Polyhedral decompositions, essential surfaces and colored Jones knot polynomials

joint with D. Futer and J. Purcell

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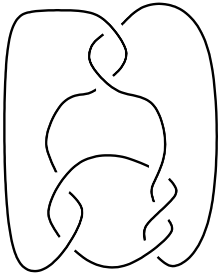
Given: Diagram of a knot or link



= 4–valent graph with over/under crossing info at each vertex.

Quantum Topology

- Knot invariants esp. colored Jones polynomials

Geometric topology

- Incompressible surfaces in knot complements
- Geometric structures and data esp. hyperbolic geometry and volume

Long term goal: Develop a setting to study both sides and relate them.

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- Setting:
  - Given knot diagram construct state graphs (ribbon graphs)..
  - Build state surfaces spanned by the knot...
  - Create polyhedral decomposition of surface complements...

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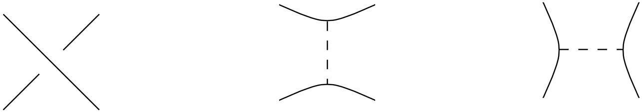
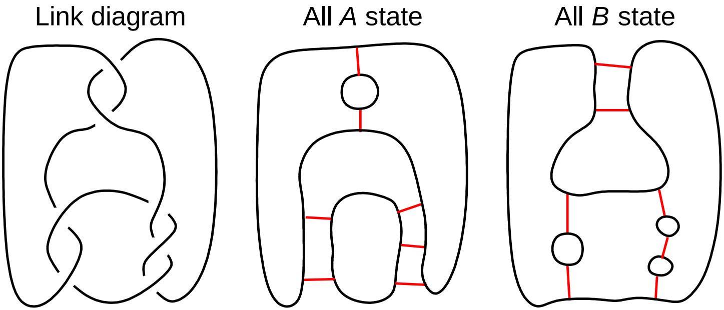
  - Guts $\rightarrow$ relate CJP and volume of hyperbolic knots.

Two choices for each crossing, *A* or *B* resolution.



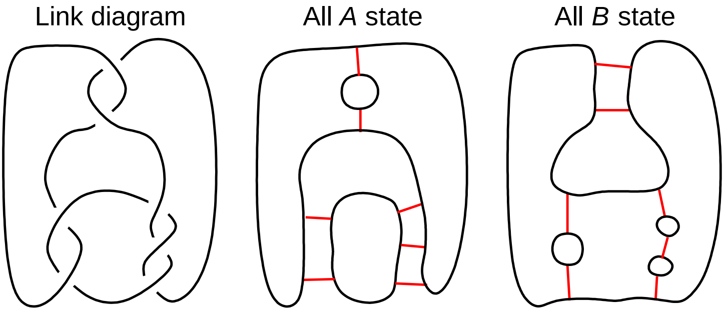
- Choice of *A* or *B* resolutions for all crossings: *state* σ.
- Result: Planar link without crossings. Components: *state circles*.
- Form a graph by adding edges at resolved crossings. Call this graph H_σ.
  (Note: n crossings $\rightarrow 2^n$ state graphs)

Example states



Above: H_A and H_B.

Example states



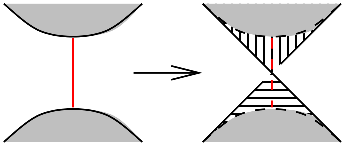
Above: H_A and H_B.

- The colored Jones polynomials of the knot can be calculated from H_A or H_B: *spanning graph expansion* arising from the Bollobas-Riordan *ribbon graph* polynomial (Dasbach-F-K-Lin-Stoltzfus , 2006).

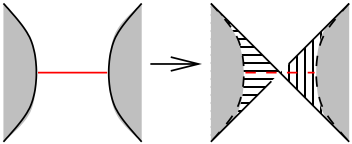
State surface

Using graph H_σ and link diagram, form the *state surface* S_σ.

- Each state circle bounds a disk in S_σ (nested disks drawn on top).
- At each edge (for each crossing) attach twisted band.



