Normal and Jones Surfaces (talk slides)

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Boundary Slopes:

- Recall $M_K = S^3 \setminus N_K$ where N_K= tubular neighborhood of K.
- $\langle \mu, \lambda \rangle$ = meridian–*canonical* longitude basis of $H_1(\partial N_K)$.
- **Defin.** $p/q \in \mathbb{Q} \cup \{1/0\}$ is called a *boundary slope* of K if there is an essential surface $(S, \partial S) \subset (M_K, \partial N_K)$, such that ∂S represents $p\mu + q\lambda \in H_1(\partial N_K)$.

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- (Hatcher, 80's) Every knot $K \subset S^3$ has finitely many boundary slopes.
- (Hatcher-Thurston, 80's) Gave algorithm to find all boundary slopes of 2-bridge knots.
- (Hatcher-Oertel) Gave algorithm to find all boundary slopes of Montesinos knots. – Algorithm allows to find all essential surfaces.
- (Jaco-Sedwick, 2003) Reproved Hatcher's finiteness result and generalized it to *normal surfaces*: There are finitely many slopes on ∂N_K that are realized by normal surfaces with respect to any " nice" (= one vertex) triangulation of M_K.
- Normal surfaces "contain" the essential ones–not every normal surface is essential.

Colored Jones Polynomials

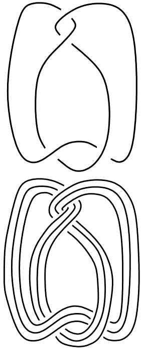
- For a knot K, the colored Jones function $J_K(n)$ is a sequence

$$J_K : \mathbb{Z} \to \mathbb{C}[t^{\pm 1}]$$

  of Laurent polynomials in t. Extended to $\mathbb{Z}$ by $J_K(n) = -J_K(-n)$.
- Normalized so that $J_{\text{unknot}}(n) = (t^{2n} - t^{-2n})/(t^2 - t^{-2})$.

  - Encodes information about the Jones polynomial of K and its parallels K^n. The Jones polynomial corresponds to $n = 2$.
  - Technically, $J_K(n)$ is the quantum invariant using the n-dimensional representation of $SU(2)$.
  - Structure of quantum invariants and representation theory of $SU(2)$ (decomposition of tensor products of representations) lead to formulae in terms of "parallel" cables:

$$J_K(1) = 1, \quad J_K(2)(t) = J_K(t),$$
$$J_K(3)(t) = J_{K^2}(t) - 1, \quad J_K(4)(t) = J_{K^3}(t) - 2J_K(t), \ \ldots$$



Colored Jones Polynomials

- (Garoufalidis- Le, 2005) The colored Jones function "*t-holonomic*": It satisfies satisfies non-trivial linear recurrence relations.
- Given K, there are polynomials $a_j(t^{2n}, t) \in \mathbb{C}[t^{2n}, t)]$, so that

$$a_d(t^{2n}, t)J_K(n+d) + \cdots + a_0(t^{2n}, t)J_K(n) = 0,$$

  for all n.
- **Example.** K=right hand side trefoil.
- Colored Jones Function

$$J_K(n) = t^{-6(n^2-1)} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{24j^2+12j} \frac{t^{8j+2} - t^{-(8j+2)}}{t^2 - t^{-2}}.$$

- Linear recurrence relation

$$(t^{8n+12} - 1)J_K(n+2) + (t^{-4n-6} - t^{-12n-10} - t^{8n+10} + t^{-2})J_K(n+1)$$

$$-(t^{-4n+4} - t^{-12n-8})J_K(n) = 0.$$

More on the structure of the degree of CJP

- Holonomicity property implies: Given K there is $N_K > 0$, such that, for $n \geq N_K$,
$$d_+[J_K(n)] = a_K(n)\, n^2 + b_K(n)n + c_K(n),$$
- where $a_K(n), b_K(n), c_K(n) : \mathbb{Z} \to \mathbb{Q}$ are periodic functions.
- $d_+[J_K(n)]$ =maximum degree of CJP
- We have finitely many cluster points $js_K = \{4b_K(n)\}'$ (Jones Slopes) and $js_K = \{2b_K(n)\}'$.

