Geometric structures and knot invariants

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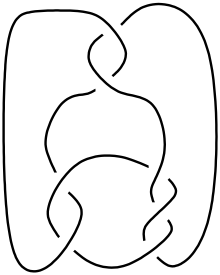
General theme

Knots: Smooth embedding $K : S^1 \to S^3$.
Equivalence: K_1, K_2 are equivalent if $f(K_1) = K_2$, f homeomorphism of S^3.



Talk Goal: Relations among the three perspectives.

Knot diagrams

- Combinatorial invariants

3-manifold topology/geometry

- $S^3 \setminus K$ is 3-manifold. Geometric structures and invariants arising from geometry

Physics originated invariants

- Jones polynomials
- Quantum invariants

Knots and 3-manifolds:

Given K remove an open tube around K to obtain the

$$\textit{Knot complement: } M_K = S^3 \setminus K$$

Compact, orientable 3-manifold with torus boundary.

Papakyriakopoulos, 1950's

- Map $\pi_1(\partial M_K) \to \pi_1(M_K)$ is injection unless K=*Trivial Knot*. Thus $\pi_1(\partial M_K)$ always contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup.

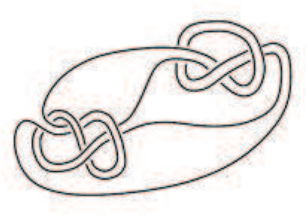
Schreirer (1920's), Schubert (1950's), Burde-Zieschang (1960'), Jaco-Shalen-Johannson (1970's), W. Thurston (1980's), ...

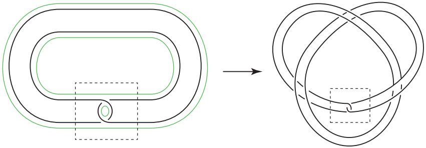
Three distinct types of knot complements according to π_1:

- *Toroidal:* $\pi_1(M_K)$ contains $\mathbb{Z} \oplus \mathbb{Z}$ subgroups not conjugate to $\pi_1(\partial M_K)$.
- *Annular:* Center of $\pi_1(M_K)$ is non-trivial (It is $\mathbb{Z}$).
- *Hyperbolic:* $\pi_1(M_K)$ has no center and contains no $\mathbb{Z} \oplus \mathbb{Z}$ subgroup not conjugate to $\pi_1(\partial M_K)$.

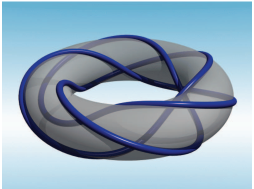
Thee types of knots:

Satellite Knots: Complement contains embedded "essential" tori carrying $\mathbb{Z} \oplus \mathbb{Z}$ subgroups of π_1. There is a *canonical* (finite) collection of such tori.



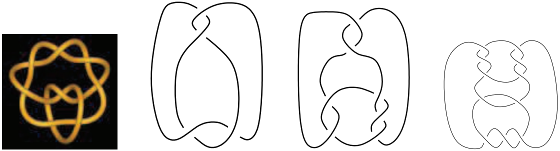


Torus knots: Complement contains embedded "essential" annulus carrying the center of π_1. Knot embeds on standard torus in T in S^3 and is determined by its class in $H_1(T)$.



Thee types of knots con't.

Hyperbolic knots: Knot complement can be given a complete Riemannian metric of constant negative curvature.



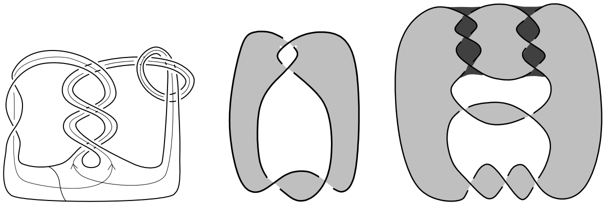
- *Mostow-Prasad Rigidity Theorem:* Hyperbolic metrics in three dimensions are essentially unique: any two are isometric. Hence, invariants of hyperbolic metric are topological invariants of complement.

Important invariant: *Volume* of a hyperbolic knot: $\mathrm{Vol}\,(S^3 \setminus K)$.

- *Volume* Can be defined for *all* knots:
- For torus knots $\mathrm{Vol}\,(S^3 \setminus K) = 0$.
- For satellite knots: Decompose $S^3 \setminus K$ along the canonical collection of tori– add the volumes of the hyperbolic pieces.