A–resolution



B–resolution

Example state surfaces

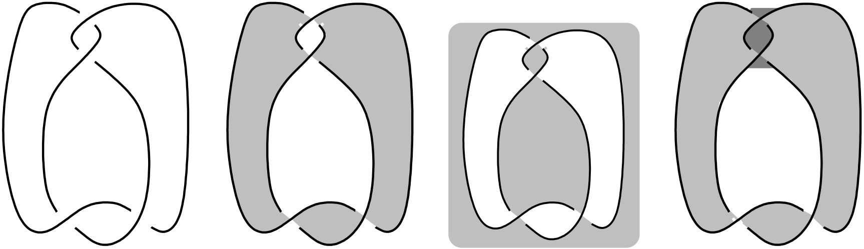


Fig-8 knot S_A S_B Seifert surface

- For alternating knots: S_A and S_B are checkerboard surfaces.

Example state surfaces

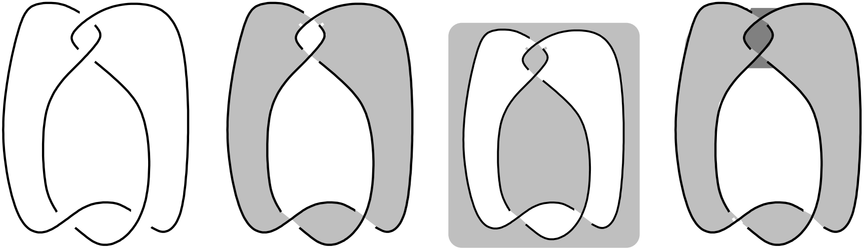
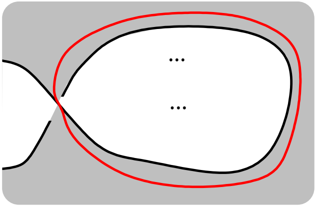


Fig-8 knot | S_A | S_B | Seifert surface

- For alternating knots: S_A and S_B are checkerboard surfaces.
- For alternating knots S_A and S_B are *essential*: incompressible, ∂-incompressible (Menasco-Thistlethwaite, Lackenby)

When are state surfaces incompressible?

Not always: If H_A has edge with both endpoints on a single state circle, then we form boundary compression disk:



That's the only thing that can go wrong.

Theorem

(Ozawa, Futer-K-Purcell)The following are equivalent:

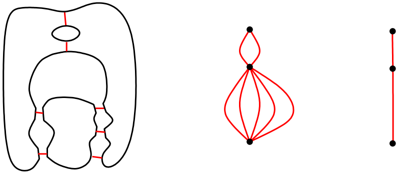
- *H_A has no edge with both endpoints on a single state circle*
- *S_A is incompressible and boundary incompressible.*

Ozawa proof is different; uses Murasugi sum arguments.— We see more about S_A.

For example: When is S_A a fiber in the complement of $K = \partial S_A$?

Recall the graph H_A.

- Collapse each state circle of H_A to a vertex to obtain the state graph $\mathbb{G}_A$.
- Remove redundant edges. Result is a graph $\mathbb{G}'_A$.



Theorem

(FKP) The complement $S^3 \setminus K$ fibers over S^1 with fiber S_A if and only if the reduced graph G'_A is a tree.

Exercise: Derive Stalling's classical result: Positive closed braids are fibered.

Geometry and topology of the state surface complement

Take JSJ-decomposition of $S^3 \backslash\backslash S_A$. There are no essential tori. Cut along essential annuli into components:

1. Solid tori,
2. I–bundles over surfaces,
3. *Simple* pieces (admitting complete hyperbolic metric).

The union of all the hyperbolic pieces is the *guts*: $\mathrm{Guts}\,(S^3 \backslash\backslash S_A)$.

- $\mathrm{Guts}\,(S^3 \backslash\backslash S_A) = \emptyset \Leftrightarrow S^3 \backslash\backslash S_A$ is a union of I-bundles and solid tori (i.e. *a book of I-bundles*).
- $\chi(\mathrm{Guts}\,(S^3 \backslash\backslash S_A))$ measures how far S_A is from being “fiber-like” (*a fibroid*).

Notation: Set $\chi_-(Y) = \max\{-\chi(Y_i), 0\}$, for Y =connected cell complex. For non-connected Y sum χ_-'s over connected components.

Theorem

Let $D(K)$ be a diagram such that H_A has no edge with both endpoints on a single state circle, and let S_A be the essential spanning surface determined by this diagram. Then

$$\chi_-(Guts(S^3 \backslash\backslash S_A)) = \chi_-(\mathbb{G}'_A) - ||E_c||,$$

where $||E_c|| \geq 0$ is a diagrammatic quantity.

