# Jones polynomials and hyperbolic geometry of knots

with

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## Setting /Notation

For $K$ a hyperbolic knot, vol$(S^3 \setminus K)$:= the volume of $K$.

**colored Jones polynomials:** Invariants

$$\{J_K^n(t) \in \mathbf{Z}[t, t^{-1}] \quad , n = 2, 3, ...\}$$

determined by the "classical" Jones polynomials of $K$ and the cables of $K$. For $n = 2$, we have the JP of $K$, denoted by $J_K(t)$.

**Question:** [ *Coarse Volume Conjecture* (**CVC**)] Are there bounded constants $C_1$, $C_2$, and $B_K$:= a function of the absolute values of the coefficients of $J_K^n$, so that

$$C_1 \; B_K - C_2 < \text{vol}(S^3 \setminus K) < C_3 \; B_K + C_4 \; ?$$

## Motivation/context

**Volume Conjecture (VC).** (Kashaev, Murakami-Murakami) For every hyperbolic knot $K$, we have

$$2\pi \lim_{n\to\infty} \frac{\log \left| J_K^n(e^{2\pi i/n}) \right|}{n} = \mathrm{vol}(S^3 \setminus K),$$

where $e^{2\pi i/n}$ is a primitive $n$-th root of unity.

If the **VC** is true, one expects co–relations between $\mathrm{vol}(S^3 \setminus K)$ and the coefficients of $J_K^n(t)$, at least for large values of $n$. For example, for $n \gg 0$ one would have

$$\mathrm{vol}(S^3 \setminus K) < C||J_K^n||,$$

where $||J_K^n||$ :=the sum of absolute values of the coefficients of $J_K^n(t)$, $C$ is a constant independent of $K$.

## Relations among colored JP's:

Garoufalidis-Le: The invariants $J_K^n$ satisfy recursive relations; given $K$ finitely many $J_K^n$'s determine the whole sequence of the colored JP's. From this point of view, one hopes for linear co–relations between the volume of $K$ and a function $B_K$ depending on the coefficients of finitely many $J_K^n$. This works particularly well for **adequate knots** (definition later):

$$J_K^n(t) = \underbrace{\alpha_n t^{s_n} + \beta_n t^{s_n - 1}} + \ldots + \underbrace{\beta'_n t^{r_n + 1} + \alpha'_n} t^{r_n}$$

Dasbach-Lin: For $K$ adequate, $\alpha_n$, $\beta_n$ (**head**) and $\alpha'_n$ $\beta'_n$ (**tail**) are independent of $n$!

**Notation.** We will write $\alpha, \beta$ for the head and $\alpha', \beta'$ for the tail coefficients.

**The quantity** $|\beta|+|\beta'|$ gives two sided linear bounds of volume.

- Alternating knots; Dasbach-Lin using work of Lackenby on volume and **twist numbers**.

– Large classes of non-alternating knots; Futer-K.-Purcell

**Strong CVC?** Can $J_K(t)$ suffice for coarse volume estimates?

– **Several results** (Alternating, adequate, tangle summations, closed 3-braids;) **and**

–**Numerical evidence** ($\leq 16$ crossings, "geometrically simple" knots in Snappea census)

suggest **Yes**.

## How to get CVC relations

1. Start with a "reduced" knot diagram $D := D(K)$ of $K$.

2. Find a diagrammatic quantity that estimates (computes) coefficients of the Jones polynomials.

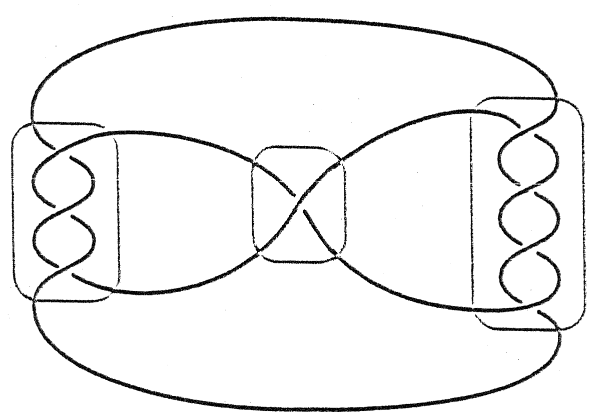
3. Prove that the same diagrammatic quantity also estimates volume.

Two approaches; two key ingredients:

– Estimating volume change under "long" Dehn filling.

