

# Sinks in Acyclic Orientations of Graphs

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Greene and Zaslavsky proved that the number of acyclic orientations of a graph  $G$  with a unique sink at a given vertex is, up to sign, the linear coefficient of the chromatic polynomial. We give three proofs of this result using pure induction, noncommutative symmetric functions, and an algorithmic bijection. We also prove their result that if  $e = u_0v_0$  is an edge of  $G$  then the number of acyclic orientations having a unique source at  $u_0$  and unique sink at  $v_0$  is Crapo's beta invariant.

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## 1. ACYCLIC ORIENTATIONS

Our primary focus will be a theorem of Greene and Zaslavsky [9] concerning acyclic orientations of a graph and its chromatic polynomial. To state it we need some definitions. For any undefined terms, we follow the terminology of Harary's book [10].

Let  $G$  be a finite graph with vertices  $V = V(G)$  and edges  $E = E(G)$ . We permit  $E$  to contain loops and multiple edges. An *orientation* of  $G$  is a digraph formed by replacing  $e \in E$  by one of the two possible directed arcs. The orientation is *acyclic* if it has no directed cycles. We let  $\mathcal{A}(G)$  be the set of acyclic orientations of  $G$ . So if  $G$  has a loop then it has no acyclic orientations and  $\mathcal{A}(G) = \emptyset$ . A *sink* of a digraph is a vertex  $v_0$  such that all

arcs incident with  $v_0$  are directed toward it. Let  $\mathcal{A}(G, v_0)$  be the set of acyclic orientations of  $G$  with a unique sink at  $v_0$ .

A *proper coloring* of  $G$  with color set  $C$  is a map  $\kappa: V \rightarrow C$  such that  $uv \in E$  implies  $\kappa(u) \neq \kappa(v)$ . Now consider the *chromatic polynomial* of  $G$  which is

$$\chi_G(n) = \# \text{ of proper } \kappa: V \rightarrow \{1, 2, \dots, n\}.$$

It is well known [11] that  $\chi_G(n)$  is a polynomial in  $n$  of degree  $d = |V|$  so we write

$$\chi_G(n) = a_0 + a_1 n + \dots + a_d n^d.$$

If we need to be specific about the graph, we will write  $a_i(G)$  for the coefficient of  $n^i$  in  $\chi_G(n)$ .

Stanley [12] was the first to connect acyclic orientations of graphs and the characteristic polynomial and give an inductive proof of the following theorem. In what follows, absolute value signs around a set denote its cardinality.

**THEOREM 1.1 (Stanley).** *For any graph  $G$*

$$|\mathcal{A}(G)| = |\chi_G(-1)|.$$

The result of Greene and Zaslavsky that will interest us can be seen as an analog of Stanley's theorem for acyclic orientations with a unique sink [9, Theorem 7.3]

**THEOREM 1.2 (Greene and Zaslavsky).** *Let  $v_0$  be any vertex of  $G$ . Then*

$$|\mathcal{A}(G, v_0)| = |a_1|. \quad (1)$$

Originally this theorem was proved using the theory of hyperplane arrangements. The purpose of this paper is to give three other proofs using different techniques.

In the next section we will give a purely inductive proof. Stanley [13] indicated that such a proof exists and we provide the details.

In the paper just cited, Stanley introduced a symmetric function analog of the chromatic polynomial and showed that it counts the number of acyclic orientations of  $G$  with  $j$  sinks,  $1 \leq j \leq d$ . Note that this is not quite the same as counting those with a given sink. In Section 3 we will show how using noncommutative variables allows us to generalize the Greene–Zaslavsky theorem to the level of symmetric functions.