In several instances $||E_c|| = 0$. Examples:

1. Alternating links,
2. Montesinos Links,
3. Closures of positive braids where each exponent is at least 3.

Guts relates to volume:

Theorem (Agol–Storm–W. Thurston 2005)

For K hyperbolic

$$Vol(S^3 \setminus K) \geq v_8 \, \chi_-(Guts(S^3 \backslash\backslash S_A)),$$

here $v_8 \approx 3.66...$ is the volume of a regular ideal octahedron.

Corollary

Let $D = D(K)$ be a prime A–adequate diagram of a hyperbolic link K. Then

$$Vol(S^3 \setminus K) \; \geq \; v_8 \, (\chi_-(\mathbb{G}'_A) - ||E_c||).$$

Application example: Volume and twist number

Theorem (Lackenby, 2005)

Let D be a reduced alternating diagram of a hyperbolic link K. Then

$$\frac{v_8}{2}(t(D) - 1) \; \leq \; Vol(S^3 \setminus K) \; < \; 10 v_3 (t(D) - 1),$$

where $v_3 = 1.0149...$ is the volume of a regular ideal tetrahedron.

We extended the list of manifolds for which we can compute explicitly the Euler characteristic of the guts and can be used to derive results analogous Lackenby's. Samples:

Theorem

Let $D(K)$ be a diagram of a hyperbolic link K, obtained as the closure of a positive braid with at least three crossings in each twist region. Then

$$\frac{2v_8}{3}\, t(D) \;\leq\; Vol\,(S^3 \smallsetminus K) \;<\; 10v_3(t(D) - 1).$$

In this case $||E_c|| = 0$. Similar results for: Montesinos links, Conway sums of alternating tangles...

Colored Jones polynomials

For a knot K we write its *n-colored Jones polynomial*:

$$J_{K,n}(t) := \alpha_n t^{m_n} + \beta_n t^{n-1} + \cdots + \beta'_n t^{m+1} + \alpha'_n t^{k_n}.$$

- Some properties:
  - $J_{K,n}(t)$ is determined by the Jones polynomials of certain cables of K.
  - The sequence $\{J_{K,n}(t)\}_n$ is *q-holonomic*: for every knot the CJP's satisfy linear recursion relations (Garoufalidis-Le , 2004). Then for every K
    1. Degrees m_n, k_n are quadratic (quasi)-polynomials in n
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    1. Degrees m_n, k_n are quadratic (quasi)-polynomials in n
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- Properties manifest themselves in strong forms for knots with *state graphs* that have no edge with both endpoints on a single state circle!

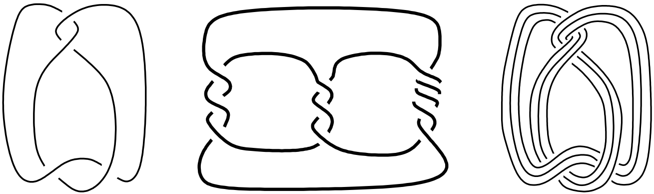
Adequate links

Lickorish–Thistlethwaite 1987: Introduced *A*–adequate links (*B*–adequate links) in the context of Jones polynomials.

Definition

A link is *A*–adequate if has a diagram with its graph H_A has no edge with both endpoints on the same state circle.

A or *B*-adequate: all alternating knots, Montesinos knots, positive braids, negative braids, “most” arborescent knots, blackboard cables of adequate knots, “most” knots on tables up to 15 crossings.



Jones polynomials and adequate links

Properties of interest:

1. The Jones polynomial detects the unknot within the class of *A*-adequate knots.
2. coefficients $|\alpha'_n| = 1$ are independent of *n*: $\alpha'_K := |\alpha_n|$.
3. $\min \deg J_{K,n}(t)$ quadratic polynomial in *n*; can be calculated explicitly.
4. (Dasbach-Lin) Coefficients $|\beta'_n|$ are independent of *n*: $\beta'_K := |\beta_n| = 1 - \chi(\mathbb{G}'_A)$.

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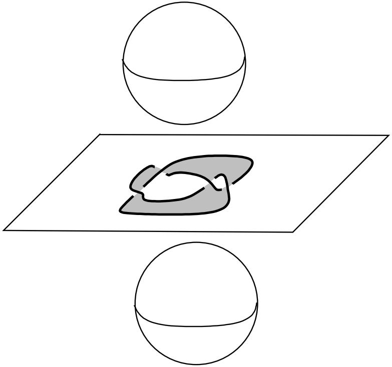
We have analogous properties for B-adequate.

