On the growth of Turaev-Viro 3-manifold invariants

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Notation/Definitions

Quantum integer: $r \geqslant 3$ odd integer and $q = e^{\frac{2i\pi}{r}}$.

$$\{n\} = q^n - q^{-n} = 2\sin(\frac{2n\pi}{r}) = 2\sin(\frac{2\pi}{r})[n], \text{ where } [n] = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{2\sin(\frac{2n\pi}{r})}{2\sin(\frac{2\pi}{r})}.$$

Quantum factorial: $\{n\}! = \prod_{i=1}^{n} \{i\}$.

Set of colors: $I_r = \{0, 2, 4 \ldots, r-3\}$ even integers less than $r-2$.

Admissible Triple: (a_i, a_j, a_k) of elements in I_r,

$$a_i + a_j + a_k \leqslant 2(r-2), \quad \textit{and}$$

$$a_i \leqslant a_j + a_k, \; a_j \leqslant a_i + a_k, \; \; a_k \leqslant a_i + a_j.$$

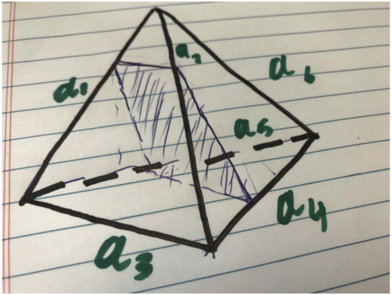
$$\Delta(a_i, a_j, a_k) = \zeta_r^{\frac{1}{2}} \left(\frac{\{\frac{a_i+a_j-a_k}{2}\}!\{\frac{a_j+a_k-a_i}{2}\}!\{\frac{a_i+a_k-a_j}{2}\}!}{\{\frac{a_i+a_j+a_k}{2}+1\}!} \right)^{\frac{1}{2}}$$

where $\zeta_r = 2\sin(\frac{2\pi}{r})$.

Admissible 6-tuple: $(a_1, a_2, a_3, a_4, a_5, a_6) \in I_r^6$ each triple is dmisibble

$$F_1 = (a_1, a_2, a_3), \quad F_2 = (a_2, a_4, a_6), \quad F_3 = (a_1, a_5, a_6) \text{ and } F_4 = (a_3, a_4, a_5).$$

Tetrahedron colorings: Given an admissible 6-tuple:



Faces : $T_1 = \dfrac{a_1 + a_2 + a_3}{2}, \quad T_2 = \dfrac{a_1 + a_5 + a_6}{2}, \quad T_3 = \ldots \text{ and } T_4 = \ldots.$

Quadrilaterals:

$$Q_1 = \frac{a_1 + a_2 + a_4 + a_5}{2}, \quad Q_2 = \frac{a_1 + a_3 + a_4 + a_6}{2} \text{ and } Q_3 = \frac{a_2 + a_3 + a_5 + a_6}{2}.$$

Quantum 6j-symbol: Given admissible 6-tuple
$\alpha := (a_1, a_2, a_3, a_4, a_5, a_6) \in I_r^6$,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{vmatrix} = \Delta(\alpha) \times \sum_{z=\max\{T_1,T_2,T_3,T_4\}}^{\min\{Q_1,Q_2,Q_3\}} \frac{(-1)^z\{z+1\}!}{\prod_{j=1}^4\{z-T_j\}!\prod_{k=1}^3\{Q_k-z\}!} \tag{1}$$

where

$$\Delta(\alpha) := (\zeta_r)^{-1}(\sqrt{-1})^{\lambda}\prod_{i=1}^{4}\Delta(F_i),$$

and

$$\lambda = \sum_{i=1}^{6} a_i,$$

and

$$\zeta_r = 2\sin(\frac{2\pi}{r}).$$

Colorings of Triangulations

Given a compact orientable 3-manifold M consider a triangulation τ of M. If $\partial M \neq \emptyset$ allow τ to be a (partially) *ideal triangulation*: some vertices of the tetrahedra are truncated and the truncated faces triangulate ∂M.