– Estimating volume via Guts theory (Agol-Storm-Thurston).

**Key quantity: the twist number.** We consider two crossings of a knot diagram $D := D(K)$ equivalent if they are connected by a string of bigons. The equivalence classes are called twist regions. The total number of twist regions is the **twist number** of $D$.



$t(D) = 3$

**Theorem.** (Lackenby- Agol-D. Thurston) For any diagram $D$ of a hyperbolic knot $K$

$$\mathrm{vol}(S^3 \setminus K) < 10 v_3 (t(D) - 1),$$

where $v_3 \approx 1.015$ is the volume of a regular ideal tetrahedron.

## Lower bounds: Dehn filling approach

## Twist number estimates volume.

Sample result: Highly twisted knots.

**Theorem.** (FKP) Suppose $D(K)$ is a prime, twist–reduced diagram of a knot $K$. Assume that $t(D) \geq 2$ and that each region contains at least 7 crossings. Then, $K$ is hyperbolic and

$$\frac{v_3}{3}\, t(D) \;<\; \mathrm{vol}(S^3 \setminus K) \;<\; 10\, v_3\, (t(D) - 1).$$

**Other classes of knots.** (1) Lackenby: Alternating knots;— FKP: (2) "long enough" closed 3-braids; (3) diagrams of two "generalized" twist regions; (4) "long enough" Conway sums of tangles.

**Key Ingredient: Volume change estimate under Dehn filling with long slopes.**

**Theorem.** Let $M$ be a complete, finite–volume hyperbolic manifold with cusps. Suppose that $M(\mathrm{s})$ is obtained by Dehn filing along some or all the cusps of $M$ with slopes each of length greater than $2\pi$. Denote the minimal slope length by $l_{\text{min}}$. Then, if $M(\mathrm{s})$ is hyperbolic*, we have

$$\text{vol}(M) > \text{vol}(M(\mathrm{s})) \geq \left(1 - \left(\frac{2\pi}{l_{\text{min}}}\right)^2\right)^{3/2} \text{vol}(M).$$

* Always true by Geometrization.

**Estimating** vol$(S^3 \setminus K)$.

- **Step 1.** Represent $S^3 \setminus K$ by Dehn filling on $S^3 \setminus L$; $L$= "nice"

"nice" = vol$(S^3 \setminus L)$ can be estimated from below in terms of twist number and the filling slopes have length $> 2\pi$.

- **Step 2.** Apply Dehn filling Theorem to bound the volume vol$(S^3 \setminus K)$ in terms of twist number.

- **Step 3.** Relate the coefficients of the Jones polynomial to twist number to establish the CVC.

**For example.** We estimate $|\beta| + |\beta'|$ in terms of $t(D)$ to obtain.

**Theorem** Let $K \subset S^3$ be a link with a prime, twist–reduced, **adequate** diagram $D(K)$. Assume that $D(K)$ has $t(D) \geq 2$ twist regions, and that each region contains at least 7 crossings. Then $K$ is a hyperbolic link, satisfying

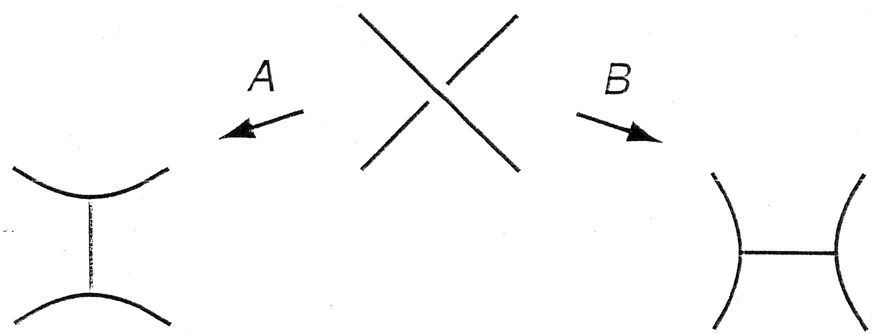
$$\frac{v_3}{6}\,(|\beta|+|\beta'|) \;<\; \mathrm{vol}(S^3\backslash K) \;<\; 30\, v_3\,(|\beta|+|\beta'|-1).$$

**Question.** Why is $|\beta| + |\beta'|$ related to hyperbolic volume?

The Guts approach provides an intrinsic explanation.