Restate Theorems proved earlier:

1. Diagram is A–adequate $\Leftrightarrow S_A$ incompressible and boundary incompressible.
2. S_A=state surface corresponding to A–adequate diagram of K. The complement $S^3 \setminus K$ fibers over S^1 with fiber $S_A \Leftrightarrow \beta'_K = 0$
3. $\beta'_K = 1 \Leftrightarrow S_A$ is a fibroid (but not a fiber!) with $\chi(S^3 \backslash\backslash S_A) = \chi(\mathbb{G}_A) - \chi(\mathbb{G}'_A)$.
4. In general, β'_K measures distance of S_A from being fiber.
$$\beta'_K - 1 = \chi_-(\text{Guts}\,(S^3 \backslash\backslash S_A)) + ||E_c||.$$
5. Volume estimates of hyperbolic knots in terms of coefficients of CJP.
$$lim_{n\to\infty} \frac{\text{min degree of } \mathrm{J}^n_K(t)}{n^2} = \text{slope of } S_A$$
6. Growth rate of degree of CJP =boundary slope of S_A
7. Relations of Jones polynomial and volume: old and new

Polyhedral decomposition prototype

Menasco (1984): Expand balloons above and below 2-sphere of alternating projection, obtain polyhedral decomposition of link complement (two 3-cells).



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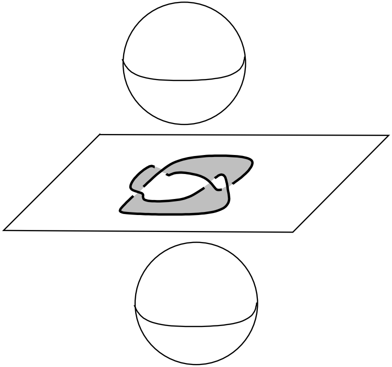
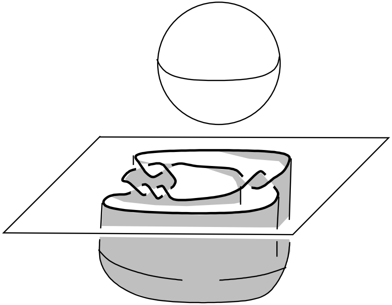


For alternating knots this gives polyhedral decomposition of checkerboard surface complement.—- This is the picture we seek to generalize to all knots.

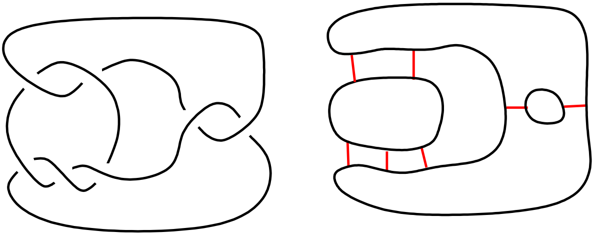
General case:

S_A (or S_B) hangs below plane of projection. Need more balloons.



3–cells:

- One "upper" 3–cell, on top of plane of projection.
- One "lower" 3–cell for each nontrivial component of complement of state circles in *A*–resolution.



Two nontrivial components

Polyhedral decomposition of complement of S_A

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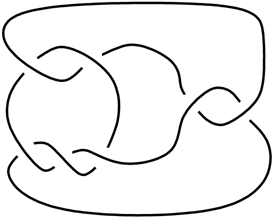

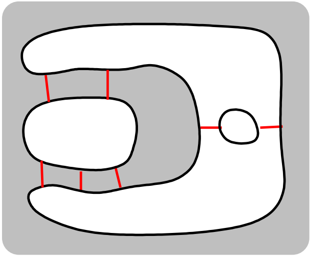

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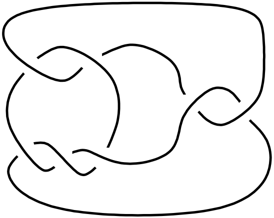
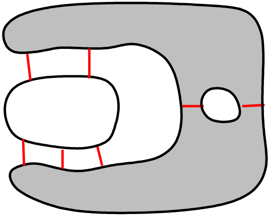


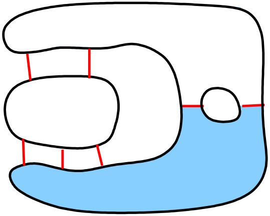


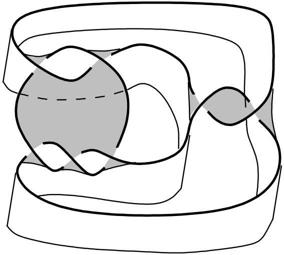
Two nontrivial components

Polyhedral decomposition of complement of S_A, continued

"Faces":

- Portions of 3–cell meeting S_A. Shade these.
- Disks lying slightly below plane of projection, with boundary on S_A.
  - One disk for each region of graph H_A.





