CROSSING NUMBERS OF CABLES

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Abstract. We use the degree of the colored Jones knot polynomials to show that the crossing number of a (p, q)-cable of an adequate knot with crossing number c is larger than $q^2\, c$. As an application we determine the crossing number of 2-cables of adequate knots.

1. Introduction

Given a knot K we will use $c(K)$ to denote the crossing number of K, which is the smallest number of crossings over all diagrams that represent K. Crossing numbers are known to be notoriously intractable. For instance their behavior under basic knot operations, such as connect sum of knots and satellite operations, is poorly understood. In particular, the basic conjecture that if K is a satellite knot with companion C then $c(K) \geq c(C)$ is sill open [8, Problem 1.68]. In this note, we prove a much stronger inequality for cables of adequate knots and we determine the exact crossing numbers of infinite families of such knots.

To state our results, for a knot K in the 3-sphere let $N(K)$ denote a tubular neighborhood of K. Given co-prime integers p, q let $K_{p,q}$ denote the (p, q)-cable of K. In other words, $K_{p,q}$ is the simple closed curve on $\partial N(K)$ that wraps p times around the meridian and q-times around the canonical longitude of K. Recall that the writhe of an adequate diagram $D = D(K)$ is an invariant of the knot K [9]. We will use $\mathrm{wr}(K)$ to denote this invariant.

Theorem 1.1. *For any adequate knot K with crossing number $c(K)$, and any coprime integers p, q, we have*

$$c(K_{p,q}) \geq q^2 \cdot c(K) + 1.$$

Theorem 1.1, combined with the results of [6], has significant applications in determining crossing numbers of prime satellite knots. We have the following:

Corollary 1.2. *Let K be an adequate knot with crossing number $c(K)$ and writhe number $\mathrm{wr}(K)$. If $p = 2\,\mathrm{wr}(K) \pm 1$, then $K_{p,2}$ is non-adequate and $c(K_{p,2}) = 4c(K) + 1$.*

The proof of Corollary 1.2 shows that when $p = 2\,\mathrm{wr}(K) \pm 1$, if we apply the $(p, 2)$-cabling operation to an adequate diagram of K the resulting diagram is a minimum crossing diagram of the knot $c(K_{p,2})$. It should be compared with other results in the literature asserting that the crossing numbers of some important classes of knots are realized by a "special type" of knot diagrams. These classes include alternating and more generally adequate knots, torus knots, Montesinos knots [7, 12, 15] and untwisted Whitehead doubles of adequate knots with zero writhe number [6]. We note that these Whitehead doubles and the cables $c(K_{p,2})$ of

Corollary 1.2 are the first infinite families of prime satellite knots for which the crossing numbers have been determined.

Corollary 1.2 allows us to compute the crossing number of $(\pm1, 2)$-cables of adequate knots that are equivalent to their mirror images (a.k.a. amphicheiral) since such knots are known have $\mathrm{wr}(K) = 0$. In particular, since for any adequate knot K with mirror image K^* the connect sum $K\#K^*$ is adequate and amphicheiral, we have the following:

Corollary 1.3. *For any adequate knot K with crossing number $c(K)$ and mirror image K^* let $K^2 := K\#K^*$. Then $c(K^2_{\pm1,2}) = 8\,c(K) + 1$.*

Our results also have an application to the open conjecture on the additivity of crossing numbers [8, Problem 1.68] under connect sums. The conjecture has been proved in the cases where each summand is adequate, [7, 12, 15] both torus knots, [4] and when one summand is adequate and the other an untwisted Whitehead doubles of adequate knot with zero writhe number [6]. To these we add the following:

Theorem 1.4. *Suppose that K is an adequate knot and let $K_1 = K_{p,2}$, where $p = 2\,\mathrm{wr}(K)\pm1$. Then for any adequate knot K_2, the connected sum $K_1\#K_2$ is non-adequate and we have*

$$c(K_1\#K_2) = c(K_1) + c(K_2).$$

It may be worth noting that out of the 2977 prime knots with up to 12 crossings, 1851 are listed as adequate on Knotinfo [11] and thus our results above can be applied to them