- V=set of vertices of τ which do not lie on ∂M.
- E= set of interior edges (thus excluding edges coming from the truncation of vertices).
- *Admissible coloring at level r:* An assignment

$$c : E \longrightarrow I_r$$

  so that edges of each tetrahedron get an *admissible 6-tuple*.
- Given a coloring c and an edge $e \in E$ let

$$|e|_c = (-1)^{c(e)}[c(e) + 1].$$

- For Δ a tetrahedron in τ let $|\Delta|_c$ be the quantum 6j-symbol corresponding to the admissible 6-tuple assigned to Δ by c.

The invariant

- $A_r(\tau)$= the set of r-admissible colorings of τ
- $\eta_r = \dfrac{2\sin(\frac{2\pi}{r})}{\sqrt{r}}$.
- Turaev-Viro invariants as a state-sum over $A_r(\tau)$.

Theorem (Turaev-Viro 1990)

Let M be a compact, connected, orientable manifold closed or with boundary. Let b_2 denote the second $\mathbb{Z}_2$-Betti number of M. Then the state sum

$$TV_r(M) = 2^{b_2-1}\ \eta_r^{2|V|} \sum_{c\in A_r(\tau)} \prod_{e\in E} |e|_c \prod_{\Delta\in\tau} |\Delta|_c, \qquad (2)$$

is independent of the partially ideal triangulation τ of M, and thus defines a topological invariant of M.

- 6j-sympols satisfy identities (Biedenharn-Elliot identity, Orthogonality relation). These identities are used to show that state sum in 2 is invariant under Pachner moves of triangulations of M. Thus invariant of M.

Turaev-Viro invariants and hyperbolic volume

- Families of of real valued invariants $TV_r(M,q)$ by Turaev-Viro.
- For this talk: $TV_r(M) := TV_r(M, e^{\frac{2\pi i}{r}})$, $r = odd$ and $q = e^{\frac{2\pi i}{r}}$.
- For experts: These correspond to the $SO(3)$ quantum group.
- (Q. Chen- T. Yang, 2015): studied the "large r" asymptotics for hyperbolic 3-manifolds experimentally. Looked at

$$\frac{2\pi}{r}\log(TV_r(M, e^{\frac{2\pi i}{r}})), \quad \text{as} \quad r \longrightarrow \infty.$$

- Gave experimental evidence supporting (volume conjecture).

Conjecture. [C-Y] For M hyperbolic 3-manifold of finite volume

$$\lim_{r\to\infty}\frac{2\pi}{r}\log(TV_r(M, e^{\frac{2\pi i}{r}})) = \mathrm{Vol}\,(M),$$

where r runs over odd integers.

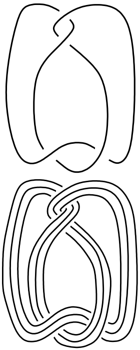
Colored Jones Polynomial connection

- For $M = S^3 \setminus L$, a link complement in S^3, the invariants $TV_r(M)$ can be expressed in terms of the colored Jones polynomial of L.
- *Colored Jones Polynomials*: Infinite sequence of Laurent polynomials $\{J_{K,n}(t)\}_n$ encoding the *Jones polynomial* of K and these of the links K^s that are the parallels of K.
- Formulae for $J_{K,n}(t)$ come from representation theory of $SU(2)$ (decomposition of tensor products of representations). For example, They look like

$$J_{K,1}(t) = 1, \qquad J_{K,2}(t) = J_K(t)\text{- Original JP}$$

$$J_{K,3}(t) = J_{K^2}(t) - 1, \quad J_{K,4}(t) = J_{K^3}(t) - 2J_K(t), \ \ldots$$

- Kashaev-Murakami-Murakami Volume Conjecture (2000).