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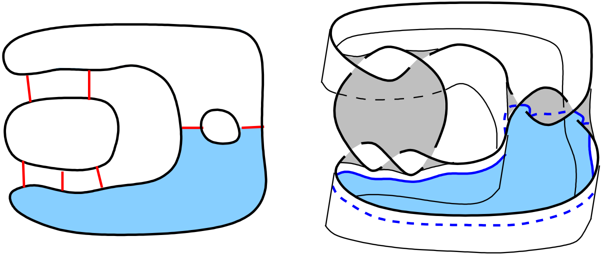
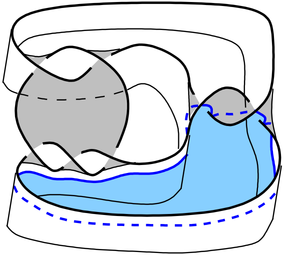


Polyhedral decomposition of complement of S_A, continued

Ideal edges:

- Run from undercrossing to undercrossing, adjacent to region of H_A.

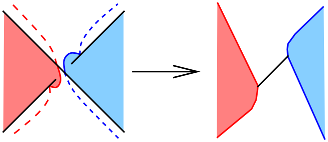


Ideal vertices:

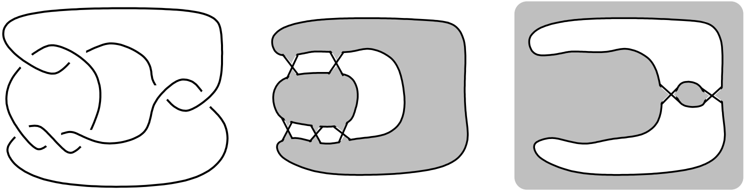
- On the link. Portions of the link visible from inside the 3–cell.

Combinatorics of lower polyhedra:

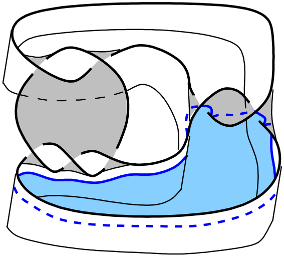
Ideal edges lie below plane of projection, so cut off view of link from below *except* at an undercrossing.



Result: Polyhedron is identical to checkerboard polyhedron of alternating sublink.



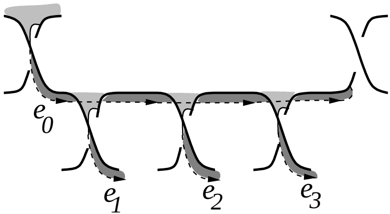
Combinatorics of upper polyhedron:

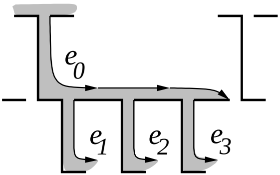


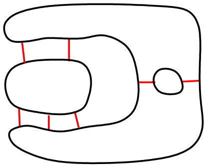
- **“ Faces”:** Shaded “faces” contain innermost disks, White faces correspond to regions of H_A.
- **Ideal edges** start and end at undercrossings, stay adjacent to single region of H_A.
- **Ideal vertices** are connected components of overcrossings = diagram components in usual diagram of link (with breaks at undercrossings).

Combinatorics of upper polyhedron, continued

- Sketch ideal edges onto usual projection of link diagram, or onto H_A.
- Edges bound *white disks*, *shaded "faces"*.
- Shaded faces: *Innermost disks*, along with *tentacles* adjacent to ideal edges.