Surfaces spanned by knots

- Homological reasons imply that every knot bounds a *Seifert surface*: an embedded, oriented 2-manifold.
- Knots also bound non-orientable surfaces



- (S, K) can be viewed as properly embedded in the knot complement M_K.
- S is *essential* if inclusion induces injection

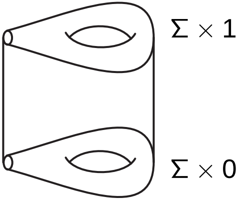
$$\pi_1(S, K) \longrightarrow \pi_1(M_K, \partial M_K).$$

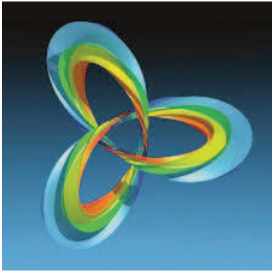
  All knots bound essential surfaces (e.g. minimum genus surfaces).
- Essential surfaces are important for geometry and topology.
- Given essential S cut the knot complement along S; the 3-manifold $M_S := S^3 \backslash\backslash S$ carries information about topology/geometry of $S^3 \setminus K$.

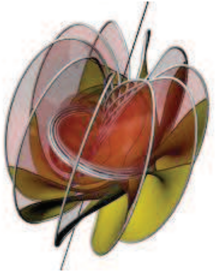
Fibered Knots

K=Knot, Σ= surface bounded by K, $M_K = S^3 \setminus K$= Knot complement.

- Σ is a *fiber* for K iff $M_K = S^3 \setminus K$ cut along Σ is a product $\Sigma \times [0,1]$.
- M_K is a fiber-bundle over S^1 with fiber Σ.
- A "fan" of surfaces around K, fills entire S^3.







Fibered knots are important in several mathematics areas: e.g. 3-manifold and 4-manifold theory, symplectic geometry, ...

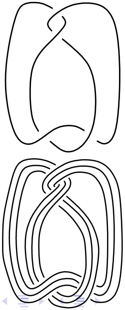
Jones Polynomial–Quantum invariants

- 1980's: Ideas originated in physics and constructions, often, inspired by representation theory led to invariants of knots and 3-manifolds. (*Quantum invariants*)
- Knots and 3-manifolds often enter the picture through their combinatorial descriptions: e.g. knot diagrams, Dehn surgery presentations..
- Knot invariants can be calculated from diagrams via "Skein theory".

Jones, Witten, Reshetikhin-Turaev, Kauffman, HOMFLY-PT,...

Of particular interest for this talk are:

- The *Colored Jones Polynomials*: Infinite sequence of Laurent polynomials $\{J_{K,n}(t)\}_n$ encoding the *Jones polynomial* of K and these of the parallels of K.
- **Key Question:** How do the *CJP* relate to geometric structures of knot complements and to incompressible surfaces in them?
- **Talk Focus.** Discuss joint work with Futer (Temple), Purcell (BYU), Lee (MSU)- state known conjectures.



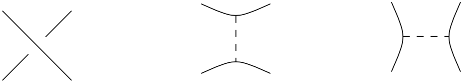
Plan of rest of talk

Given a diagram $D(K)$ construct a certain graph $\mathbb{G}$ (*state graph*) such that,

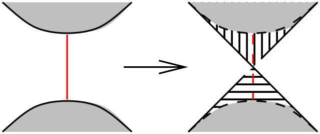
- $\mathbb{G}$ encodes information about the (colored) Jones polynomial of K.
- $\mathbb{G}$ embeds "canonically" on a surface $S_{\mathbb{G}}$ spanned by K.
- Combinatorics of $\mathbb{G}$ determine when $S_{\mathbb{G}}$ is essential in $S^3 \setminus K$ and the *geometric decomposition* of surface complement $M_S := S^3 \backslash\backslash S_{\mathbb{G}}$.
- M_S carries a lot of geometric information about $S^3 \setminus K$. Use this to relate Jones polynomials to topology/geometry of $S^3 \setminus K$.
- **CPJ encode information about:**
  - *Boundary slopes* of knots
  - Fibers in knot complement.
  - volume of knot complements.
- **Tools:** Ideal polyhedral decompositions- Normal surface theory.
- **Conjectures/Motivation:**
- Slopes Conjecture
- Volume conjecture

State Graphs

Two choices for each crossing, of knot diagram D: A or B resolution.