## The Guts approach for Volume bounds

**State graphs:** A Kauffman state $\sigma$ of a knot diagram is a choice of $A-$ or $B-$ splicing at every crossing of D:

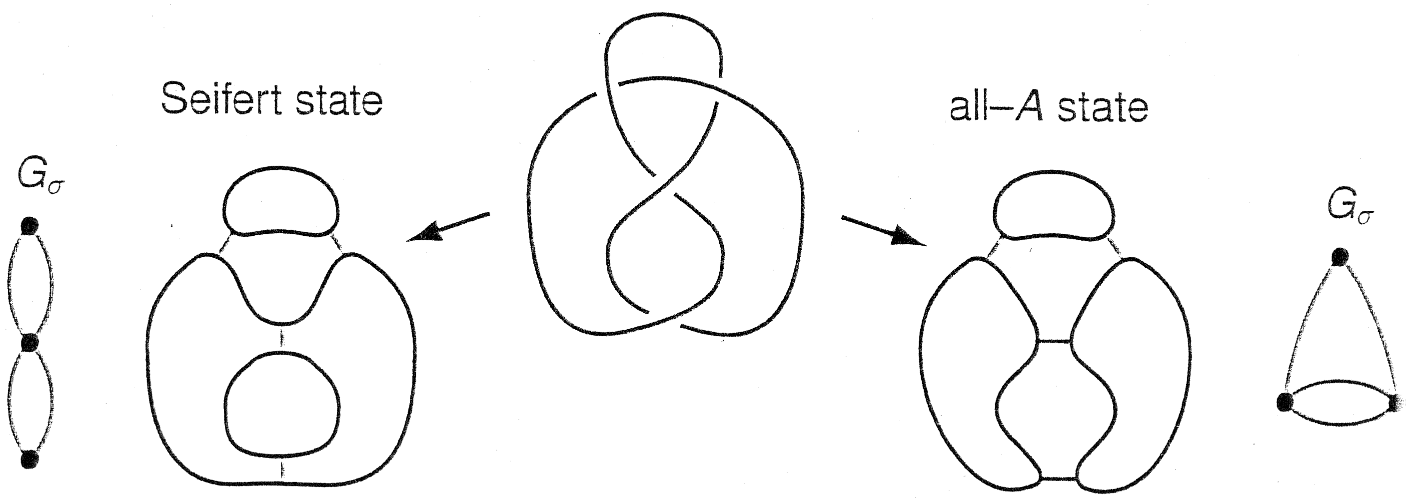



Resolving each crossing yields a collection of circles in the projection plane. As a result, the state determines a state graph $\mathbf{G}_\sigma$:

– vertices of $\mathbf{G}_\sigma \longleftrightarrow$ circles from $D$

– edges of $\mathbf{G}_\sigma \longleftrightarrow$ former crossings of $D$

**Adequate states:** A state $\sigma$ is called adequate if its graph $\mathbf{G}_\sigma$ is loopless. For example,both of the states below are adequate:

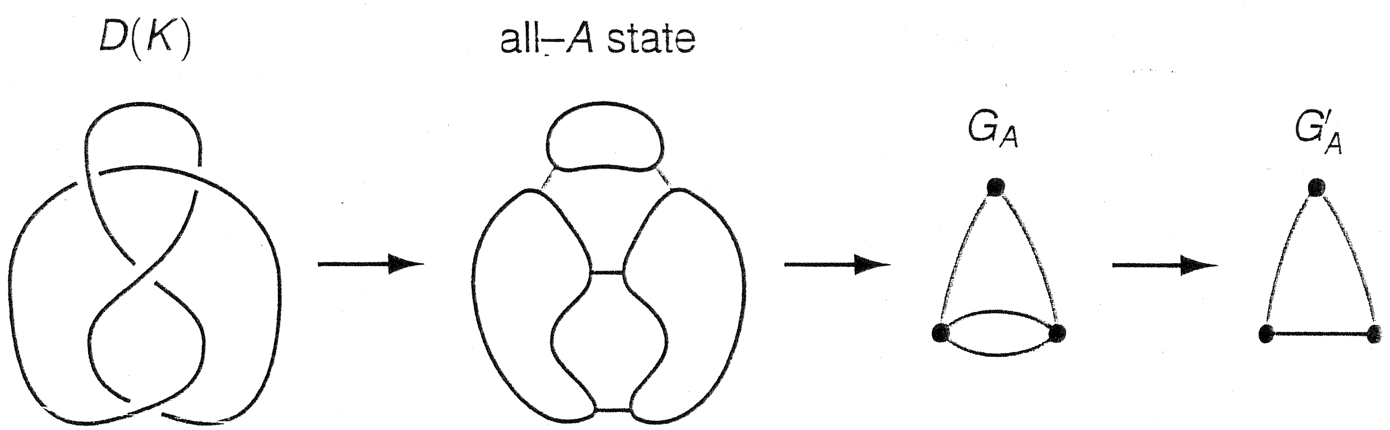



A diagram $D$ is **adequate** if its all $-A$ and all $-B$ states are both adequate.