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2. Crossing numbers of cables of adequate knots

2.1. **Preliminaries.** Given a knot diagram D, a *Kauffman state* is a choice of either the A-resolution or the B-resolution for each crossing of D as shown in Figure 1. Applying a Kauffman state σ to a diagram leads to a collection $\sigma(D)$ of disjoint simple closed curve called *state circles*. The *all-A state* on a knot diagram D, denoted by σ_A, is the state where the A-resolution is chosen at every crossing of D. Similarly, the *all-B state*, denoted by σ_B, is the state where the B-resolution is chosen at every crossing of D.

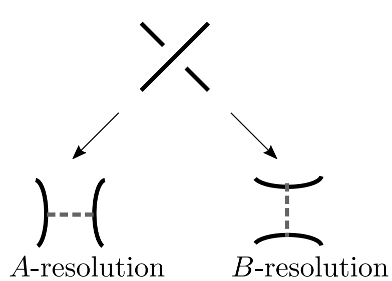

Figure 1. The A- and B-resolution at a crossing. The dashed segments indicate the original location of the crossing.

We will use the following notation:

- $c(D)$ is the number of crossings of the knot diagram D.
- With an orientation on D, $c_+(D)$ and $c_-(D)$ are respectively the number of positive crossings and the number of negative crossings in the knot diagram D following the conventions of Figure 2.
- $v_A(D)$ is the number of state circles in the all-A state, and $v_B(D)$ is the number of state circles in the all-B state.
- The state graphs $\mathbb{G}_A(D)$ and $\mathbb{G}_B(D)$ have vertices the state circles of the all-A and all-B state respectively, and edges segments recording the original location of the crossing. In Figure 1, these segments are indicated in dashed line.
- The *writhe* of a knot diagram D, denoted by $\mathrm{wr}(D) := c_+(D) - c_-(D)$.
- The Turaev genus of D is defined by $2g_T(D) := 2 - v_A(D) - v_B(D) + c(D)$ [16, 3].

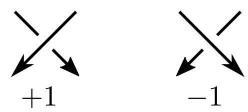

FIGURE 2. A positive crossing and a negative crossing.

Definition 2.1. A knot diagram $D = D(K)$ is called *A-adequate* (resp. *B-adequate*) if $\mathbb{G}_A(D)$ (resp. $\mathbb{G}_B(D)$) has no one-edged loops. A knot is *adequate* if it admits a diagram $D = D(K)$ that is both A- and B-adequate [10, 9].

We recall that if $D = D(K)$ is an adequate diagram the quantities $c(D)$, $c_\pm(D)$ [9, 7, 12, 15] as well as the Turaev genus $g_T(D)$ [1] are minimal over all diagrams representing K. As a result the writhe $\mathrm{wr}(D)$ of adequate diagrams is also constant for K. Thus they are invariants of K and we will denote them by $c(K)$, $c_\pm(K)$, $g_T(K)$, and $\mathrm{wr}(K)$ respectively.

Given a knot K let $J_K(n)$ denote its n-th colored Jones polynomial, which is a Laurent polynomial in a variable t. Let $d_+[J_K(n)]$ and $d_-[J_K(n)]$ denote the maximal and minimal degree of $J_K(n)$ in t and set

$$d[J_K(n)] := 4d_+[J_K(n)] - 4d_-[J_K(n)].$$

Lemma 2.2. [9] *Given a knot diagram $D = D(K)$, for all $n \in \mathbb{N}$, we have the following.*

(a) $d_+[J_K(n)] \leq \frac{c_+(D)}{2}n^2 + O(n)$ and we have equality if D is B-adequate.

(b) $d_-[J_K(n)] \geq -\frac{c_-(D)}{2}n^2 + O(n)$ and we have equality if D is A-adequate.