Definition. A Jones surface of K is an essential surface $S \subset M_K = S^3 \smallsetminus K$

- ∂S represents a Jones slope $a/b \in js_K$, $b > 0$ and $\gcd(a, b) = 1$, and
$$\frac{\chi(S)}{|\partial S| b} \in jx_K.$$

- **Conjecture.** (Garoufalidis, K+Tran) All Jones slopes are realized by Jones surfaces.
- **Question.** How do we recognize Jones surfaces?

Some data on the degree of the CJP

K	js_K	$\{2b_K(n)\}'$	$\chi(S)$	$\lvert\partial S\rvert$
8_{19}	$\{12\}$	$\{0\}$	0	2
8_{20}	$\{8/3\}$	$\{-1, -5/3\}$	-3	1
8_{21}	$\{1\}$	$\{-2\}$	-4	2
9_{42}	$\{6\}$	$\{-1\}$	-2	2
9_{43}	$\{32/3\}$	$\{-1, -5/3\}$	-3	1
9_{44}	$\{14/3\}$	$\{-2, -8/3\}$	-6	1
9_{45}	$\{1\}$	$\{-2\}$	-4	2
9_{46}	$\{2\}$	$\{-1\}$	-2	2
9_{48}	$\{11\}$	$\{-3\}$	-6	2

Table: Non-alternating Montesinos knots up to nine crossings.

- s= denominator of Jones slope, $|\partial S|$ = # of boundary components.

$$\frac{\chi(S)}{s|\partial S|} \in \{2b_K(n)\}'.$$

$s|\partial S|$ is called *the number of sheets* of S.

What is known

The Strong Slope Conjecture is known for the following classes of knots.

- Alternating knots (Garoufalidis)
- Adequate knots (Futer-K.-Purcell)
- Iterated torus knots (K.-Tran)
- Families of 3-tangle pretzel knots (Lee-van der Veen)
- Knots with up to 9 crossings (Garoufalidis, K.-Tran, Howie)
- Graph knots (Motegi-Takata)
- An infinite family of arborescent non-Montesinos (Do-Howie)
- “Near-adequate” knots constructed by taking Murasugi sums of an alternating diagram with a non-adequate diagram (Lee)
- Knots obtained by iterated cabling and connect sums of knots from any of the above classes, since the conjecture was shown to be closed under these operations (K.- Tran)
- Jones slopes (but no Jones surfaces) found for family of 2-fusion knots (Garoufalidis-Veen)

Jones period/vs number of sheets of Jones surfaces

- The least common multiple of the periods of all the coefficient functions of degrees of CJP is called the *Jones period* p of K.

Lemma (K.-Lee) Suppose that $K \subset S^3$ is a knot of Jones period p. Let $a/b \in js_K \cup js_K^*$ be a Jones slope and let S be a corresponding Jones surface. Then b divides p^2 and $b|\partial S|$ divides $2p\chi(S)$.

- Numerical data suggests stronger relation:
- We call a Jones surface S of a knot K *characteristic* if the number of sheets of S divides the Jones period of K.
- In all all examples where Jones surfaces have been found, we have characteristic ones
- **Two Interesting Examples** The pretzel knot $P(-1/101, 1/35, 1/31)$ has a Jones slope $s = 1345/8$ and realized by a Jones surface with number of sheets 32 and 4 boundary components. The Jones period is also 32 !
- $P(-1/101, 1/61, 1/65)$, which has $p = 62$. It has a Jones slope 4280/31 from a Jones surface with number of sheets 31, which divides the Jones period 62, but is not equal to it.
- **Question.** Is it true that for every Jones slope of a knot K we can find a characteristic Jones surface?

Recognizing Jones surfaces

- **Starting point** Since Jones surfaces are essential they can be isotoped to be in normal form with respect to any triangulation of the knot complement!

Theorem. (Lee-K.) Given a knot K with known sets $js_K \cup js_K^*$, $jx_K \cup jx_K^*$ and Jones period p, there is a normal surface theory algorithm that decides whether K satisfies the Strong Slope Conjecture.