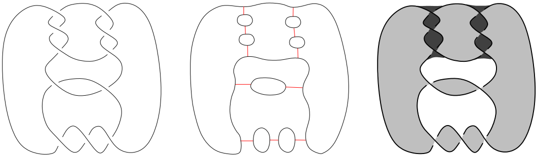


- A Kauffman *state* $\sigma(D)$ is a choice of A or B resolutions for all crossings.
- $\sigma(D)$: *state circles*
- Form a *fat graph* H_σ by adding edges at resolved crossings.
- Get a *state surface* S_σ: Each state circle bounds a disk in S_σ (nested disks drawn on top).
- At each edge (for each crossing) attach twisted band.
- Contract state circles to vertices to get *state graph* $\mathbb{G}_\sigma$: surface is orientable iff the state graph is bipartite.

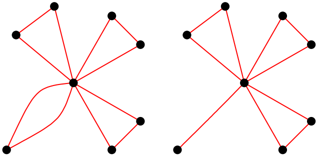


Example: Two component link

Working with the all A-state:
Diagram $D(K)$ of a two-component link, and graphs H_A, the surface S_A.



State graph $\mathbb{G}_A$ and reduced graph $\mathbb{G}'_A$.



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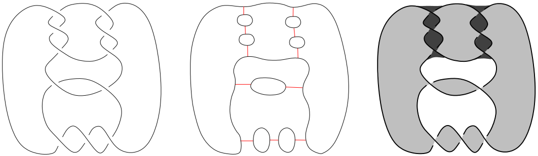

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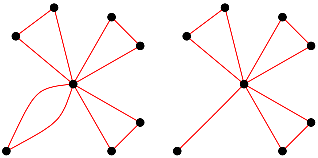

- The Jones polynomial of the knot can be calculated from H_A: *spanning graph expansion* arising from the Bollobas-Riordan *fat graph* polynomial (Dasbach-Futer-K-Lin-Stoltzfus, 2006).

The CJP and state graphs and surfaces

- As said, given any link diagram $D(K)$ the Jones polynomial $J_{K,2}(t)$ can be computed from the fat graph H_A.
- The *n-colored Jones polynomial* $J_{K,n}(t)$, is expressed as a function that, roughly speaking, counts *spanning* subgraphs of H_A and of A-state graphs of certain parallels of $D(K)$.
- (K.-Lee) Studied asymptotic behavior of this function ($n \to \infty$) and obtained a linear polynomial (invariant of K)

$$\tau_K(t) = \alpha' + \beta' t,$$

detecting exactly when the state surface $S_A(D)$ is essential in $S^3 \setminus K$!

Theorem (K.-Lee, 2013)

We have, $\tau_K(t) \neq 0$ iff K admits a diagram $D(K)$ such that the state surface $S_A(D)$ is essential in the complement $S^3 \setminus K$.

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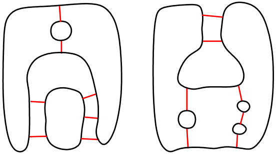
So what?

When is S_A essential?

- (Ozawa, Futer-K.-Purcell) The surface S_A is essential in $S^3 \setminus K$ iff the corresponding the state graph H_A has no 1-edge loops.
- Lickorish–Thistlethwaite 1980's: Introduced *A–adequate* links in the context of Jones polynomials.
- **Definition.** A link is A–adequate if has a diagram where H_A has no 1-edge loops.



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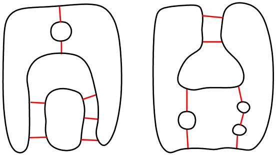

- All, but two, prime knots up to 11 crossings.
- Torus knots: all
- hyperbolic: non-torus, alternating, Montesinos Knots, positive knots, closed 3-braids, "complicated" plat closures...
- Satellites: planar cables, Whitehead doubles

Colored Jones polynomial prelims

For a knot K, and $n = 1, 2, \ldots,$ we write its *n-colored Jones polynomial*:

$$J_{K,n}(t) := \alpha_n t^{m_n} + \beta_n t^{m_n - 1} + \cdots + \beta'_n t^{k_n + 1} + \alpha'_n t^{k_n} \in \mathbb{Z}[t, t^{-1}]$$

Some properties:

- $J_{K,1}(t) = 1$ and $J_{K,2}(t)$ is the ordinary Jones polynomial of K.
- $J_{K,n}(t)$ is determined by the Jones polynomials of certain cables of K.
- (Garoufalidis-Le, 2004): The sequence $\{J_{K,n}(t)\}_n$ is *q-holonomic*. This implies, that for every K the sequence $\{J_{K,n}(t)\}_n$ is determined by finitely many terms.
- Degrees m_n, k_n grow quadratically in n. Furthermore, each of the two sequences

  $$\{\frac{-4}{n^2} k_n\}_n \quad \{\frac{-4}{n^2} m_n\}_n,$$

  has finitely many cluster points.
- Each of $\alpha'_n, \beta'_n \ldots$ satisfies a linear recursive relation in n, with integer coefficients . (e. g. $\alpha'_{n+1} + (-1)^n \alpha'_n = 0$).