(c) $d[J_K(n)] \leq 2c(D)n^2 + (4 - 4g_T(D) - 2c(D))n + (4g_T(D) - 4)$, and we have equality if D is adequate.

The set of cluster points $\left\{n^{-2}d[J_K(n)]\right\}'_{n\in\mathbb{N}}$ is known to be finite and the point with the largest absolute value, denoted by dj_K, is called the *Jones diameter* of K.

We recall the following.

Theorem 2.3. [6]*Let K be a knot with Jones diameter dj_K and crossing number $c(K)$. Then,*

$$dj_K \leq 2\,c(K),$$

with equality $dj_K = 2\,c(K)$ if and only if K is adequate.

In particular, if K is a non-adequate knot admitting a diagram $D = D(K)$ such that $dj_K = 2(c(D)-1)$, then we have $c(D) = c(K)$.

Next we recall a couple of results from that give the extreme degrees of the colored Jones polynomials for knots where the degrees $d_{\pm}[J_K(n)]$ are quadratic polynomials.

Proposition 2.4. [5, 2] *Suppose that K is a knot such that $d_+[J_K(n)] = a_2n^2+a_1n+a_0$ and $d_-[J_K(n)] = a_2^*n^2+a_1^*n+a_0^*$ are quadratic polynomials for all $n>0$. Suppose, moreover, that $a_1 \leq 0$, $a_1^* \geq 0$ and that $\frac{p}{q} < 4a_2$ and $\frac{-p}{q} < 4a_2^*$.*

Then for n large enough,

$$4d_+[J_{K_{p,q}}(n)] = q^2 4a_2 n^2 + (q4a_1 + 2(q-1)(p-4qa_2))n + A,$$

$$4d_-[J_{K_{p,q}}(n)] = -q^2 4a_2^* n^2 - (q4a_1^* + 2(q-1)(p-4qa_2^*))n + A^*,$$

where $A, A^* \in \mathbb{Q}$ depend only on K and p, q.

Proof. The first equation is shown in [5] (see also [2]). To obtain the second equation we use the fact that, since $K^*_{-p,q} = (K_{p,q})^*$, we have $d_-[J_{K_{p,q}}(n)] = -d_+[J_{K^*_{-p,q}}(n)]$ and apply the first equation to $K^*_{-p,q}$.

□

Now we recall the second result promised earlier.

Lemma 2.5. [5, 2]*Let the notation and setting be as in Proposition 2.4.*

If $\frac{p}{q} > 4a_2$, then

$$4d_+[J_{K_{p,q}}(n)] = pqn^2 + B,$$

where $B \in \mathbb{Q}$ depends only on K and p, q.

Similarly, if $\frac{-p}{q} > 4a_2^*$, then

$$4d_-[J_{K_{p,q}}(n)] = -pqn^2 + B^*,$$

where $B^* \in \mathbb{Q}$ depends only on K and p, q.

2.2. **Lower bounds and admissible knots.** We will say that a knot K is *admissible* if there is a diagram $D = D(K)$ such that we have

$$dj_K = 2\,(c(D)-1).$$

Our interest in admissible knots comes from the fact that if K is admissible and non-adequate, then by Theorem 2.3, D is a minimal diagram (i.e. $c(D) = c(K)$).

Theorem 2.6. *Let K be an adequate knot and let $c(K)$, $c_{\pm}(K)$ and $\mathrm{wr}(K)$ be as above.*

(a) For any coprime integers p, q, we have

$$c(K_{p,q}) \geq q^2 \cdot c(K). \tag{1}$$

(b) The cable $K_{p,q}$ is admissible if and only if $q = 2$ and $p = q\,\mathrm{wr}(K) \pm 1$.

Proof. Since K is adequate, by Lemma 2.2,

$$4d_+[J_K(n)] - 4d_-[J_K(n)] = 2c(K)n^2 + (4 - 4g_T(K) - 2c(K))n + 4g_T(K) - 4, \tag{2}$$

for every $n \geq 0$.

We distinguish three cases.

