

1. Consider the closed unit ball $\bar{B}_1(0)$ in \mathbf{R}^n . Let p and q be distinct points in the open unit ball $B_1(0)$. Prove that there is a C^∞ diffeomorphism g from \mathbf{R}^n to \mathbf{R}^n such that $g(p) = q$ and $g(x) = x$ for all x such that $|x| \geq 1$. (Hint: Find g as the time-one map of a suitable C^∞ vector field X).
2. Construct a C^∞ vector field X in the plane with exactly eight critical points such that half of them are hyperbolic saddle points and half of them are hyperbolic sinks.
3. Let $X(x) = \text{grad}(\psi)(x)$ be the gradient vector field of a C^∞ function $\psi : \mathbf{R}^n \rightarrow \mathbf{R}$. Assume that ψ is bounded above; i.e., there is a constant $K > 0$ such that $\psi(x) \leq K$ for all $x \in \mathbf{R}^n$. Assume that X has exactly two hyperbolic critical points. Prove that they cannot both be sinks.
4. Let X be a C^1 vector field in \mathbf{R}^n . Prove that every hyperbolic critical point is isolated in the set of all critical points.
5. Let X be a C^1 planar vector field such that $X(x) \cdot x > 0$ for all x with $|x| \geq 3$. Assume that X has only hyperbolic critical points. Prove that they cannot all be saddles.
6. Let $\mathcal{X}^1(\mathbf{R}^n)$ be the set of C^1 vector fields defined on all of \mathbf{R}^n . For $X \in \mathcal{X}^1(\mathbf{R}^n)$, define

$$\|X\|_1 = \sup_{x \in \mathbf{R}^n} \max(|X(x)|, |DX(x)|)$$

This satisfies the usual properties of a norm except that it may be infinite.

Let $r \geq 0$.

We say that two vector fields X and Y on \mathbf{R}^n are *topologically equivalent* if there is a homeomorphism h from \mathbf{R}^n onto itself carrying orbits of X to orbits of Y .

If X and Y are complete vector fields (i.e., solutions defined for all time), with flows ϕ, η , respectively, we say that X and Y are topologically conjugate if there is a homeomorphism $h : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that, for each $t \in \mathbf{R}$, we have

$$h\phi_t h^{-1} = \eta_t$$

We also use the terms C^0 -equivalent or C^0 -conjugate for topological equivalent or topologically conjugate, respectively. If the maps h can be taken to be C^r with $r \geq 1$, then we use the terms C^r -equivalent or C^r -conjugate, respectively. The maps h are then called a topological (C^r) equivalence or topological (C^r) conjugacy.

We say that X is *structurally stable* if there is an $\epsilon > 0$ such that if $Y \in \mathcal{X}^1(\mathbf{R}^n)$ and $\|Y - X\|_1 < \epsilon$, then Y is topologically equivalent to X .

7. Prove that if X is structurally stable, then there is an $\epsilon > 0$ such that if $\|Y - X\|_1 < \epsilon$, then Y is also structurally stable.
8. Suppose that X is structurally stable. Prove that every isolated critical point of X is hyperbolic.

Remark. This is actually true without the assumption that the critical point is isolated.

9. Give an example of two C^∞ planar vector fields X and Y which are topologically equivalent, but not topologically conjugate.
10. Let X be a complete C^∞ vector field in \mathbf{R}^n , and let ρ be a C^∞ diffeomorphism from \mathbf{R}^n onto itself. Prove that the push-forward vector field $\rho_*(X)$ is C^∞ conjugate to X .
11. Sketch the solutions of each of the following differential equations, indicating whether the critical points are sources, sinks, saddles, or centers.

$$\begin{aligned} \text{(a)} \quad x' &= y \\ y' &= x(x-1)(x-2) \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad x' &= 3y \\ y' &= -x(x-1)(x-2)(x-3)(x-4) - y \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad x' &= 3y \\ y' &= -(x^3 - x)(x-3) + 2y \end{aligned}$$