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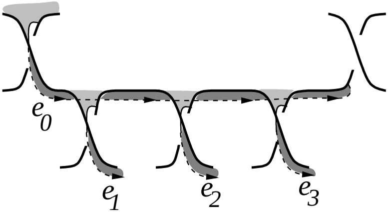

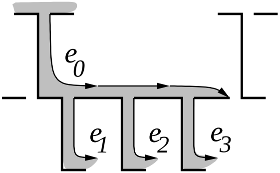

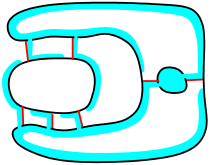

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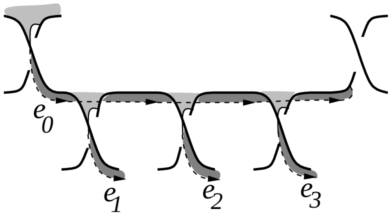
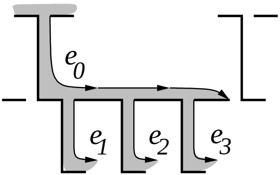
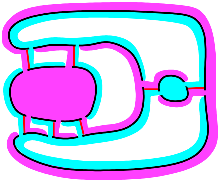


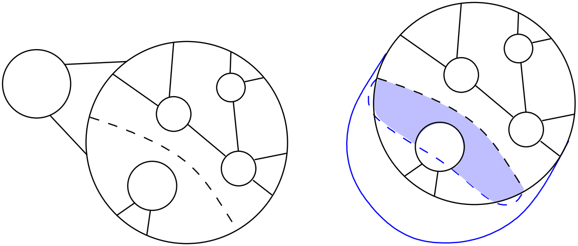




One additional issue

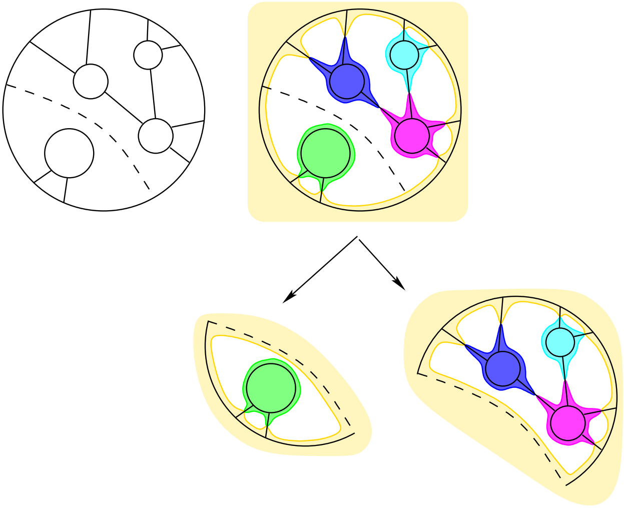
Lower polyhedra may not give *prime* alternating links.

Example:

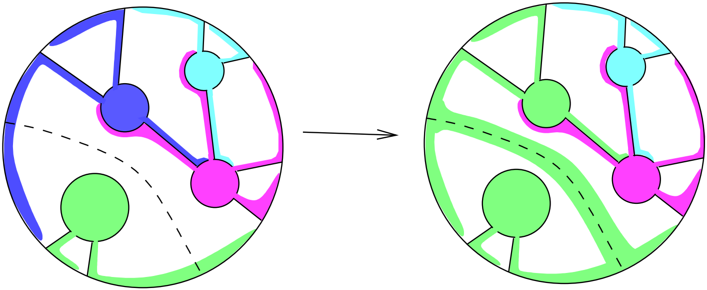


- See bigon in polyhedral decomposition.
- Fix: Modify polyhedra — surger along bigon.
  - Splits 3–cell into two.
  - Splits white disk into two.
  - In upper polyhedron: Connects two shaded “faces” along arc.

Example: Lower polyhedron splits in two



Example: Upper polyhedron



Generic form of Upper polyhedron

Summary: Ideal polyhedral decomposition

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The above procedure gives an ideal polyhedral decomposition of $S^3 \backslash\backslash S_A$ if "faces" are simply connected. This happens when H_A has no edge with both endpoints on a single state circle!

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Use these polyhedra and normal surface theory to study the topology of $S^3 \backslash\backslash S_A$.

Smallest complexity normal surfaces: Normal bigons

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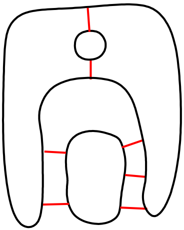

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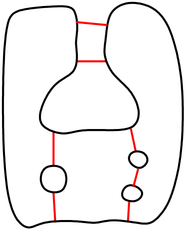
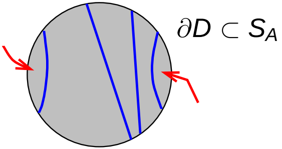


Non-example

Application: A-adequate $\Rightarrow S_A$ is essential

Suppose H_A has no such edge but is compressible or ∂-compressible: Put compressing disk, boundary compressing disk into normal form.
A compressing disk D for S_A would meet white faces of polyhedra in arcs.
Outermost arc on D forms a *normal bigon*. Contradiction.