Case 1. Suppose that $\frac{p}{q} < 2c_+(K)$ and $\frac{-p}{q} < 2c_-(K)$. Then, $d_+[J_K(n)]$ satisfies the hypothesis of Proposition 2.4 with $4a_2 = 2c_+(K) > 0$ and $d_-[J_K(n)] = -d_+[J_{K^*}(n)]$, where $d_+[J_{K^*}(n)]$ satisfies that hypothesis of Proposition 2.4 with $4a_2^* = 2c_+(K^*) = 2c_-(K)$. The requirement that $a_1 \leq 0$ is satisfied since for adequate knots the linear terms of the degree of $J_K^*(n)$ are multiples of Euler characteristics of spanning surfaces of K. See [5, Lemmas 3.6, 3.7]. Now Proposition 2.4 implies that for sufficiently large n we have that $d_\pm[J_{K_{p,q}}(n)]$ is a quadratic polynomial and the Jones diameter of $K_{p,q}$ is $dj_K = q^2 c(K)$. Hence by Theorem 2.3 we get $c(K_{p,q}) \geq q^2 \cdot c(K)$ which proves part (a) of Theorem 1.1 in this case.

For part (b), we recall that a diagram $D_{p,q}$ of $K_{p,q}$ is obtained as follows: Start with an adequate diagram $D = D(K)$ and take q parallel copies to obtain a diagram D^q. In other words, take the q-cabling of D following the blackboard framing. To obtain $D_{p,q}$ add *t-twists* to D^q, where $t := p - q\,\mathrm{wr}(K)$ as follows: If $t < 0$ then a twist takes the leftmost string in D^q and slides it over the $q-1$ strings to the right; then we repeat the operation $|t|$-times. If $t > 0$ a twist takes the rightmost string in D^q and slides it over the $q-1$ strings to the left; then we repeat the operation $|t|$-times. Now

$$c(D_{p,q}) = q^2\, c(K) + |t|(q-1) = q^2\, c(K) + |p - q\,\mathrm{wr}(K)|(q-1),$$

while $dj_K = 2q^2\, c(K)$. Now setting $2c(D_{p,q}) - 2 = dj_K$, we get $|p - q\,\mathrm{wr}(K)|(q-1) = 1$ which gives that $q = 2$ and $p = q\,\mathrm{wr}(K) \pm 1$. Similarly, if we set $p = q\,\mathrm{wr}(K) \pm 1$ we find that $2c(D_{p,q}) - 2 = dj_K$ must also be true. Hence in this case both (a) and (b) hold.

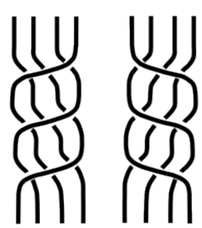

FIGURE 3. Left: 3 positive twists on four strands Right: 3 negative twists on four strands.

Case 2. Suppose that $\frac{p}{q} > 2c_+(K)$. Then by Lemma 2.5, $4d_+[J_{K_{p,q}}(n)] = pqn^2 + B$, where $B \in \mathbb{Q}$ depends only on K and p, q. Since $\frac{p}{q} > 2c_+(K)$, we get $pq > 2c_+(K)q^2$. On the other hand, since $\frac{-p}{q} < 0$, we clearly have $\frac{-p}{q} < 2c_-(K)$, and Proposition 2.4 applies to $d_+[J_{K^*_{-p,q}}(n)]$ for $4a_2^* = 2c_-(K)$. Then

$$4d_+[J_K(n)] - 4d_-[J_K(n)] = d_+[J_K(n)] + 4d_+[J_{K^*_{-p,q}}(n)] > q^2 \cdot c(K),$$

as desired. This finishes the proof for part (a) of the theorem. In this case, we don't get any admissible knots: first note that $p > 2qc_+(K) > q\,\mathrm{wr}(K)$. As in Case 1 we get a diagram $D_{p,q}$ of $K_{p,q}$ with

$$c(D_{p,q}) = q^2\, c(K) + (p - q\,\mathrm{wr}(K))(q-1),$$

while $dj_K = 2q^2\, c_-(K) + p\, q$. Now setting $2c(D_{p,q}) - 2 = dj_K$, and after some straightforward algebra, we find that in order for $K_{p,q}$ to be admissible we must have

$$2(q^2 - q)\, c_-(K) + 2q\, c_+(K) + p\,(q-2) - 2 = 0.$$

However, since $p, c(K) > 0$ and $q \geq 2$, above equation is never satisfied.