- **KEY POINT.** Fix a Jones slope $a/b \in js_K$, with $b > 0$ and $\gcd(a, b) = 1$, and suppose that we have Jones surfaces corresponding to it. Let S be such a surface with $\beta := \dfrac{\chi(S)}{|\partial S| b} \in jx_K$.
- Since $|\partial S| b$ divides $2p\chi(S)$, where p is the Jones period of K. Thus

$$2p\chi(S) - \lambda|\partial S|b = 0 \text{ where } \lambda = 2p\beta \in \mathbf{Z}. \quad (1)$$

Algorithm

- There is an algorithm to determine whether $M_K = S^3 \setminus K$ is a solid torus and thus if K is the unknot (Jaco-Tollefson, Haken) If K is the unknot then the Strong Slope Conjecture is known and we are done.
- If K is not the unknot apply an algorithms of Jaco and Rubinstein to obtain a "NICE" triangulation $\mathcal{T}$ of the complement M_K. NICE, allows to control how far normal Jones surfaces are from being fundamental.
- There are finitely many fundamental surfaces in $\mathcal{T}$ and there is an algorithm to find them $\mathcal{F} = \{F_1, \ldots, F_k\}$
- There is an algorithm to compute $\chi(F)$ for all surfaces $F \in \mathcal{F}$, and to compute their boundary slopes of the ones with boundary. Let

  $$\mathcal{A} = \{a_1/b_1, \ldots, a_s/b_s\}$$

  denote the list of distinct boundary slopes in $\mathcal{F}$.
- Check whether $js_K \subset \mathcal{A}$ and $js^*_K \subset \mathcal{A}$. If one of the two inclusions fails then K does not satisfy the Slope Conjecture. (This uses work of Jaco-Sedgwick)

Algorithm cont'

- If $\mathcal{F}$ contains no closed surfaces move to the next step. If we have closed surfaces we need to find any incompressible ones among them. There is an algorithm that decides whether a given 2-sided surface is incompressible and boundary incompressible if the surface has boundary. Apply the algorithm to each closed surface in $\mathcal{F}$ to decide whether they are incompressible. Let $\mathcal{C} \subset \mathcal{F}$ denote the set of incompressible surfaces found.
- For every $s := a/b \in js_K \subset \mathcal{A}$ consider the set $\mathcal{F}_s \subset \mathcal{F}$ that have boundary slope a/b. By of Jaco-Sedgwick we know that $\mathcal{F}_s \neq \emptyset$. Decide whether $\mathcal{F}_s$ contains essential surfaces and find them. Call the set found $\mathcal{E}\mathcal{F}_s$.
- For every $\lambda \in 2pjx_K$ and every $F \in \mathcal{E}\mathcal{F}_s$ calculate the quantity

$$x(F) := 2p\chi(F) - \lambda b|\partial F|.$$

Suppose that there is $F \in \mathcal{E}\mathcal{F}_s$ with $x(F) = 0$. Then any such F is a Jones surface corresponding to s.

Algorithm cont'

- Suppose $\mathcal{E}F_s := \{\Sigma'_1, \ldots, \Sigma'_r\} \neq \emptyset$ and that we have $x(F) \neq 0$, for all $F \in \mathcal{E}\mathcal{F}_s$. Then consider

$$x(\Sigma'_1)n_1 + \ldots x(\Sigma'_r)n_r + 2p\chi(C_1)m_1 + \ldots + 2p\chi(C_t)m_t = 0,$$

where C_i runs over all the surfaces in $\mathcal{C}$.
Find and enumerate all the fundamental solutions of the equation. Among these solutions pick the *admissible* ones: That is solutions for which, for any incompatible pair of surfaces in $\mathcal{C} \cup \mathcal{E}F_s$, at most one of the corresponding entries in the solution should be non-zero. Hence pairs of non-zero numbers correspond to pairs of compatible surfaces. Every admissible fundamental solution represents a normal surface. If a surface in this set is essential, then it is a Jones surface, otherwise, K fails the Strong Slope Conjecture.