Every (prime, reduced) alternating diagram is adequate. Almost all Montesinos knot (more generally, arborescent knots) have $A-$ or $B-$ adequate diagrams.

## Reduced graphs give colored JP coeffs

We take the graph $G_\sigma$ and throw away double ("redundant") edges. This gives the **reduced graph** $\mathrm{G}'_\sigma$




**Theorem.** (Stoimenow, Dasbach-Lin) Let $D(K)$ be an adequate diagram and let $\mathrm{G}'_A$, $\mathrm{G}'_B$ be the reduced graphs of the all-$A$ and all-$B$ states. Then

$$|\alpha| = |\alpha'| = 1, |\beta| = 1 - \chi(\mathrm{G}'_A), |\beta'| = 1 - \chi(\mathrm{G}'_B)$$

## Reduced graphs estimate knot volume

**Theorem.** (Futer-K-Purcell) Let $D(K)$ be a reduced, adequate diagram and $\sigma$ its all-$A$ or all-$B$ state. Assume $D := D(K)$ is a "Conway sum of alternating tangles"' (e.g. alternating, Montesinos, arborescent.) Then,

$$-v_8\ \chi(\mathbf{G'}_\sigma)\ \leq\ \mathrm{vol}(S^3 \setminus K),$$

where $v_8 \approx 3.664$ is the volume of a regular ideal octahedron. Moreover,

$$t(D)\ \leq\ 2(1 - \chi(\mathbf{G'}_A) - \chi(\mathbf{G'}_B)).$$

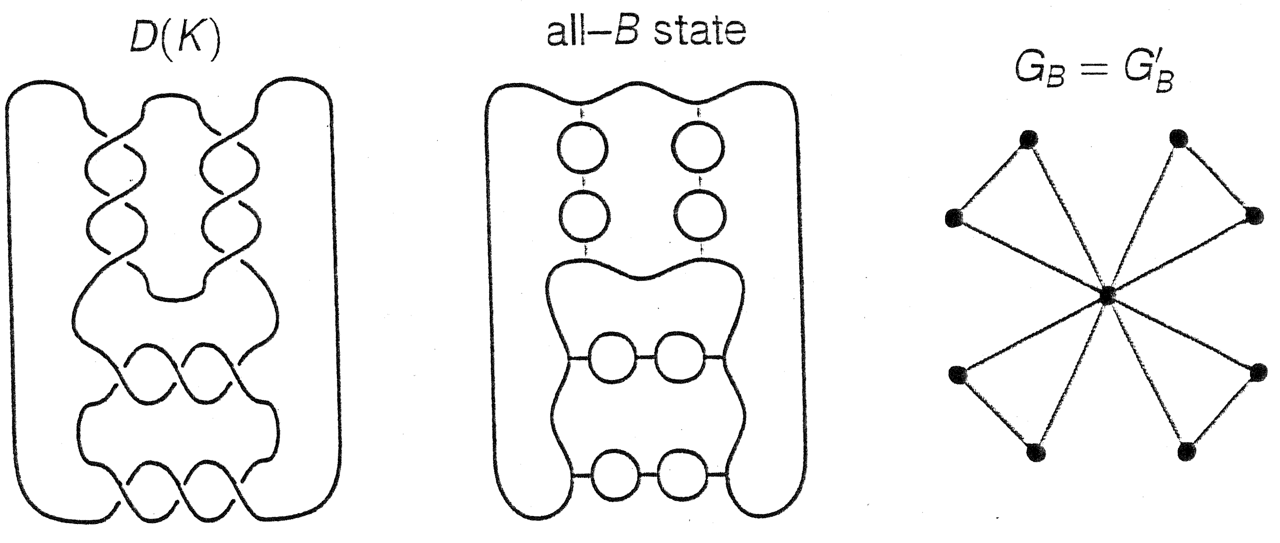
**Corollary.** For $K$ as above

$$1.83(|\beta| + |\beta'| - 2) \leq \mathrm{vol}(S^3 \setminus K) < 20.3(|\beta| + |\beta'| - 1).$$