Case 3. Suppose that $\frac{-p}{q} > 2c_-(K) > 0$, in which case $\frac{p}{q} < 0 \leq 2c_-(K)$. This case is similar to Case 2 above.

□

Remark 2.7. In [13] inequality (1) is also verified, for some choices of p and q, using crossing number bounds obtained from the ordinary Jones polynomial in [14] and also from the 2-variable Kauffman polynomial. Theorem 1.1 shows that the colored Jones polynomial and the results of [6] provide better bounds for crossing numbers of satellite knots, allowing in particular exact computations for infinite families.

3. Non-adequacy results

To prove the stronger version of inequality (1), stated in Theorem 1.1, we need to know that the cables $K_{p,q}$ are not adequate. This is the main result in this section.

Theorem 3.1. *Let K be an adequate knot with crossing number $c(K) > 0$ and suppose that $\frac{p}{q} < 2c_+(K)$ and $\frac{-p}{q} < 2c_-(K)$. Then, the cable $K_{p,q}$ is non-adequate.*

To prove Theorem 3.1 we need the following lemma:

Lemma 3.2. *Let K be an adequate knot with crossing number $c(K) > 0$ and suppose that $\frac{p}{q} < 2c_+(K)$ and $\frac{-p}{q} < 2c_-(K)$. If $K_{p,q}$ is adequate, then $c(K_{p,q}) = q^2\, c(K)$.*

Proof. By our earlier discussion, for n large enough,

$$4d_+[J_{(K_{p,q}}(n)] - 4d_-[J_{K_{p,q}}(n)] = d_2 n^2 + d_1 n + d_0,$$

with $d_i \in \mathbb{Q}$. By Proposition 2.4, we compute $d_2 = q^2(4a_2 + 4a_2^*) = 2q^2 c(K)$. Now if $K_{p,q}$ is adequate, since $d_2 = 2c(K_{p,q})$, we must have $c(K_{p,q}) = q^2 c(K)$. □

We now give the proof of Theorem 3.1:

Proof. First, we let K, p, and q such that $t := p - q\,\mathrm{wr}(K) < 0$.

Recall that if K has an adequate diagram $D = D(K)$ with $c(D) = c_+(D) + c_-(D)$ crossings and the all-A (rep. all-B) resolution has $v_A = v_A(D)$ (resp. $v_B = v_B(D)$) state circles, then

$$4\,d_-[J_K(n)] = -2c_-(D)n^2 + 2(c(D) - v_A(D))n + 2v_A(D) - 2c_+(D), \tag{3}$$

$$4\,d_+[J_K(n)] = 2c_+(D)n^2 + 2(v_B(D) - c(D))n + 2c_-(D) - 2v_B(D). \tag{4}$$

Equation (3) holds for A-adequate diagrams $D = D(K)$. Thus in particular the quantities $c_-(D), v_A(D)$ are invariants of K (independent of the particular A-adequate diagram). Similarly, Equation (4) holds for B-adequate diagrams $D = D(K)$ and hence $c_+(D), v_B(D)$ are invariants of K. Recall also that $c(D) = c(K)$ since D is adequate.

Now we start with a knot K that has an adequate diagram D then $\mathrm{wr}(D) = \mathrm{wr}(K)$. Hence we have $c_+(D) = c_-(D) + \mathrm{wr}(K)$. Since D is B-adequate and $t < 0$, the cable $D_{p,q}$ is a B-adequate diagram of $K_{p,q}$ with $v_B(D_{p,q}) = qv_B(D)$ and $c_+(D_{p,q}) = q^2c_+(D)$. See Figure 4. Furthermore, since as said above these quantities are invariants of $K_{p,q}$, they remain the same for all B-adequate diagrams of $K_{p,q}$.

