Polynomial Invariants for Links in 3-manifolds

by

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Polynomial invariants for links in $\mathbb{R}^3$ (or S^3) discussed in this talk: Jones polynomial, HOMFLY polynomial, Kauffman polynomial.

L oriented link

–Jones polynomial: $J_L(t) \in \mathbf{C}[t^{\pm\frac{1}{2}}]$.

–HOMFLY polynomial: $P_L(v, z) \in \mathbf{C}[v^{\pm 1},\ z^{\pm 1}]$.

–Kauffman polynomial $F_L(a, z) \in \mathbf{C}[a^{\pm 1},\ z^{\pm 1}]$.

Note: $J_L(t) = P_L(t, \quad \sqrt{t} - \frac{1}{\sqrt{t}})$

• Determined by crossing change formulae and certain "initial conditions": Values on the trivial knot together with rule for eliminating trivial components.

• Constructed in late 80's-early 90's: Initial definitions heavily depend on link projections and skein moves only valid in $\mathbb{R}^3$ (and S^3).

For example, $P(L) := P_L(v,z)$ is defined by

- $v^{-1}P(L_+) - vP(L_-) = z\ P(L_0)$
- $P(L \coprod U) = (v^{-1} - v)z^{-1}\ P(L)$
- $P(U) = 1$

L_+, L_-, L_0 oriented links that form a **Skein triple**: The are identical except in a 3-ball in $\mathbf{R}^3$ where they differ by a "positive" , a "negative" and an orientation consistent splicing of the crossing.

- This scheme works because every link in $L \subset \mathbf{R}^3$ has a finite binary "skein tree" with root L and leaves a collection of trivial knots.

- $P_L(v,z) \longleftrightarrow \{J_n(L) := P_L(t^{\frac{n+1}{2}},\ \sqrt{t} - \frac{1}{\sqrt{t}})\}_{n \in \mathbf{N}}$

From the **Quantum group** viewpoint: $J_n(L)$ is Built from the fundamental representation of $sl(N,C)$.

Applications to contact topology: (x, y, z) Cartesian coordinates in $\mathbf{R}^3$.

$\xi :=$ span$\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\}$, $[\xi = \text{Ker}(dz - ydy)]$

the **standard contact structure**.

Legendrian link: An embedding $L(\coprod S^1)$ *always* tangent to ξ.

Thurston-Bennequin number: L_l=oriented, Legendrian link; S=a Seifert surface bounded by L_l. There is a non-zero* vector field over S in ξ, say $v \in \xi$, such that along $L_l = \partial S$, the tangent vector of L in ξ is transverse to v. Let L_l' be a push-off of L along v. $TB(L_l) := \text{lk}(L_l, L_l')$.

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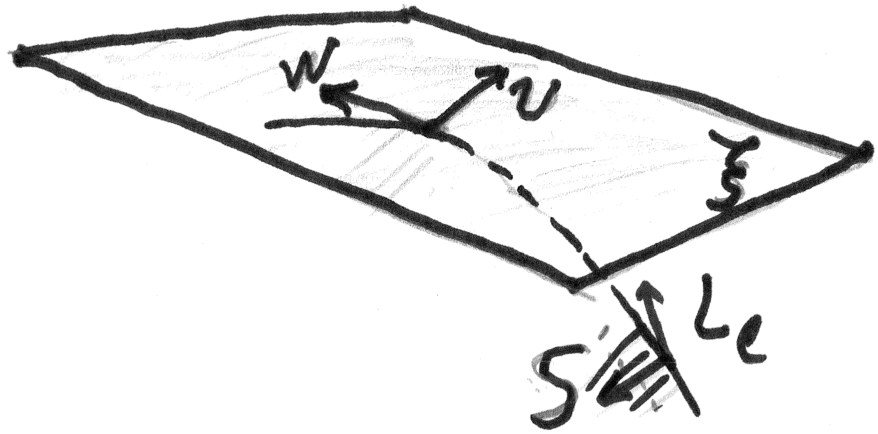
* The restriction of ξ on S is a trivial 2-dimensional bundle.



• There is a *Seifert framing* along L; TB measures the difference between "contact" and Seifert framing along L_l.

rotation number: There is a non-zero vector field $w \in \xi$ tangent to L_l pointing the same direction with the orientation of L_l.

$r(L_l)$:=the obstruction of extending w to a non-zero vector field along S.

Note. Any **topological link type** has Legendrian representatives. The topological link type, TB are r are know as the *classical* Legendrian isotopy invariants.

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Transversal link: An embedding $L(\coprod S^1)$, *always* transverse to ξ. Any **topological link type** has transverse representatives.

Self-linking number: (*invariant of transversal isotopy*) For a transverse link L_t let S a Seifert surface bounded by L_t. Let $v \in \xi$ a non-zero vector field over S in ξ and let $L"_t$ be a push off of L_t in the direction of v.

$sl(L_t) := \mathrm{lk}(L_t, L"_t)$.

• *Classical* transversal link invariants: Topological link type and $sl(L_t)$.

• A Legendrian link L has two transversal push-offs $L_t^{\pm}$. Then

$sl(L_t^{\pm}) = TB(L_l) \mp r(L_l)$

Remark: All goes through for contact 3-manifolds (M, ξ), M a (rational) homology 3-sphere and $\xi =$"tight"; adapts to rational homology spheres (e.g. Lens spaces).