Remark. Properties manifest themselves in strong forms when K is *A-adequate* (next).

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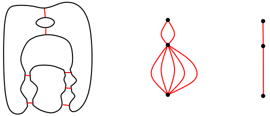
CJP of A-adequate links facts

State graph $\mathbb{G}_A$; remove multiple edges to get simple graph $\mathbb{G}'_A$.



Lickorish-Thistlethwaite (80's), Dasbach-Lin (2006)
Armond (2011), Armond-Dasbach (2011), Garoufalidis-Le (2011)..

$$J_{K,n}(t) := \alpha_n t^{m_n} + \beta_n t^{m_n - 1} + \cdots + \beta'_n t^{k_n+1} + \alpha'_n t^{k_n}.$$

- Last two coefficients $\alpha'_K = |\alpha'_n| = 1$, $\beta'_K := |\beta_n| = 1 - \chi(\mathbb{G}'_A)$, $n > 1$.
- Invariant studied by K.-Lee now becomes $\tau_K(t) = 1 + \beta'_K t$.
- Minimum degree $k_n = -sn^2 + O(n)$, s is an integer.
- (the abs. values of) m-th to last coefficients of $J_{K,n}(t)$ is independent on n, for $n \geq m$. They get *stable coefficients* for all. They define the *Tail* of JCP.
- Stable coefficients depend only on $\mathbb{G}'_A$!

CJP and the surface S_A: Boundary slopes

- The class $[\partial S_A]$ in $H_1(\partial(S^3 \setminus K))$ is determined by an element in $\mathbf{Q} \cup \{\infty\}$, called *a boundary slope of K*.
- (Hatcher, 1980) Every knot has finitely many boundary-slopes.

Theorem (Futer-K-Purcell, 2010)

For an A–adequate diagram,

$$s = \textit{bdry slope of } S_A = \lim_{n\to\infty} \frac{-4}{n^2} k_n,$$

$k_n :=$ *min* $\deg J_{K,n}(t)$.

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- **Slopes Conjecture.** (Garoufalidis, motivated by work of Garoufalidis-Le and Frohman- Gelca- Lofaro) For every knot K each of the finitely many cluster points $\{\frac{-4}{n^2} k_n\}_n$ is a boundary slope of K.
- (Dunfield-Garoufalidis) Verified conjecture for class of knots that are not A-adequate. (Degree of CJP was found by computer calculation).

CJP and the surface S_A: Coefficients

For an A-adequate link, β'_K is the stabilized penultimate coefficient of CJP.

Theorem (Futer–K–Purcell)

For an A–adequate diagram $D(K)$, the following are equivalent:

1. *The penultimate coefficient is $\beta'_K = 0$.*
2. *S_A is a fiber in $S^3 \setminus K$.*

Exercise. Derive Stalling's classical result: *positive* closed braids are fibered with fiber obtained from Seifert's algorithm to the braid diagram.

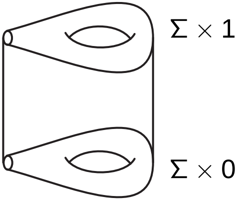
Next:

- What about when $\beta_K > 0$?
- When β'_K is large, S_A is far from being a fiber, in a sense we will specify below.
- This, combined with work of Agol- W. Thurston- Storm, gives that large β'_K implies large $\mathrm{Vol}(S^3 \setminus K)$.

Topology of complement of S_A

- $M_A = S^3 \backslash\backslash S_A$ is obtained by removing a neighborhood of S_A from S^3.
- On ∂M_A we have
  parabolic locus P = remains from $\partial(S^3 \setminus K)$ after cutting along S_A.
- We work with pair (M_A, P).
- You may think as if ∂M_A is "decorated"; decompositions of M_A below do not disturb decorations.
- There is a version of Jaco-Shalen-Johannson decomposition theory for paired 3-manifolds that assures that M_A cut along a canonical collection of essential annuli results in three kinds of pieces:

1. I–bundles (think of $\Sigma \times I$ for $\Sigma \subset S_A$, although $\Sigma \widetilde{\times} I$ can also occur),
2. Seifert fibered solid tori,
3. *Guts* $(S^3 \setminus K, S_A)$. By Thurston's theory the guts admits hyperbolic structure.