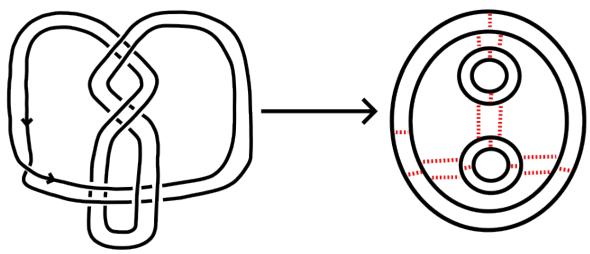

Figure 4. Left: the -1,2 cabling of the figure eight knot. Right: the B-state graph showing that the cabling is B-adequate.

Now assume, for a contradiction, that $K_{p,q}$ is adequate: Then, it has a diagram $\bar{D}$ that is both A and B-adequate. By above observation we must have $v_B(\bar{D}) = v_B(D_{p,q}) = qv_B(D)$ and $c_+(\bar{D}) = c_+(D_{p,q}) = q^2c_+(D)$.

By Lemma 3.2, $c(\bar{D}) = c(K_{p,q}) = q^2c(K)$.

Write

$$4\, d_+[J_{K_{p,q}}(n)] = xn^2 + yn + z,$$

for some $x, y, z \in \mathbb{Q}$.

For sufficiently large n we have two different expressions for x, y, z. On one hand, because $\bar{D}$ is adequate, we can use Equation (4) to determine x, y, z.

On the other hand, using $4\, d_+[J_{K^*_{-p,q}}(n)]$, x, y, z can be determined using Proposition 2.4 with a_2 and a_1 coming from Equation (4) applied to D.

We will use these two ways to find the quantity y. Applying Equation (4) to $\bar{D}$ we obtain

$$y = 2(v_B(\bar{D} - c(\bar{D}))) = 2qv_B(D) - 2q^2c(D) \tag{5}$$

On the other hand, using Proposition 2.4 with a_2 and a_1 coming from Equation (4) we have: $4a_2 = 2c_+(D) = c(D) + \mathrm{wr}(K)$. Also, we have $4a_1 = 2v_B(D) - 2c(D)$.

We obtain

$$y = q(4a_1) - 2q(q-1)(4a_2) + 2(q-1)p = 2qv_B(D) - 2q^2c(D) + 2(q-1)p - 2q(q-1)\mathrm{wr}(K). \tag{6}$$

It follows for the two expressions derived for y from Equations (5) and (6) to agree we must have

$$2q((q-1)2\mathrm{wr}(K) + p) - 2p = 0.$$

However this is impossible since $q > 1$ and p, q are coprime. This contradiction shows that $K_{p,q}$ is non-adequate.

To deduce the result for $K_{p,q}$, with $t(K, p, q) := p - q\,\mathrm{wr}(K) > 0$, let K^* denote the mirror image of K. Note that f $(K_{p,q})^* = K^*_{-p,q}$ and since being adequate is a property that is preserved under taking mirror images, it is enough to show that $K^*_{-p,q}$ is non-adequate. Since $t(K^*, -p, q) := -p - q\,\mathrm{wr}(K^*) = -t(K, p, q) < 0$, the later result follows from the argument above. □

3.1. **Proof of Theorem 1.1 and Corollary 1.2.** By Theorem 2.6, we have

$$c(K_{p,q}) \geq q^2 c(K).$$

We need to show that this inequality is actually strict. Recall that by the proof of Theorem 2.6, if $\frac{p}{q} > 2c_+(K)$ or $\frac{-p}{q} > 2c_-(K)$, then the above inequality is strict so we need to only consider when $\frac{p}{q} < 2c_+(K)$ and $\frac{-p}{q} < 2c_-(K)$. By Theorem 3.1, $K_{p,q}$ is non-adequate. Hence by Theorem 2.3 again we have $2c(K_{p,q}) \neq dj_K$ and the strict inequality follows. □