Our final proof of Theorem 1.2 is an algorithmic bijection. To explain it, we need to recall Whitney's Broken Circuit Theorem [15]. A *circuit* in a

graph  $G$  will be the same as a *cycle*, i.e., a closed walk with distinct vertices. If we fix a total order on  $E(G)$ , a *broken circuit* is a circuit with its largest edge (with respect to the total order) removed. Let the *broken circuit complex*  $B_G$  of  $G$  denote the set of all  $S \subseteq E(G)$  which do *not* contain a broken circuit. The Broken Circuit Theorem asserts:

**THEOREM 1.3 (Whitney).** *For any finite graph,  $G$ , on  $d$  vertices we have*

$$\chi_G(n) = \sum_{S \in B_G} (-1)^{|S|} n^{d-|S|}.$$

It follows immediately from Theorems 1.1 and 1.3 that  $|\mathcal{A}(G)| = |B_G|$ . This result was given a bijective algorithmic proof by Blass and Sagan [1]. It is also clear from the previous theorem that

$$|a_1| = |\{S \in B_G : |S| = d-1\}|. \quad (2)$$

So to prove the Greene–Zaslavsky theorem bijectively it suffices to find a bijection between  $\mathcal{A}(G, v_0)$  and  $\{S \in B_G : |S| = d-1\}$ . This will be done in Section 4 by modifying the Blass–Sagan algorithm.

In the penultimate section, we will give an inductive proof of another theorem of Green and Zaslavsky [9] which states that Crapo's beta invariant [3] enumerates graphs with a unique source and sink connected by a given edge. Finally, we end with some open questions.

## 2. PURE INDUCTION

We will show that both sides of Eq. (1) satisfy the same recurrence relation and boundary conditions. We begin with the well-known Deletion–Contraction Rule for the chromatic polynomial [11]. If  $e \in E(G)$  we will let  $G \setminus e$  be  $G$  with  $e$  deleted. We also let  $G/e$  be  $G$  with  $e$  contracted to a point and any resulting multiple edges *not* identified. So  $|E(G \setminus e)| = |E(G/e)| = |E(G)| - 1$ . We will also use this notation for directed graphs.

**THEOREM 2.1 (Deletion–Contraction Rule).** *For any  $e \in E(G)$*

$$\chi_G(n) = \chi_{G \setminus e}(n) - \chi_{G/e}(n).$$

From this result it is easy to prove inductively that the coefficients of  $\chi_G(n)$  alternate in sign with  $a_d = +1$ . Using Theorem 2.1 again, we see that if  $e$  is not a loop then

$$|a_1(G)| = |a_1(G \setminus e)| + |a_1(G/e)|.$$

We now show bijectively that  $|\mathcal{A}(G, v_0)|$  satisfies the same recursion.

LEMMA 2.2. *Consider any vertex  $v_0$ , and any edge  $e = uv_0$ ,  $u \neq v_0$ , with the corresponding arc  $a = \overrightarrow{uw_0}$ . The map*

$$D \rightarrow \begin{cases} D \setminus a \in \mathcal{A}(G \setminus e, v_0) & \text{if } D \setminus a \in \mathcal{A}(G \setminus e, v_0) \\ D/a \in \mathcal{A}(G/e, v_0) & \text{if } D \setminus a \notin \mathcal{A}(G \setminus e, v_0), \end{cases}$$

is a bijection between  $\mathcal{A}(G, v_0)$  and  $\mathcal{A}(G \setminus e, v_0) \uplus \mathcal{A}(G/e, v_0)$ , where the vertex of  $G/e$  formed by contracting  $e$  is labeled  $v_0$ .

*Proof.* We first prove that this map is well-defined by showing that in both cases we actually obtain an acyclic orientation with unique sink at  $v_0$ . This is clear in the first case by definition. In the second, where  $D \setminus a \notin \mathcal{A}(G \setminus e, v_0)$ , it must be true that  $D \setminus a$  has sinks both at  $u$  and at  $v_0$  (since deleting a directed edge of  $D$  will not introduce a cycle, nor will it cause us to lose the sink at  $v_0$ ). So the orientation  $D/a$  will be in  $\mathcal{A}(G/e, v_0)$ : since  $u$  and  $v_0$  were the only sinks in  $D \setminus a$  the contraction must have a unique sink at  $v_0$ , and no new cycles will be formed. Hence this map is well-defined.

To see that this is actually a bijection, we need only exhibit the inverse. This is obtained by simply orienting all edges of  $G$  as in  $D \setminus a$  or  $D/a$  as appropriate, and then adding in  $a$ . It should be clear that this map is also well-defined. ■

For the boundary conditions, we will need the following well-known result.