Topology of Guts and Volume

Guts serve as an indication that a surface S_A is far from being a fiber.

1. If S_A is a fiber of $M_A = S_A \times I$: no guts. (Recall, β'_K=0)
2. Guts $(S^3 \setminus K, S_A) = \emptyset$ M_A is union I-bundles and solid tori. – S_A is "almost fiber" .
3. We want to calculate $\chi(\text{Guts}(M, S))$ because it estimates volume via the following theorem:

Theorem (Agol–Storm–W. Thurston, 2007)

Let M be a compact 3–manifold with hyperbolic interior of finite volume, and $S \subset M$ an embedded essential surface. Then

$$Vol(M) \geq -v_8\, \chi(Guts(M, S)),$$

where $v_8 \approx 3.6638$ is the volume of a regular ideal octahedron.

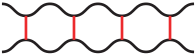
$D(K)$ =an A-adequate diagram with S_A the corresponding all-A state surface.

Theorem (FKP, 2011)

Let $D(K)$ be an A–adequate diagram such that every 2–edge loop in G_A comes from a twist region. Then the surface S_A satisfies

$$\chi(Guts\,(S^3 \setminus K,\, S_A)) \;=\; 1 - \beta'_K$$

twist region



In General ,

$$\chi(\mathrm{Guts}\,(S^3 \setminus K,\, S_A)) \;=\; 1 - \beta'_K + \text{explicit correction term}$$

Corollary

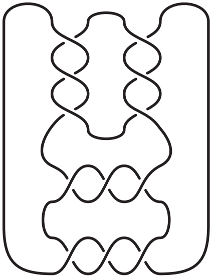
Under the hypotheses of theorem,

$$Vol\,(S^3 \setminus K) \geq v_8\,(\beta'_K - 1).$$

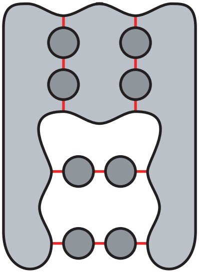
Alternating knots: follows from work of Lackenby and Dasbach–Lin.
A. Giambrone: large families of non-alternating knots satisfying hypothesis.

A worked example

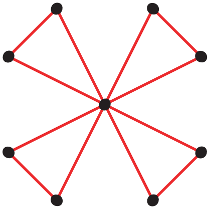
$D(K)$



all–A state



$G_A = G'_A$



$$1 - |\beta'| = \chi(G_A) = \chi(S_A)$$

A worked example

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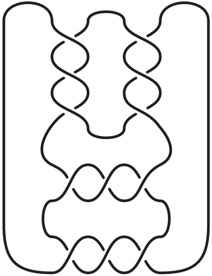

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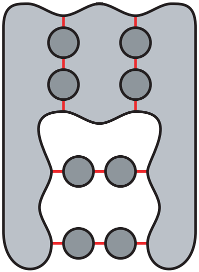

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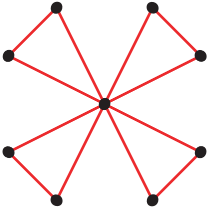


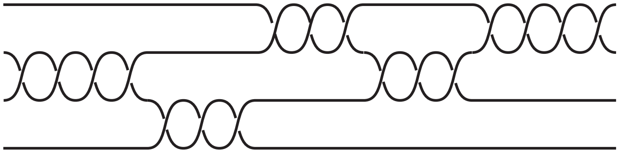
$$1 - |\beta'| = \chi(G_A) = \chi(S_A) = \chi(S^3 \backslash\backslash S_A) = \chi(\text{Guts}) = -3$$

$$v_8\,(|\beta'| - 1) = -v_8\,\chi(G'_A) = 10.99...$$

$$\text{Vol}\,(S^3 \setminus K) = 13.64...$$

Exercise. Above diagram is also B-adequate and the reduced state graph S_B is a tree. Thus K is fibered knot with fiber the state surface S_B.