Next we discuss how to deduce Corollary 1.2:

Proof. If $q = 2$ and $p = q\,\mathrm{wr}(K) \pm 1$, then by Theorem 2.6 $K_{p,q}$ is admissible. Thus by Theorem 2.3, the diagram $D_{p,2}$ constructed in the proof of Theorem 2.6 is minimal. That is $c(K_{p,2}) = c(D_{p,2}) = 4\,c(K) + 1$. □

4. Composite non-adequate knots

Here we give an application of Theorem 1.1 to the question on additivity of crossing numbers under the connected sum of knots. [8, Problems 1.67]. As already mentioned, for adequate knots the crossing number is additive under connected sum. The next result proves additivity for families of knots where one summand is adequate while the other is not.

Theorem 1.4. *Suppose that K is an adequate knot and let $K_1 := K_{p,2}$, where $p = 2\,\mathrm{wr}(K) \pm 1$. Then for any adequate knot K_2, the connected sum $K_1 \# K_2$ is non-adequate and we have*

$$c(K_1 \# K_2) = c(K_1) + c(K_2).$$

Before we proceed with the proof of the theorem we need some preparation. Given a knot K, such that for n large enough the degrees of the colored Jones polynomials of K are quadratic polynomials with rational coefficients, we will write

$$4\,d_+[J_K(n)] = x(K)n^2 + y(K)n + z(K) \text{ and } -4\,d_-[J_K(n)] = x^*(K)n^2 + y^*(K)n + z^*(K).$$

We also write

$$4d_+[J_K(n)] - 4d_+[J_K(n)] = d_2(K)n^2 + d_1(K)n + d_0(K).$$

Now let K_1, K_2 be as in the statement of Theorem 1.4. By assumption and Proposition 2.4, for n large enough the degrees of the colored Jones polynomials of both K_1 and K_2 are quadratic polynomials. For the proof we need the following well known lemma:

Lemma 4.1. [9] *For large enough n, the degrees $d_{\pm}[J_{K_1\#K_2}(n)]$ are polynomials, and we have the following.*

(a) $x(K_1\#K_2) = x(K_1) + x(K_2)$ and $x^*(K_1\#K_2) = x^*(K_1) + x^*(K_2)$.

(b) $y(K_1\#K_2) = y(K_1) + y(K_2) - 2$ and $y^*(K_1\#K_2) = y^*(K_1) + y^*(K_2) - 2$.

(c) $d_2(K_1\#K_2) = d_2(K_1) + d_2(K_2)$.

The second ingredient we need for the proof of Theorem 1.4 is the following lemma.

Lemma 4.2. *Let K be a non-trivial adequate knot, $p = 2\,\mathrm{wr}(K) \pm 1$ and let $K_1 := K_{p,2}$. Then for any adequate knot K_2, the connected sum $K_1\#K_2$ is non-adequate.*

Proof. The claim is proven by applying the arguments applied to $K_1 = K_{p,2}$ in the proofs of Lemma 3.2 and Theorem 3.1 to the knot $K_1\#K_2$ and using the fact that the degrees of the colored Jones polynomial are additive under connected sum.

First we claim that if $K_1\#K_2$ were adequate then we would have

$$c(K_1\#K_2) = 4c(K) + c(K_2) \tag{7}$$

Note that as $p = 2\,\mathrm{wr}(K) \pm 1$, we have $\frac{p}{2} < 2c_+(K)$ and $\frac{-p}{2} < 2c_-(K)$. Hence Proposition 2.4 applies to K_1. Now write

$$4d_+[J_{K_1\#K_2}(n)] - 4d_-[J_{K_1\#K_2}(n)] = d_2(K_1\#K_2)n^2 + d_1(K_1\#K_2)n + d_0(K_1\#K_2).$$

Since we assumed that $K_1\#K_2$ is adequate, we have $d_2(K_1\#K_2) = 2c(K_1\#K_2)$. On the other hand by Lemma 4.1, $d_2(K_1\#K_2) = d_2(K_1) + d_2(K_2) = 2 \cdot 4c(K) + 2c(K_2)$ which leads to (7).