Normality implies boundary arc of boundary compressing disk E lies in a single polyhedron. Outermost intersection of E with white face cuts off a disk E' which cannot be a normal bigon, so contains boundary arc. But then $E \setminus E'$ is a disk meeting white faces, obtain normal bigon. Contradiction. $\square$

A closer look at the topology of $S^3\backslash\backslash S_A$

- Alexander duality: $\chi(S^3\backslash\backslash S_A) = \chi(S_A) = \chi(H_A)$

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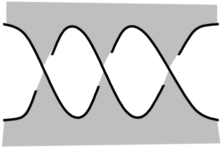
- Relate these I–bundles to combinatorial properties of state graph H_A; they relate to *2-edge loops* of H_A.
- What is the difference $||E_c|| = |\beta'| - |\chi(\text{Guts}\,(S_A))|$? What obstructs to equality in general?

Example: I–bundles

A *twist region* is a non-empty string of bigons arranged end to end.



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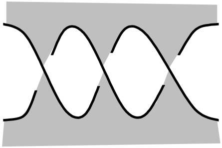

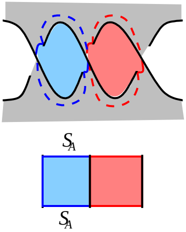

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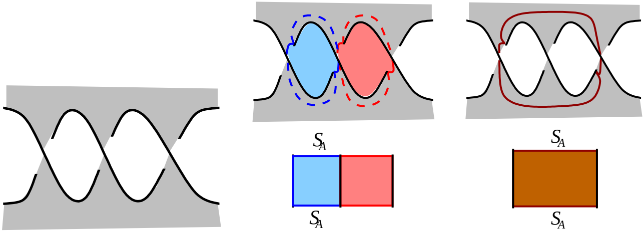

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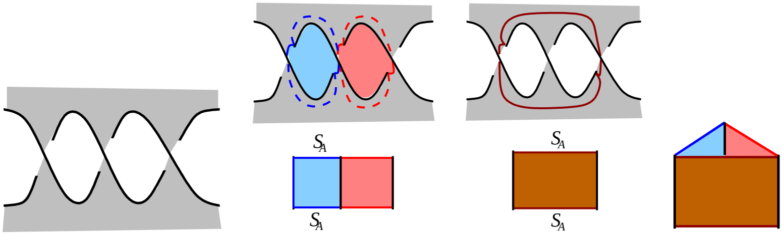

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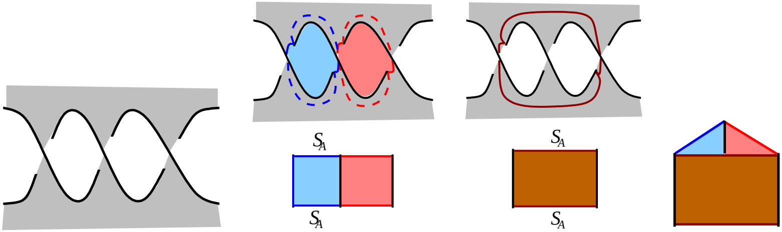

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An *essential product disk (EPD)* is a normal disk with boundary consisting of two on S_A connecting two ideal vertices (we view these as arcs on parabolic locus=knot).

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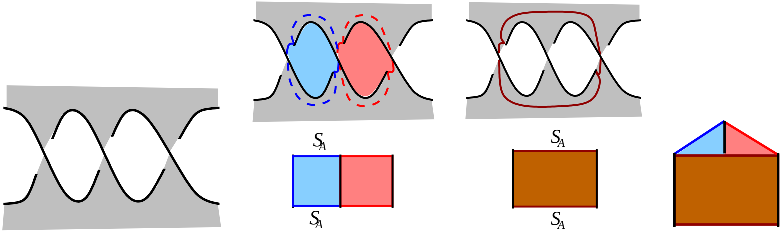