Relation-calculations

Theorem (Detcherry-K.-Yang, 2016)

Let L be a link in S^3 with n components.

$$TV_r(S^3 \smallsetminus L, q) = 2^{n-1}(\eta_r')^2 \sum_{1 \leqslant \mathbf{i} \leqslant \frac{r-1}{2}} |J_{L,\mathbf{i}}(q^2)|^2.$$

- Verified the Chen-Yang conjecture for some examples: Borromean rings, Figure-eight knot.
- Numerical evidence: Asymprotics of TV_r behave well under “cutting/glueing” along boundary tori—Recover *Gromov norm* (simplicial volume) State “simplicial volume conjecture” that is compatible with disjoint unions of links and connect sums (Warning: Original volume conjecture is not!).
- Observe “new” exponential growth phenomena of the colored Jones polynomial at values that are not predicted by the Kashaev-Murakami-Murakami conjecture or generalizations.

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Remark C-Y conjecture holds for.3-manifolds obtained by integer surgery on figure-8 (Ohtsuki, 2017), infinite family of hyperbolic links in $S^1 \times S^2$ (DKY).

Geometric decomposition

Theorem (Knesser, Milnor 60's, Jaco-Shalen, Johanson 1970, Thurston 1980 + Perelman 2003)

M=oriented, compact, with empty or toroidal boundary.

1. *There is a unique collection of 2-spheres that decompose M*

$$M = M_1 \# M_2 \# \ldots \# M_p \# (\# S^2 \times S^1)^k,$$

   where $M_1, \ldots, M_p$ are compact orientable irreducible 3-manifolds.

2. *For M=irreducible, there is a unique collection of disjointly embedded essential tori $\mathcal{T}$ such that all the connected components of the manifold obtained by cutting M along $\mathcal{T}$, are either Seifert fibered manifolds or hyperbolic.*

- *Seifert fibered manifolds:* For this talk, think of it as

$$S^1 \times \text{surface with boundary} + \text{union of solid tori.}$$

- *Hyperbolic:* Interior admits complete, hyperbolic metric of finite volume. Hyperbolic metric is essentially unique (Mostow rigidity).

Gromov Norm/Volume highlights:

- Recall M uniquely decomposes along spheres and tori into disjoint unions of Seifert fibered spaces and hyperbolic pieces $M = S \cup H$,
- Gromov, Thurston, 80's:
- *Gromov norm of M:* $||M|| = v_{\mathrm{tet}} \mathrm{Vol}\,(H)$, $\mathrm{Vol}\,(H)$ is the sum of the hyperbolic volumes of components of H and v_{tet} is the volume of the regular hyperbolic tetrahedron.
- $||M||$ is additive under disjoint union and connected sums of manifolds.
- If M hyperbolic $||M|| = v_{\mathrm{tet}} \mathrm{Vol}\,(M)$.
- If M Seifert fibered then $||M|| = 0$
- If M contains an embedded torus T and M' is obtained from M by cutting along T then

  $$||M|| \leqslant ||M'||.$$

  Moreover, the inequality is an equality if T is incompressible in M.

Turaev-Viro invariants and Gromov norm?

- M compact, orientable 3-manifold with empty or toroidal boundary. $TV_r(M) := TV_r(M, e^{\frac{2\pi i}{r}})$, $r = odd$ and $q = e^{\frac{2\pi i}{r}}$. One can ask

$$\lim_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M, e^{\frac{2\pi i}{r}})) = v_{\text{tet}}||M||?$$

- Let

$$LTV(M) = \limsup_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M)), \quad ITV(M) = \liminf_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M)).$$

- What can we say? $LTV(M) < \infty$? $ITV(M) \geq 0$? $ITV(M) > 0$?
- Perhaps more robust conjecture
- **Question.** (*Coarse Volume Conjecture?*) Are there universal constants $A, B > 0$ such that for every M we have

$$A \cdot ||M|| \leqslant ITV(M) \leqslant LTV(M) \leqslant B \cdot ||M||?$$

Results:

Theorem (Detcherry-K., 2017)

There exists a universal constant $B > 0$ such that for any compact orientable 3-manifold M with empty or toroidal boundary we have

$$LTV(M) \leqslant B \cdot ||M||,$$

where the constant B is about 8.3581×10^9.