L=topological type,

$p_v(L)$=minimum degree in v of the HOMFLY polynomial $P_L(v,\ z)$,

$f_a(L)$=minimum degree in a of the Kauffman polynomial $F_L(a,\ z)$

By work of: Franks-Williams, Morton, Fuchs-Tabachnikov, Tabachnikov, Rudolph...

Theorem 1. (a) For every Legendrian representative L_l of L we have

$TB(L_l) \leq p_v(L) - 1$ and $TB(L_l) \leq f_a(L) - 1$.

(b) For every transversal representative L_t of L we have $sl(L_t) \leq p_v(L) - 1$

• Bennequin's inequality (80's) : sl$(L_t) \leq 2g(S) - 1$ where S is any Seifert surface of a the transversal link L_t.

• Bennequin's bound is non-negative while TB and sl are often negative. Theorem 1 gave the *first* "negative" upper bounds. The bounds are sharp in many important cases: (e.g The HOMFLY bound is sharp for positive links and the Kauffman bound is sharp for alternating links)

• More recent upper bounds from Ozsvath-Szabo and Khovanov invariants.

• Bennequin's inequality generalizes in "tight" contact 3-manifolds (Eliashberg, '92).

Questions:

Do the HOMFLY and Kauffman polynomials exist for links in 3-manifolds other than S^3?

Are there similar applications to Legendrian and transversal links in other contact 3-manifolds?

Digression/Motivation: Perturbations of Polynomials. Recall that

$$P_L(v,z) \longleftrightarrow \{J_n(L,\ t) := P_L(t^{\frac{n+1}{2}},\ \sqrt{t} - \tfrac{1}{\sqrt{t}})\}_{n\in\mathbf{N}}$$

Perturb each $J_n(L,\ t)$ by setting $t := e^x := 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$. Then

$$J_n(L,\ t)) \longleftrightarrow J_n(x) := \Sigma_{i=0}^{\infty} v_i(L) x^i$$

where $J_n(x)$ is a power series satisfying the HOMFLY skein relation. This approach comes from the theory of Vassiliev (*a. k. a.* finite type invariants). In fact, $\{v_i(L)\}_{n\in\mathbf{N}}$ are integer valued Vassiliev invariants. This theory can be approached by intrinsically 3-dimensional topology methods and the perturbative versions of the link polynomials in S^3 generalize for links in other 3-manifolds!

Power series in 3-manifolds: For the rest of the talk, M = oriented, irreducible, rational homology 3-sphere that contains no sub-manifolds that are Seifert fibered spaces (SFF) over non-orientable surfaces.

P:= collection of disjoint, unordered, oriented S^1's

Link: A tame embedding $L : P \longrightarrow M$.

Singular link of order n **:** A tame immersion $L^n : P \longrightarrow M$ with only singularities n *transverse* double points. Given a double point of L^n we can resolve in two ways; there is a well defined notion of **positive** (L_+^{n-1}) and **negative** (L_-^{n-1}) resolution and an **oriented splice** (L_0^{n-1}). HOMFLY type **skein link triple** : L_+, L_-, L_0.

Link crossing change: A link homotopy with only singularity a singular link with one double point $L_+ \longrightarrow L_\times \longrightarrow L_-$

Declaring "Trivial links": In every "free homotopy class of links" we fix, once and for all, a link TL and call it a *trivial link*.

Let $\Lambda := \mathbf{C}[v^{\pm 1}, z^{\pm 1}]$. Let $z := \sqrt{t} - \frac{1}{\sqrt{t}}$ and let $\mathbb{J}$ be the ideal of $\Lambda[t]$ generated by $v - v^{-1}$ and t. Let $\widehat{\Lambda}$ be the pro-$\mathbb{J}$ completion of $\Lambda[t]$.

Theorem 2. Let $\mathbb{L}$ the set of isotopy classes of links in M. There exists a unique map $P_M : \mathbb{L} \to \widehat{\Lambda}$ satisfying the HOMFLY skein relation

$$v^{-1} P_M(L_+) - v P_M(L_-) = z P_M(L_0)$$

and with given values on the *trivial links* .

- K. [Topology vol 37(3), 1998], Lin-K . [Topology , vol 38 (1), 1999)] with additional hypothesis– Cornwell-K. removed hypothesis. We also obtained a Kauffman type power series.

Can we choose the "trivial links" so that $P_M(L)$ is polynomial for every link? Yes in some cases.

Warm-up: Theorem 2 also leads to another construction of the HOMFLY Polynomial for links S^3.

The crossing changes unlinking any link L in S^3 can be done on any projection of L. Having the power series of Theorem 2 at hand we work with projections on a fixed S^2 in S^3: For a link projection L; $c(L)=$ # of crossings in L and $u(L)$=number of crossing changes that trivialize L. Order $(c(L),\ u(L))$ lexicographically and choose the values P(trivial links) to be polynomials. The HOMFLY skein formula allows to write any link $P(L)$ as sum of polynomials of links of *less complexity* (done!).

• I don't know of an intrinsic complexity that will work in arbitrary 3-manifolds.