Sample family: positive braids



$\sigma_2^4\,\sigma_1^3\,\sigma_3^3\,\sigma_2^3\,\sigma_3^4$

Theorem (FKP)

Suppose that K is the closure of a positive braid $b = \sigma_{i_1}^{r_1}\sigma_{i_2}^{r_2}\cdots\sigma_{i_k}^{r_k}$, where $r_j \geq 3$ for all j. In other words, there are k twist regions, each with at least 3 crossings.
Then K is hyperbolic, and

$$\frac{2v_8}{3}k \;\leq\; Vol(S^3 \setminus K) \;<\; 10v_3(k-1)\,.$$

Similarly,

$$v_8\,(\beta'_K - 1) \;\leq\; Vol(S^3 \setminus K) \;<\; 15v_3\,\beta'_K - 25v_3\,.$$

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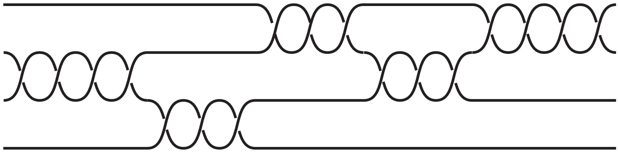

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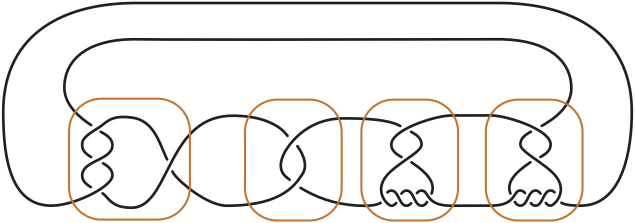
Here, $v_3 = 1.0149...$ is the volume of a regular ideal tetrahedron and $v_8 = 3.6638...$ is the volume of a regular ideal octahedron.

The gap between the upper and lower bounds is a factor of 4.155...

Sample family: Montesinos links

A Montesinos knot or link is constructed by connecting *n* rational tangles in a cyclic fashion.



Every Montesinos link is either *A*– or *B*–adequate.

Theorem (FKP + Finlinson)

Let K be an A–adequate Montesinos link. Then

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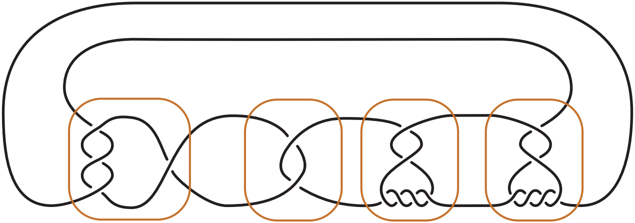

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If K has length at least four we get two-sided volume estimates:

$$v_8\,(\max\{\beta_K, \beta'_K\} - 2) \;\leq\; Vol\,(S^3 \setminus K) \;<\; 4v_8\,(\beta'_K + \beta_K - 2) + 2v_8\,(\#K),$$

where $\#K$ is the number of link components of K.

Volume Conjecture

Results and experimental evidence prompt (*A coarse Volume conjecture?*):

Question. Does there exist function $B(K)$ of the coefficients of the colored Jones polynomials of a knot K, that is easy to calculate from state graphs such that for hyperbolic knots, $B(K)$ is coarsely related to hyperbolic volume $\mathrm{Vol}(S^3 \setminus K)$?
Are there constants $C_1 \geq 1$ and $C_2 \geq 0$ such that

$$C_1^{-1}B(K) - C_2 \leq \mathrm{Vol}(S^3 \setminus K) \leq C_1 B(K) + C_2,$$

for all hyperbolic K?

- Results and stabilization properties of CJP prompt more guided speculations as to where one might look for $B(K)$.
- *Volume Conjecture* (Kashaev 1990's, H. Murakami-J. Murakami, 2001) predicts relations between volume and coefficients of CJP.– The entire JCP should determine the volume exactly.

$$2\pi \lim_{n\to\infty} \frac{\log \left|J_K^n(e^{2\pi i/n})\right|}{n} = \mathrm{Vol}(S^3 \setminus K).$$

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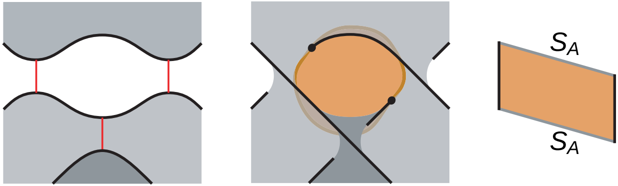
2–edge loops and I–bundles of $S^3 \backslash\backslash S_A$

Every 2–edge loop in G_A gives rise to a disk D that intersects K twice — a *essential product disk (EPD)* in the complement of the state surface S_A.