## An Example

Replacing $|\beta| + |\beta'|$ with $\max(|\beta|, |\beta'|)$ gives a better lower bound. Here

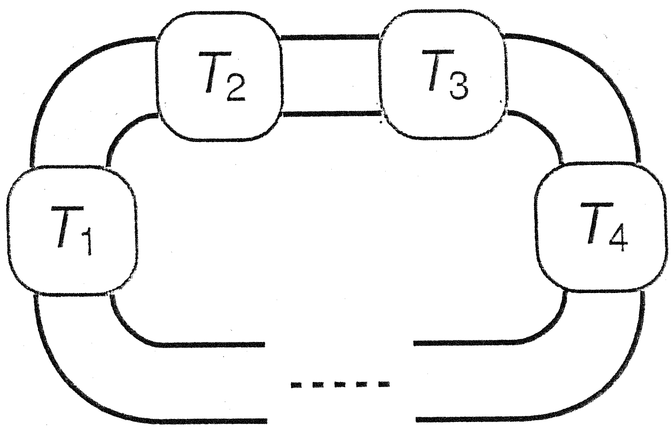
$$v_8(|\beta| - 1) = -v_8 < 0$$




$$v_8\,(|\beta'| - 1) \;=\; -v_8\,\chi(G'_B) \;=\; 10.99...$$

$$\mathrm{Vol}\,(S^3 \setminus K) \;=\; 13.64...$$

## Typical sum of alternating tangles




Mild conditions on tangles assure adequacy.

Theorem holds for more general adequate knots. In fact:

**Conjecture:** Theorem holds for all adequate knots. (**Work in progress**)

**Volume estimates from Guts:** Let $M$ be a hyperbolic 3–manifold and $S$ an essential surface in $M$. Cut $M$ along $S$ to obtain a manifold with boundary, $N$.

(**JSJ-decomposition**) $N$ cut along essential annuli, yields three kinds of pieces:

(1). Seifert fibered pieces; (2). I-bundles;

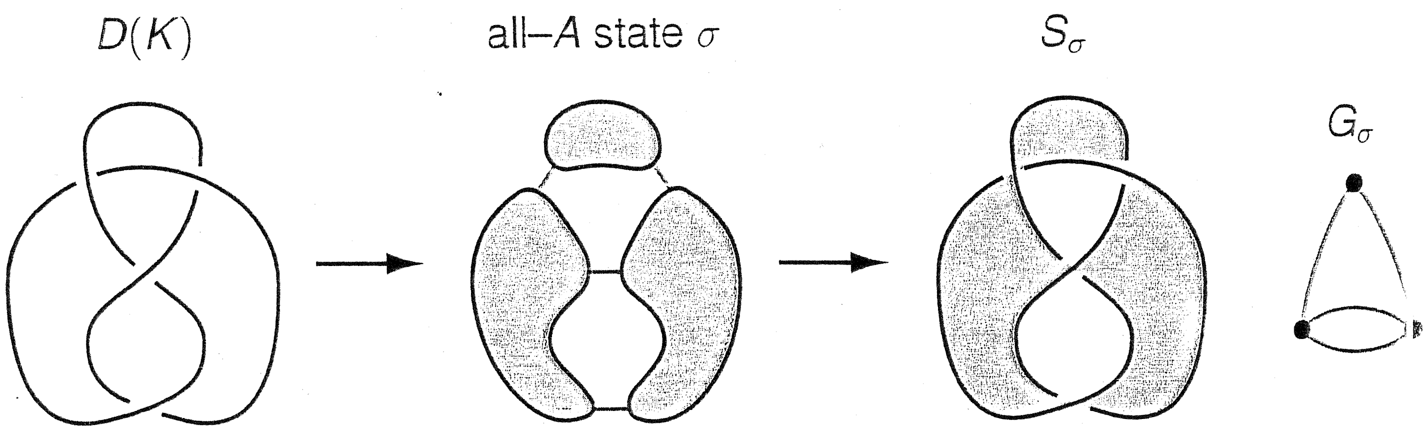
(3). Guts($M, S$). By Thurston, hyperbolic.