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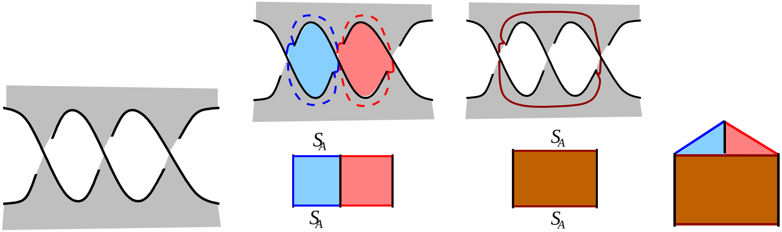

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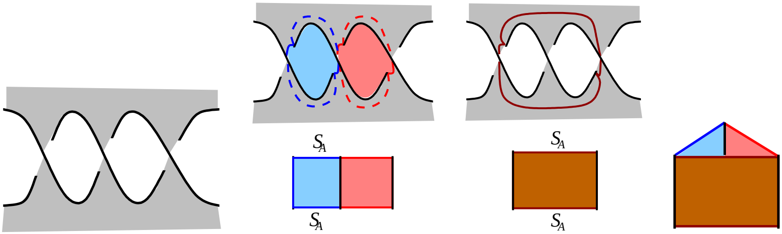

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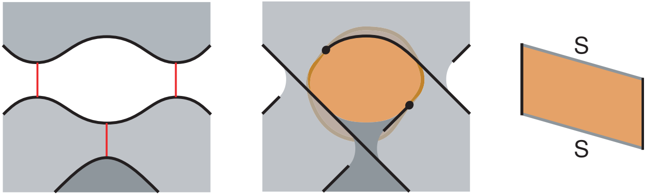
- (Lackenby) These are the only EPDs in (reduced) alternating links.

EPDs span I–bundles



A “non-twist region” EPD

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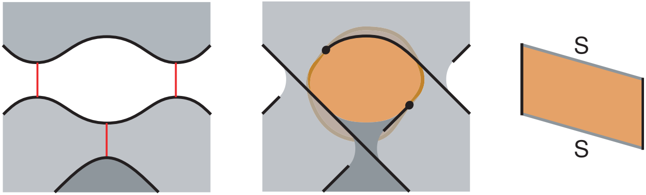

A "non-twist region" EPD

Theorem (FKP)

Let B be an I–bundle component of the JSJ decomposition of $S^3 \backslash\backslash S_A$, with $\chi(B) < 0$. Then B is spanned by EPDs, each embedded in a single polyhedron of the decomposition.

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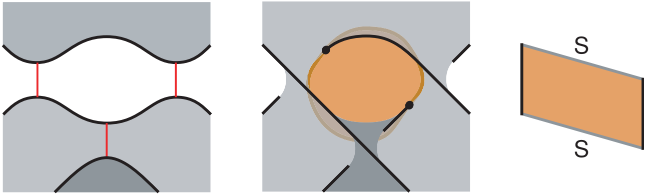

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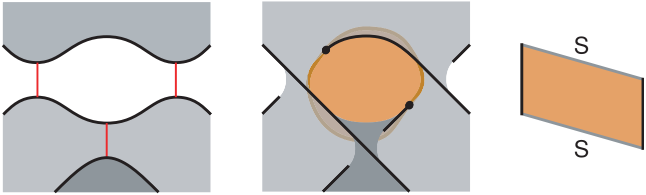

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- **Goal:** search for EPDs in polyhedra.
- Lower polyhedra: Correspond to alternating links.
  Lackenby result $\Rightarrow$ EPDs occur only at twists.

EPDs and "Upper" Polyhedron

In general, an EPD **MUST** run over a 2–edge loop in state graph $\mathbb{H}_A$. The loop either:

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In general, an EPD **MUST** run over a 2–edge loop in state graph H_A. The loop either:

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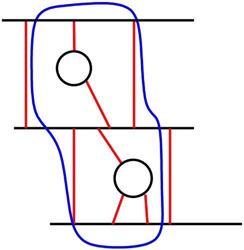
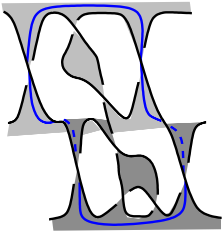
2. Does not.

   Complex EPD. It may bound "non-trivial" parts of H_A on both sides.





The correction term $||E_c||$ discussed earlier "counts" complex EPDS (In the "upper" polyhedron that do not *prabolically compress* ("simplify") to EPDs in "lower" polyhedra.)