Case 1. Suppose that $p - 2\,\mathrm{wr}(K) = -1 < 0$.

Start with $D = D(K)$ an adequate diagram and let $D_1 := D_{p,2}$ be constructed as in the proof of Theorem 2.6. Also let D_2 be an adequate diagram of K_2. As in the proof of Theorem 3.1 conclude that $D_1\#D_2$ is a B-adequate diagram for $K_1\#K_2$ and that the quantities $v_B(D_1\#D_2) = 2v_B(D) + v_B(D_2) - 1$ and $c_+(D_1\#D_2) = 4c_+(D) + c_+(D_2)$ are invariants of $K_1\#K_2$.

Let $\bar{D}$ be an adequate diagram. Then

$$v_B(\bar{D}) = v_B(D_1\#D_2) = 2v_B(D) + v_B(D_2) - 1 \text{ and } c_+(\bar{D}) = 4c_+(D) + c_+(D_2).$$

Next we will calculate the quantity $y(K_1\#K_2)$ of Lemma 4.1 in two ways: Firstly, since we assumed that $\bar{D}$ is an adequate diagram for $K_1\#K_2$, applying Equation (4), we get

$$y(K_1\#K_2) = 2(v_B(\bar{D}) - c(\bar{D})) = 2(2v_B(D) + v_B(D_2) - 1 - 4c(D) - c(D_2)).$$

Secondly, using by Proposition 2.4 we get $y(K_1) = 2(2v_B(D) - 4c(D) + p + 2\,\mathrm{wr}(K))$.

Then by Lemma 4.1,

$$y(K_1\#K_2) = y(K_1) + y(K_2) - 1 = 2(2v_B(D) - 4c(D) + p - 2\,\mathrm{wr}(K)) + v_B(D_2) - c(D_2)) - 1.$$

Now note that in order for the two resulting expressions for $y(K_1\#K_2)$ to be equal we must have $2(p - 2\,\mathrm{wr}(K)) = 1$ which contradicts our assumption that $p - 2\,\mathrm{wr}(K) = -1$. We conclude that $K_1\#K_2$ is non-adequate.

Case 2. Assume now that $p - 2\,\mathrm{wr}(K) = 1$. Since $(K_{p,2})^* = K^*_{-p,2}$ and since being adequate is a property that is preserved under taking mirror images, it is enough to show that $K^*_{-p,2} \# K^*_2$ is non-adequate. Since $-p - 2\,\mathrm{wr}(K^*) = -(p - 2\mathrm{wr}(K))) = -1$, the later result follows from the argument above.

□

Now we give the proof of Theorem 1.4.

Proof. Note that if K is the unknot then so is $K_{p,2}$ and the result follows trivially. Suppose that K is a non-trivial knot. Then by Lemma 4.2 we obtain that $K_1 \# K_2$ is non-adequate.

As discussed above $dj_{K_1} = 2(4c(K)) = 2(c(D_{\pm 1,2}) - 1)$. On the other hand, $dj_{K_2} = 2c(D_2) = 2c(K)$ where D_2 is an adequate diagram for K_2. Hence, by Lemma 4.1, $dj_{K_1 \# K_2} = 2(c(D_1 \# D_2) - 1)$, where $D_1 = D_{\pm 1,2}$. By Theorem 2.3,

$$c(K_1 \# K_2) = c(D_1 \# D_2) = c(D_1) + c(D_2) = c(K_1) + c(K_2),$$

where the last equality follows since, by Theorem 1.1, we have $c(K_1) = c(D_1) = c(D_{p,2})$. □

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