**Theorem:** (Agol–Storm- W. Thurston)

$$\text{vol}(M) \geq \ -v_8 \ \chi(Guts(M, S)).$$

This estimate relies on Perelmans volume estimates for Ricci flow with surgery. Weaker estimate was earlier obtained by Agol.

**For us, $S$ is the state surface:** We use the Kauffman state $\sigma$ to construct a state surface $S_\sigma$.



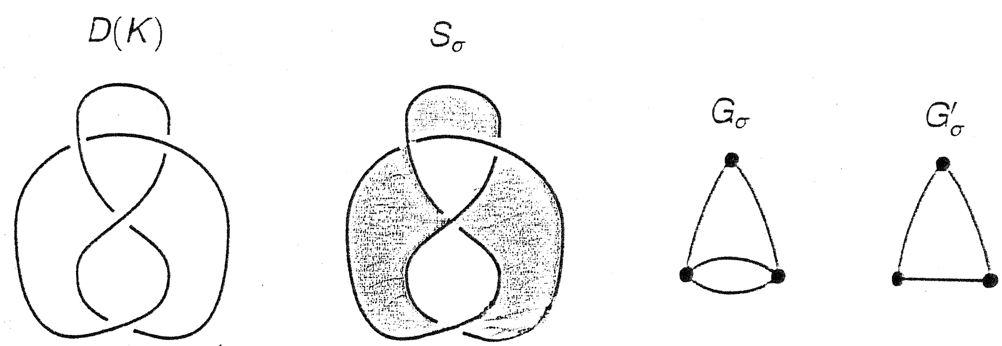

A spine of $S_\sigma$ is the graph $\mathbf{G}_\sigma$.

We have $\chi(S^3 \setminus S_\sigma) = \chi(S_\sigma) = \chi(\mathbf{G}_\sigma)$.

**Theorem:**(Ozawa) Suppose $\sigma$ is the all–$A$ or all-$B$ state of $D = D(K)$, and is adequate. Then $S_\sigma$ is essential in $M := S^3 \setminus K$.

## The guts of state surfaces

**Step 1:** All the I-bundle pieces of the JSJ-decomposition of $N := S^3 \setminus S_\sigma$ are of the form ( **pair of pants**)$\times I$ and come from redundant (double) edges in $\mathbf{G}_\sigma$. Removing such a piece reduces each of $-\chi(N)$ and $-\chi(\mathbf{G}_\sigma)$ by one.




**Step 2:** All Seifert pieces are solid tori; dont affect Euler characteristic.

**Conclusion:**

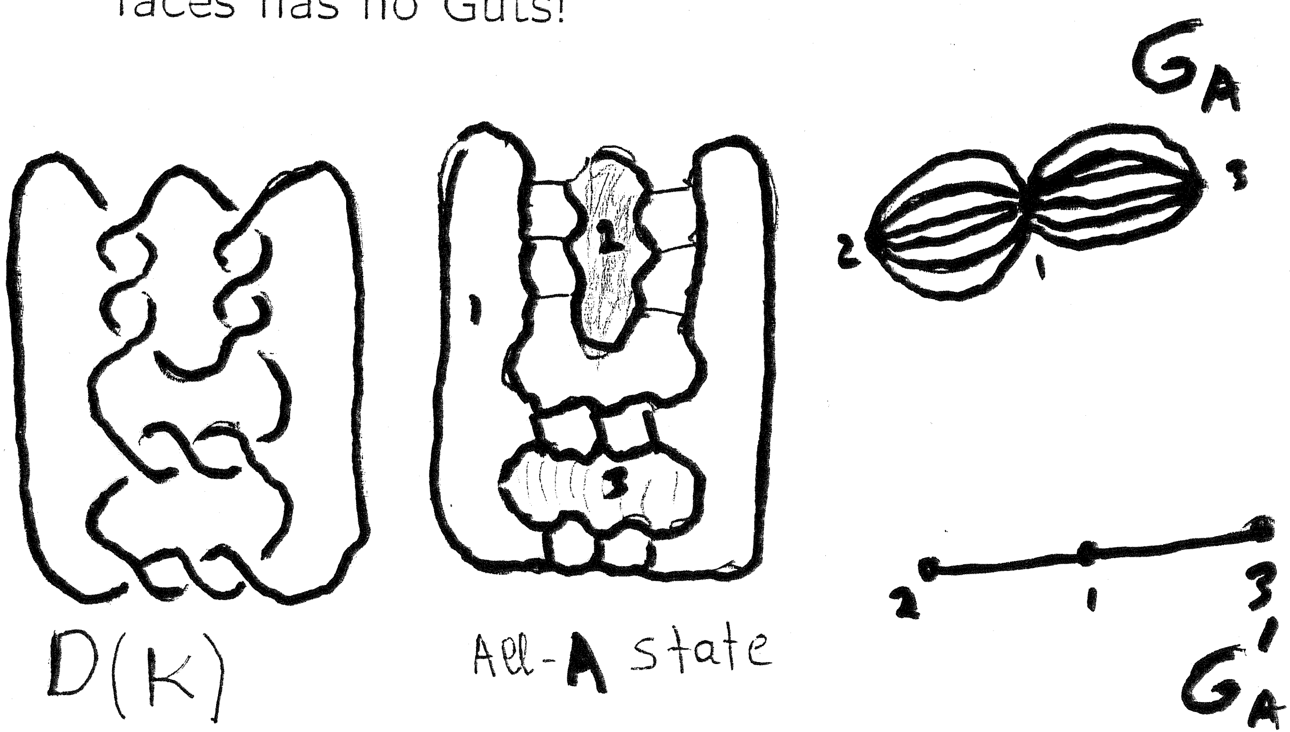
vol$(S^3 \backslash K) \geq -v_8 \chi(Guts(M, S)) = -v_8 \chi(\mathbf{G}'_\sigma)$.