- Better bounds in special cases— For "most" links in S^3, $LTV(S^3 \smallsetminus L) \leqslant 10.5 \, \mathrm{Vol}\,(S^3 \smallsetminus L)$
- If $M = S^3 \smallsetminus L$ hyperbolic link complement with $LTV(M) = \mathrm{Vol}\,(M)$ (e.g. Borromean rings). We have

$$LTV(M(s_1, \ldots, s_k)) \leqslant B(\ell_{\min}) \, \mathrm{Vol}\,(M(s_1, \ldots, s_k)),$$

where $B(\ell_{\min})$ is a function that approaches 1 as $\ell_{\min} \to \infty$.

Manifolds with $||M|| = 0$

- LT is subadditive under connect sum:

$$LTV(M_1 \# M_2) \leq LTV(M_1) + LTV(M_2),$$

- and under disjoint unions of 3-manifolds.
- Suppose M compact, orientable with empty or toroidal boundary and such that $||M|| = 0$ (e.g. M a Seifert fibered manifold.) Then

$$LTV(M) \leq 0.$$

Corollary

For any link $K \subset S^3$ (or $K \subset S^1 \times S^2$) with $||S^3 \smallsetminus K|| = 0$, we have

$$LTV(M) = \lim_{r \to \infty} \frac{2\pi}{r} \log(TV_r(M \smallsetminus K)) = v_{\text{tet}}.||S^3 \smallsetminus K|| = 0,$$

where r runs over all odd integers.

Lower bounds?

- We want: There is $A > 0$, such that for $r >> 0$, $\log|TV_r(M)| \geqslant A||M||\ r$, or

$$ITV(M) = \limsup_{r\to\infty} \frac{2\pi}{r}\log(TV_r(M)) \geqslant A||M||.$$

- Weaker: Establish *exponential growth:* If $||M|| > 0$, then $ITV(M) > 0$.

Results:

Corollary (Detcherry-K, 2017)

Let M the complement of the Figure-8 knot. Then for every link $L \subset M$, we have

$$ITV(M \smallsetminus L) > 2v_{\text{tet}}, \quad \text{where} \quad v_{\text{tet}} = 1.0149.$$

- Let M the complement of the Figure-8 or the Borromean rings. Then for any link $L \subset M$, we have $ITV(M \smallsetminus L) > 2v_{\text{oct}}$, where $v_{\text{oct}} = 3.6638$.
- (w. Detcherry-Yang) Any closed 3-manifold N contains a hyp. link K s.t.

$$LTV(N \smallsetminus K) = ITV(N \smallsetminus K) = vol(N \smallsetminus K).$$

- For $M := N \smallsetminus K$ and any $L \subset M$, we have $ITV(M \smallsetminus (L \cup K)) > \text{Vol}(N \smallsetminus K)$.

Key point: Cutting along tori

- Invariants $ITV(M)$ and $LTV(M)$ do not increase under glueing along tori.

