$x' = ax + by$

$y' = cx + dy$. $\cancel{\text{char}}$ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

char poly: $r^2 - (a+d)r + ad - bc$.

$b = c = 0$

$\Rightarrow x' = ax$

$y' = dy$.

Gen soln

$x(t) = e^{ax} c_1$

$y(t) = e^{dx} c_2$

1) Roots r_1, r_2.

Case 1: $r_1 \neq r_2$ real, $b \neq 0$.

Eigenpairs: $\left(r, \begin{pmatrix} 1 \\ \frac{r-a}{b} \end{pmatrix} \right)$, $r = r_1, r_2$.

Gen soln: $\underline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{r_1 t} \begin{pmatrix} 1 \\ \frac{r_1 - a}{b} \end{pmatrix} + c_2 e^{r_2 t} \begin{pmatrix} 1 \\ \frac{r_2 - a}{b} \end{pmatrix}$

2) Roots $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, $\beta \neq 0$.

$\underline{x}_c(t) = e^{(\alpha + i\beta)t} \begin{pmatrix} 1 \\ \frac{\alpha - i\beta - a}{b} \end{pmatrix}$

Gen soln: $\underline{x}(t) = c_1 \operatorname{Re} \underline{x}_c(t) + c_2 \operatorname{Im} \underline{x}_c(t)$

3) $r_1 = r_2$ ~~mora~~

Gen soln: $\underline{x}(t) = c_1 e^{r_1 t} \begin{pmatrix} 1 \\ \frac{r_1 - a}{b} \end{pmatrix} + c_2 \left[t e^{r_1 t} \begin{pmatrix} 1 \\ \frac{r_1 - a}{b} \end{pmatrix} + e^{r_1 t} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right]$

where $(A - r_1 I)\xi = \begin{pmatrix} 1 \\ (r-a)/b \end{pmatrix}$

Elimination Method:

$b \neq 0$.

$x' = ax + by, \quad x'' = ax' + by'$

$$= ax' + b(cx + dy)$$

$$= ax' + bcx + bd\left(\frac{x' - ax}{b}\right).$$

or $x'' - (a+d)x' + ad - bc = 0$.

$\cancel{\text{Case 1: } r^2 - tr(A) r + \det A, \ r_1, r_2}$

Take roots r_1, r_2.

Case 1: $r_1 \neq r_2$, real.

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$y(t) = \frac{x' - ax}{b} \text{ — plug in.}$$

Case 2: $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, $\beta \neq 0$.

$$x(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

$$y = \frac{x' - ax}{b}.$$

Case 3: $r_1 = r_2$ real.

$$x = c_1 e^{r_1 t} + c_2 t e^{r_1 t}.$$

$$y = \frac{x' - ax}{b}.$$

Similar if $\cancel{c \neq 0}$, $b = 0$, $c \neq 0$, Get ~~use~~ $y(t)$

$$x(t) = \frac{y' - \cancel{dy}\, d \cdot y}{c}.$$

Example: 1) $x' = 3x + 4y$

$y' = x + 2y$.

$$r^2 - 5r + 2, \quad r = \frac{5 \pm \sqrt{25-8}}{2} = \frac{5}{2} \pm \frac{\sqrt{17}}{2}$$

Matrix method:

Eigpair: $\left(\frac{5}{2} + \frac{\sqrt{17}}{2}, \begin{pmatrix} 1 \\ \frac{\frac{1}{2} + \frac{\sqrt{17}}{2}}{4} \end{pmatrix}\right)$

$\left(\frac{5}{2} - \frac{\sqrt{17}}{2}, \begin{pmatrix} 1 \\ \frac{\frac{1}{2} - \frac{\sqrt{17}}{2}}{4} \end{pmatrix}\right)$.

Gen soln: $$\underline{x}(t) = c_1 e^{\left(\frac{5}{2} + \frac{\sqrt{17}}{2}\right)t} \begin{pmatrix} 1 \\ \frac{1}{8} + \frac{\sqrt{17}}{8} \end{pmatrix} + c_2 e^{\left(\frac{5}{2} - \frac{\sqrt{17}}{2}\right)t} \begin{pmatrix} 1 \\ \frac{1}{8} - \frac{\sqrt{17}}{8} \end{pmatrix}$$

Elim method: $$x = c_1 e^{\left(\frac{5}{2} + \frac{\sqrt{17}}{2}\right)t} + c_2 e^{\left(\frac{5}{2} - \frac{\sqrt{17}}{2}\right)t}$$

$$y = \frac{x' - 3x}{4}$$

$$= \frac{1}{4}\left[c_1 \left(\frac{5}{2} + \frac{\sqrt{17}}{2}\right) e^{\left(\frac{5}{2} + \frac{\sqrt{17}}{2}\right)t} + c_2 \left(\frac{5}{2} - \frac{\sqrt{17}}{2}\right) e^{\left(\frac{5}{2} - \frac{\sqrt{17}}{2}\right)t} - 3\left[c_1 e^{r_1 t} + c_2 e^{r_2 t}\right] \right]$$

$r_1 = \frac{5}{2} + \frac{\sqrt{17}}{2}$

$r_2 = \frac{5}{2} - \frac{\sqrt{17}}{2}$

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

$$y = \frac{1}{4}\left[c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} - 3c_1 e^{r_1 t} - 3c_2 e^{r_2 t}\right]$$

$$= \frac{x' - 3x}{4}$$

2) $x' = \cancel{3}\,3x + y.$

$y' = \cancel{3y}.\ 3y$

$r^2 - 6r + 9$

$(r-3)^2$

$$x = c_1 e^{3t} + c_2 t e^{3t}$$

$$y = \underline{x' - 3x} = 3\cancel{c_1 e^{3t}} + c_2 e^{3t} + 3\cancel{c_2 t e^{3t}}$$

$$- 3\cancel{c_1 e^{3t}} - 3\cancel{c_2 t e^{3t}}$$

$$= c_2 e^{3t}$$