**The choice of state matters!**

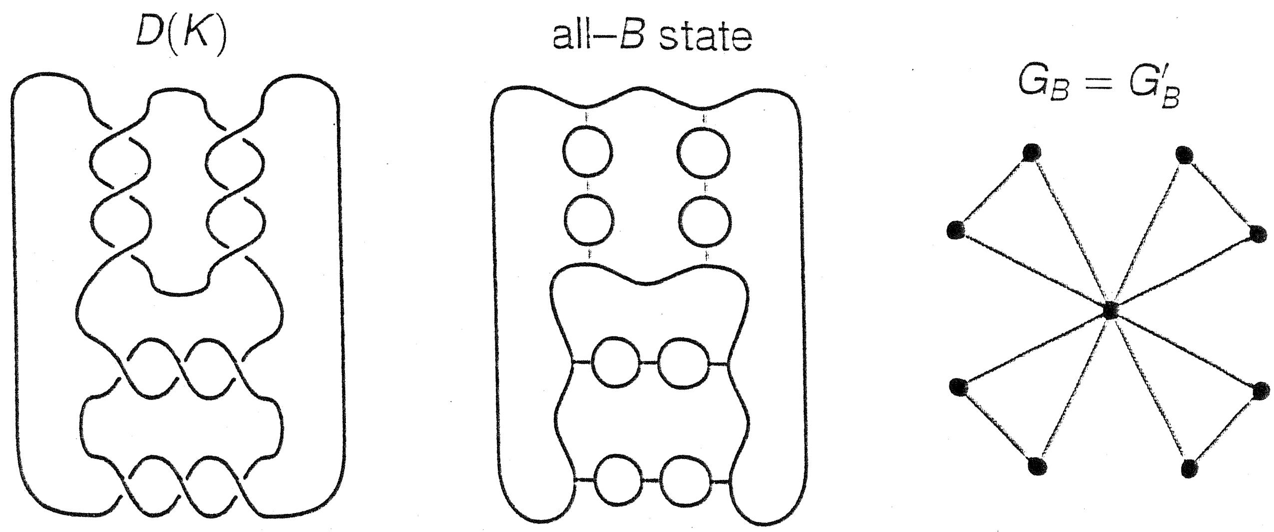
The last equality should mean:

$\chi(Guts(M, S)) = \min(\chi(\mathbf{G}'_\sigma), 0)$.

In the earlier example we have $\chi(\mathbf{G}'_A) = 1$ and $Guts(M, S) = \emptyset$. The all–$A$ state surfaces has no Guts!



The surface of the $B$-state, $S_B$, has a lot of Guts! There are no $I$-bundles in the characteristic sub manifold of $M$ $S_B$.




$$\mathcal{V}_8\,(|\beta'|-1) = -\mathcal{V}_8\,\chi(G'_B) = 10.99...$$

$$\mathrm{Vol}\,(S^3 \setminus K) = 13.64...$$