Theorem (Detcherry-K)

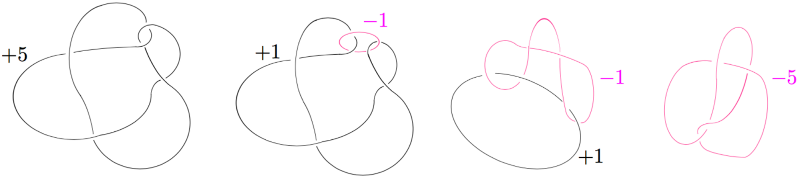
Let M be a compact oriented 3-manifold with empty or toroidal boundary. Let $T \subset M$ be an embedded torus and let M' be the manifold obtained by cutting M along T. Then

$$ITV(M) \leqslant ITV(M') \text{ and } LTV(M) \leqslant LTV(M').$$

- In particular, $ITV(M),\ LTV(M)$ do not increase under Dehn filling! (compare, Gromov norm)
- Ingredients:[Roberts, Beneditti-Petronio, 90's] T-V invariants can be computed as part of a Topological Quantum Field Theory (TQFT); this involves by cutting and gluing 3-manifolds along surfaces.
- For experts: The TQFT is the $SO(3)$- Reshetikhin-Turaev and Witten TQFT as constructed by Blanchet, Habegger, Masbaum and Vogel (1995)

An example: Knot 5_2 and parents

- $K(p)$= 3-manifold obtained by p-surgery on M.
- $LTV(4_1(-5)) = \mathrm{Vol}\,(4_1(-5)) \simeq 0.9813688 > 0$ [Ohtsuki, 2017]
- Observe $5_2(5)$ is homeomorphic to $4_1(-5)$.



- Dehn filling result implies $ITV(S^3 \smallsetminus 5_2) \geqslant ITV(5_2(5)) = ITV(4_1(-5)) > 0$
- But Dehn filling result also implies that for any link containing 5_2 as a component we have exponential growth

$$ITV(S^3 \smallsetminus L) \geqslant ITV(S^3 \smallsetminus 5_2) > 0.$$

Outline of proof of main result: (Upper bound)

1. Study the large-r asymptotic behavior of the quantum 6j-symbols, and using the state sum formulae for the invariants TV_r, to give a linear upper bound of $LTV(M)$ in terms of the number of tetrahedra in any triangulation of M. In particular, $LTV(M) < \infty$.
2. involves analytical estimates of quantum 6j-sympols.
3. Combine with work of W. of Thurston to establish the hyperbolic case:
4. **Hyperbolic Case:** There is a constant $B > 0$ such that for any hyperbolic 3-manifold M, then $LTV(M) \leq B||M||$.
5. Use TQFT properties to show that if M is a Seifert fibered manifold, then $LTV(M) \leq 0 = ||M||$.
6. Use the geometric decomposition of 3-manifolds and the compatibility properties (subadditivity) of the Gromov norm and the invariant LTV with respect to this decomposition.

Bounding LTV

- Quantum factorials estimates with Lobachevsky function give: For any r-admissible 6-tuple (a, b, c, d, e, f), we have that

$$\frac{2\pi}{r} \log \left(ev_r \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{vmatrix} \right) \leqslant v_8 + 8\Lambda(\frac{\pi}{8}) + O(\frac{\log r}{r}).$$

- Optimal estimate: Upper bound should be v_8. (Related work: Constantino, Chen-J. Murakami, Detcherry-K.-Yang).
- This gives

Theorem

Suppose that M is a compact, oriented manifold with a triangulation consisting of t tetrahedra. Then, we have

$$LTV(M) \leqslant 2.08\ v_8\ t,$$

where $v_8 \simeq 3.6638..$ is the volume of a regular ideal octahedron.

Bounding LTV, cont'

- The analysis in the proof of the so called Jorgensen-Thurston Theorem in Thurston's Notes gives the following:

Theorem (Thurston, 80's)

There exists a universal constant C_2, such that for any complete hyperbolic 3-manifold M of finite volume, there exists a link L in M and a partially ideal triangulation of $M \setminus L$ with less than $C_2||M||$ tetrahedra.

- Proof comes from the thick-thin decomposition of hyperbolic manifolds. The constant C_2 in this theorem can be explicitly estimated, $C_2 = 1.101 \times 10^9$.
- This Theorem combined with bound of *LTV* in terms of tetrahedra (last theorem) finishes the Hyperbolic Case.