}$ "Cyclic surgery thm of Culler-Gordon-Luecke-Shalen (1985)".

Torisu also reduced the problem to prime knots and conjectured Yes in general.

3) (2006): Kalfagianni Yes for $\boxed{\text{Fibered}}$ knots

$\boxed{\text{New Ingredient}}$: A result D. Kotschick about ~~surf~~ mapping class groups of surfaces. He proved this result using techniques from gauge theory (4-dimensional manifolds / Seiberg-Witten invariants)

4) * The general problem seems to be out of reach for the moment.

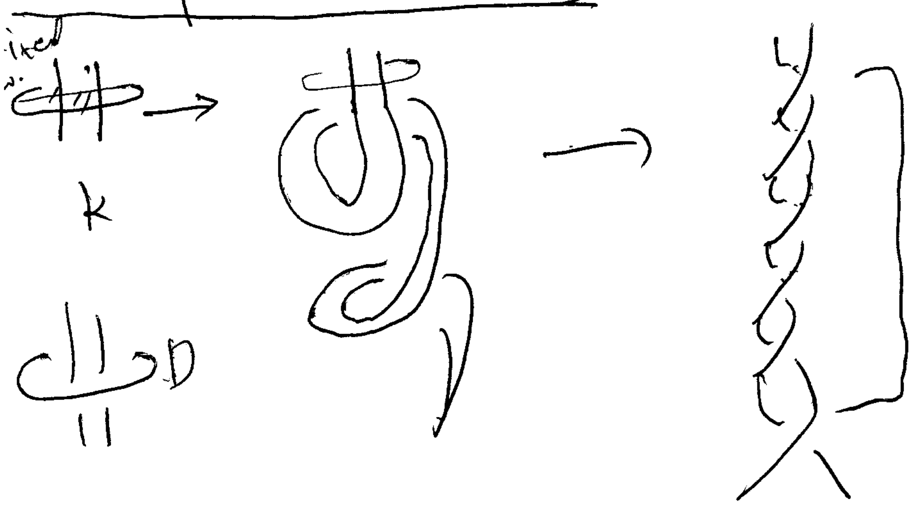
* The question fits into the general framework of the question " When does surgery in a 3-mfd give the same manifold". Cosmetic surgery

…swath-Szabo have made progress recently
n ~~the~~ cosmetic surgery issues but their
…ethods don't seem to ~~…~~ apply
…o the nugatory ~~…~~ crossing problem.

Generalized crossing changes

View it as twisting: (crossing of order 1)

Add q full twists



twist of
order 3 along L.

order 3.

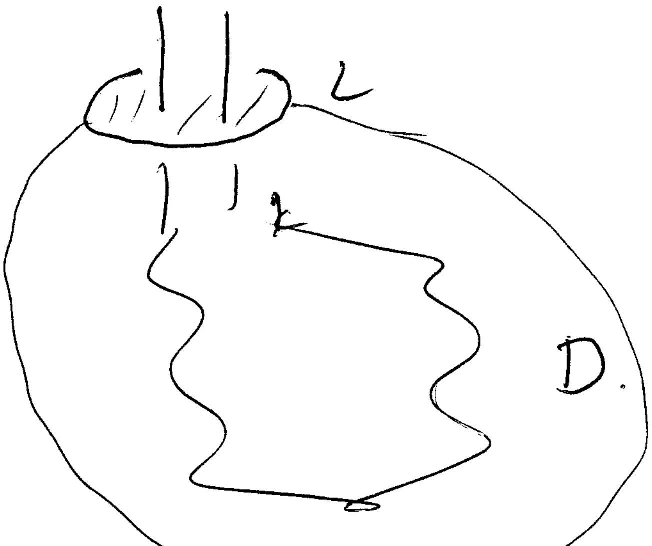
$k(q)$

$k(3)$

Crossing of
order 3 supported
on L.

Oldds-

A generalized crossing supported on L is called nugatory if ∂L bounds a disc in the complement of K



K = knot, L = crossing link of K

$K(q)$ = obtained from K by generalized crossing of order q.

Q when we have $K(q) = K$ for infinitely many q?

Theorem (K – Xiao-Song Lin, 2004)
If $K(q) = K$ for infinitely many q's then L bounds a disc in the complement of K. Every crossing change supported on L is nugatory

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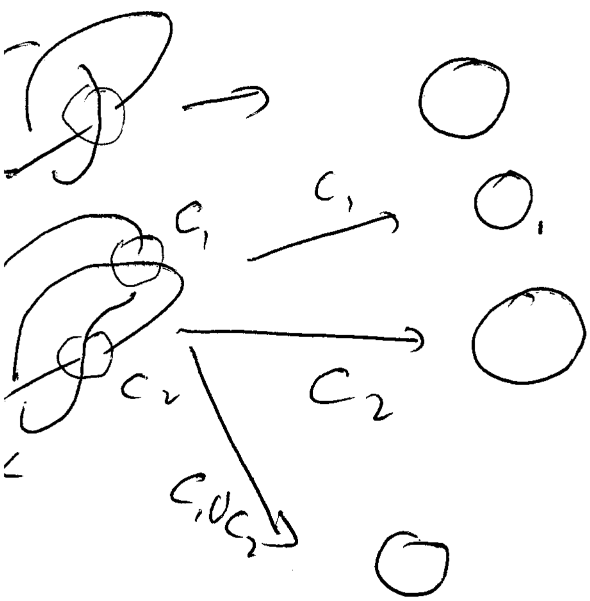
[Dehn surgery techniques / sutured manifold techniques]

Dehn surgery of S^3 along L with slope $1/q$.