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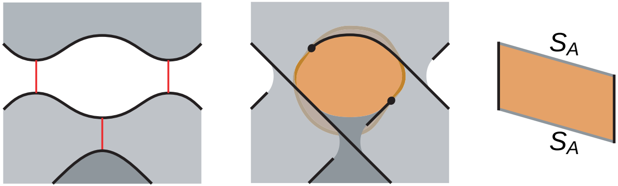

- To find Guts $(S^3\backslash\backslash S_A)$, start with $S^3\backslash\backslash S_A$ and remove I–bundle pieces.
- When we remove and EPD from $S^3\backslash\backslash S_A$, Euler number $\chi(S^3\backslash\backslash S_A)$ goes up by 1. Removing a redundant edge from G_A also increases $\chi(G_A)$ by 1.
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- We prove that the *maximal I–bundle* of $S^3\backslash\backslash S_A$ is spanned by EPD's that correspond to 2–edge loops in G_A.

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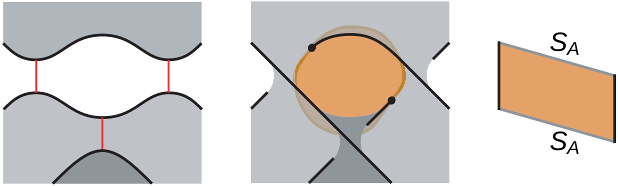

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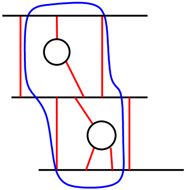
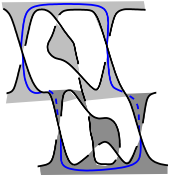
$$\chi(\text{Guts}) = \chi(S_A) + \#\text{EPDs} = \chi(G_A \setminus \text{extra edges}) = \chi(G'_A).$$

Topology of β'_K: most general form

A 2–edge loop in G_A may correspond to multiple product disks, some of which are *complex*. The number of complex disks is $||E_c|| \geq 0$.

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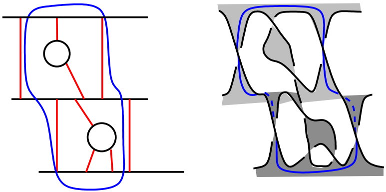

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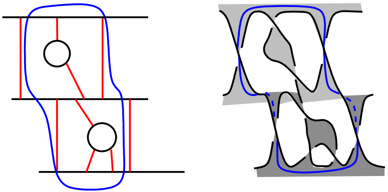

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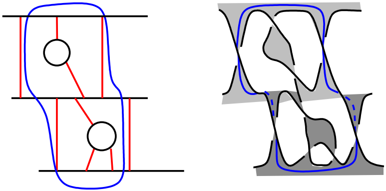

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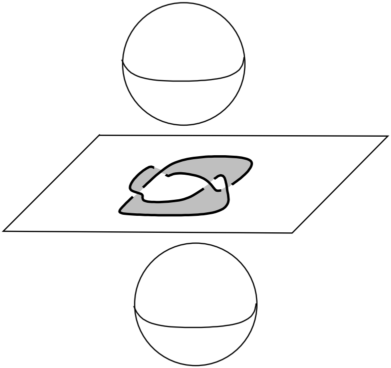
Question: For each A–adequate link, is there a diagram with $||E_c|| = 0$?

Tool for the proof: a nice polyhedral decomposition

Our results are proved using *normal surface theory* in a suitable polyhedral decomposition of the surface complement $S^3 \backslash\backslash S_A$.

For alternating links, this is Menasco's polyhedral decomposition:



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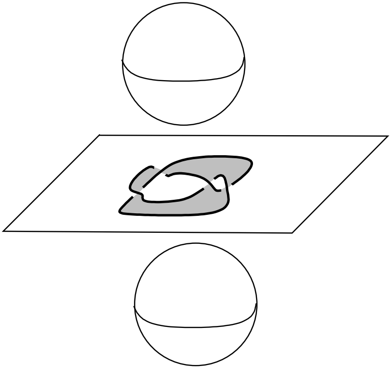

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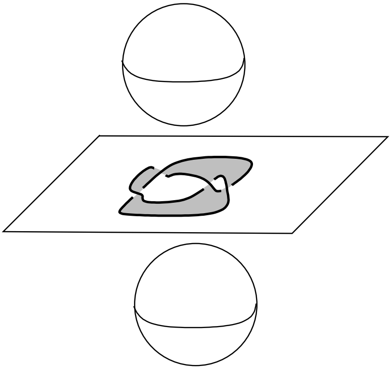

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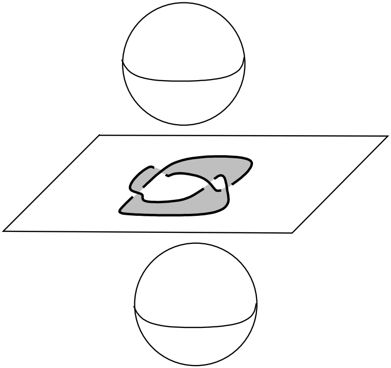

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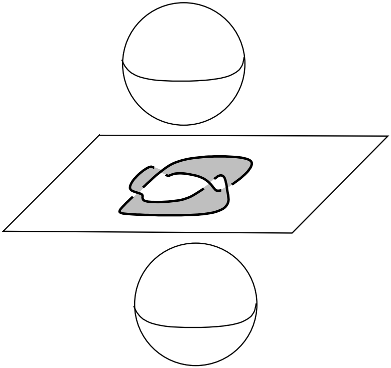

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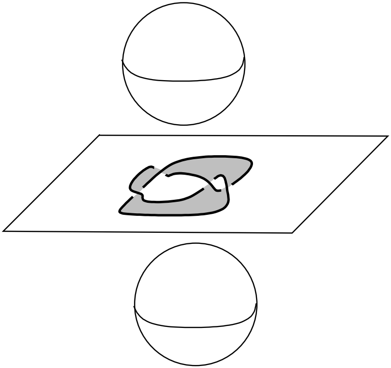

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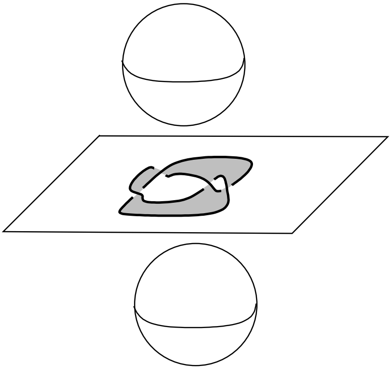
- The union of all the shaded faces is a checkerboard surface S_A.



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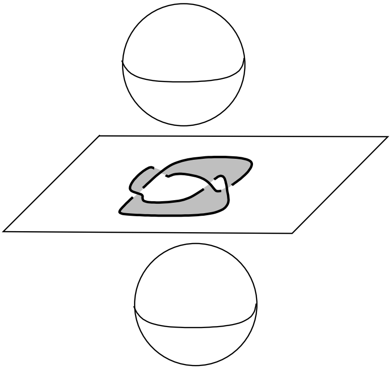
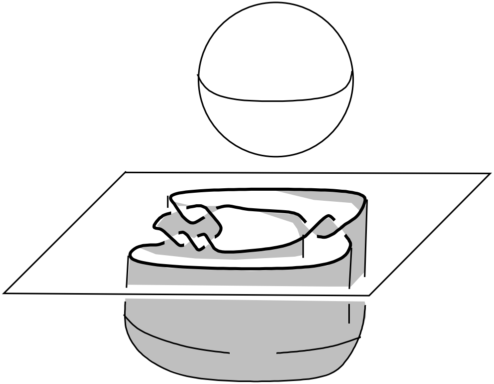
- Hence, gluing along white faces only produces a decomposition of $S^3 \backslash\backslash S_A$.



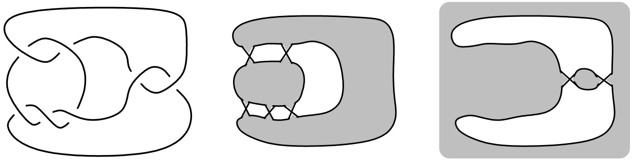
Polyhedral decomposition of the surface complement

Our surface S_A is layered below the plane of projection. We need more balloons to subdivide $S^3 \backslash\backslash S_A$.



Combinatorial descriptions of Polyhedra

Lower polyhedra are identical to checkerboard polyhedra of alternating sublinks.



Upper polyhedron: Ideal edges and shaded faces are sketched by *tentacles* on projection of H_A