Simultaneous crossing changes

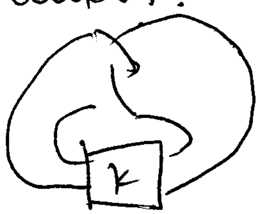


1-adjacent
(unknotting # 1).

K is 2-adjacent to the unknot.

is every unknotting # 1 knot 2-adjacent to the unknot.

No



Whitehead double.
all unknotting # 1

but only 3 such knots are 2-adjacent to the unknot.

Def (n-adjacent) K is called n-adjacent to K' IFF K admits n-crossings on a projection so that changing any $0<m<n$, of them yields a projection of ~~the unknot~~. K'

where does it come from?
How can we tell if $K \xrightarrow{m} K'$
for given K, K'?

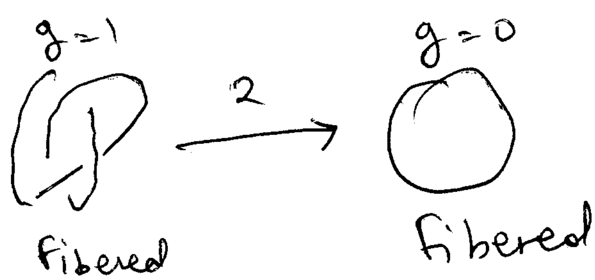
invariants: classical → Alex poly. / genus of a knot
Newer → Vassiliev invariants (finite type)

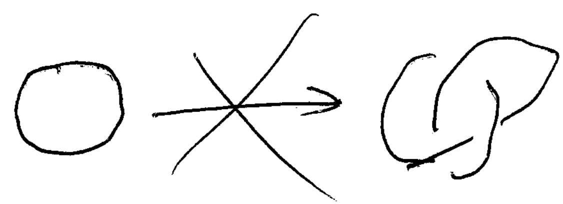
$g(K)$ = smallest genus over all embedded, oriented surfaces spanned by K.

Schematically 

$g(K) = 0 \iff K =$  trivial

Thm (—, 2006) $K \xrightarrow{n} K'$. If K' = fibered then $g(K) \geq g(K')$.

Ex 



assiliev invariants (Finite type invariants)

amilies of integer valued invariants
each coming with an order (a natural #).

Typical examples (but more ..)

$$K \quad J_K(t) = \text{Jones-type polynomial for } K$$

$$\downarrow t = e^h$$

Taylor expansion $\displaystyle\sum_{n=0}^{\infty} \boxed{a_n(k)}\, h^n$

A vassiliev invariant of order n.

Suppose $K \xrightarrow{n} K'$ then K and K' cannot be distinguished by any Vassiliev invariant of order $< n$

~~The ... trees~~ V-invariants are calculable in polynomial time in n crossings

$\therefore$

K, K' ~~...~~ If $\exists$ v-invariant of order n s.t. $v(K) \neq v(K')$ then $K \overset{n}{\not\longrightarrow} K'$

.) Vassiliev invariants grow
polynomially on the # of crossings.
Atractive For calculations).

out our real interest.

onjecture (V. Vassiliev, 1990.

The set of all finite type invariants
of all orders ~~classifies~~ is a complete
set of invariants!

That is: Given K, K' IF $K \neq K'$
$\exists$ an invariant v of some order ~~that~~
such that $v(K) \neq v(K')$

Q Can we have $K \xrightarrow{n} K'$ $\forall n \in \mathbb{N}$?
IF yes then we have a counter example
to Vassiliev's conj.

But

Thm (K — Lin, 2004)
Suppos $K \xrightarrow{n} K'$ For infinitely may $n \in \mathbb{N}$
Then $K = K'$ (isotopic).

Have we proved the conjecture?

No! but we gave plenty of evidence!

Gussarov

K and K' are not distinguishable by any Vassiliev invariant $\Longleftrightarrow$ K is n-equivalent to K'

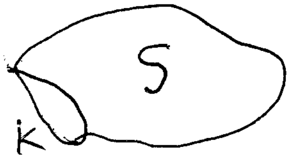
K admits projection with n sets of crossings s.t. changing all crossings in any $0 < m \leq n$ subsets yields K'

If each set is of the form

we verified Vassiliev conjecture in Gussarov's language

Fibered knots

$S^3 - K$ is constructed as follows.



• Start with surface S spanned by K.

• $f: S \to S$ orientation preserving hom. with $f|\partial S = id$

$$S \times [0,1] \Big/ (x,0) \simeq (f(x),1) \cong S^3 - K.$$

$S \times I/f$

f = the monodromy (up to isotopy on S)

f ∈ mapping class group of S.

<u>Clasicall</u> $\quad S_1 \times I/f \cong S_2 \times I/g$

if $hfh^{-1} = g$ | up to isotopy on S_2, $h: S_1 \to S_2$

K = fibered

$K = K'$ $\quad$ K' obtained by crossing change from K.

$S^3 - K = S \times I/f$ $\qquad$ $S^3 - K' = S \times I/fT$

T = Dehn twist on $\partial N(S)$ along L

Roughly speaking

twist extends in handlebody $N(S)$ ~~..... bounded by~~ $L \Rightarrow L$ bounds disc.

K isotopic to K' $\Longrightarrow$ $fT = gfg^{-1}$ or

or $T = f^{-1}gfg^{-1} = [f^{-1}, g]$ $\longleftarrow$ Kotschik say this only can happen when $T = 1$

which in our setting means crossing change nugatory!