LEMMA 2.3. *If  $G$  is connected, then any  $D \in \mathcal{A}(G)$  has at least one sink. So if  $G$  is arbitrary then for any  $D \in \mathcal{A}(G)$ , the number of sinks is greater than or equal to the number of components of  $G$ .*

We can now complete the first proof of the Greene–Zaslavsky theorem by inducting on the number of edges of  $G$ . We have already verified the recurrence relation, so we need only worry about the boundary conditions. If  $G$  has more than one component, then  $n^2$  must divide  $\chi(G)$  yielding  $|a_1| = 0$ . From Lemma 2.3 we see that  $|\mathcal{A}(G, v_0)| = 0$  as well, so we need only consider connected graphs. Also, if  $G$  has loops then clearly  $\chi_n(G) = 0 = |\mathcal{A}(G, v_0)|$ . So it suffices to do the base case where  $G = K_1$ , the graph with a single vertex and no edges, giving  $a_1 = 1 = |\mathcal{A}(G, v_0)|$ . Thus the boundary conditions match and we are done. ■

## 3. CHROMATIC SYMMETRIC FUNCTIONS

Using his symmetric function generalization,  $X_G$ , of the chromatic polynomial, Stanley [13] proved a result related to, but not quite implying, the one of Greene and Zaslavsky. (See Theorem 3.7 at the end of this section.) In [7] we introduced an analogue of  $X_G$  using noncommutative variables. This allows us to use deletion–contraction techniques on symmetric functions to prove a generalization of Greene–Zaslavsky at this level.

We begin with some background on symmetric functions in noncommuting variables. Much of this follows from the work of Doubilet [4] (although he does not explicitly mention such functions in his paper). These noncommutative symmetric functions will be indexed by *set* partitions (as opposed to integer partitions in the commutative case). Note that they differ from the noncommutative symmetric functions considered by Gelfand *et al.* [8] and Fomin and Greene [5].

We will write  $\pi = B_1/B_2/\cdots/B_k$  to denote a partition of  $[d] := \{1, 2, \dots, d\}$ , i.e.,  $\biguplus_{i=1}^k B_i = [d]$ . The  $B_i$  are called *blocks*. The set of all partitions of  $[d]$  form a lattice  $\Pi_d$  under the partial order of refinement. We will let  $\wedge$  denote the meet operation (greatest lower bound) in  $\Pi_d$ .

Now let  $\mathbf{x} = \{x_1, x_2, x_3, \dots\}$  be a set of *noncommuting* variables. We define the *noncommutative monomial symmetric function*,  $m_\pi$ , by

$$m_\pi = m_\pi(\mathbf{x}) = \sum_{i_1, i_2, \dots, i_d} x_{i_1} x_{i_2} \cdots x_{i_d}, \quad (3)$$

where the sum is over all sequences  $i_1, i_2, \dots, i_d$  of positive integers  $\mathbb{P}$  such that  $i_j = i_k$  if and only if  $j$  and  $k$  are in the same block of  $\pi$ . For example,

$$m_{124/3} = x_1 x_1 x_2 x_1 + x_2 x_2 x_1 x_2 + x_1 x_1 x_3 x_1 + x_3 x_3 x_1 x_3 + \cdots$$

is the monomial symmetric function in noncommuting variables corresponding to the partition  $\pi = 124/3$ . The  $m_\pi$  are clearly linearly independent over  $\mathbb{C}$  and we call the span of  $\{m_\pi : \pi \in \Pi_d, d \geq 0\}$  the *algebra of noncommutative symmetric functions*.

The other basis we will be interested in is given by the *noncommutative elementary symmetric functions*

$$e_\pi = e_\pi(\mathbf{x}) = \sum_{\sigma: \sigma \wedge \pi = \hat{0}} m_\sigma = \sum_{i_1, i_2, \dots, i_d} x_{i_1} x_{i_2} \cdots x_{i_d}, \quad (4)$$

where the second sum is over all sequences  $i_1, i_2, \dots, i_d$  of  $\mathbb{P}$  such that  $i_j \neq i_k$  if  $j$  and  $k$  are both in the same block of  $\pi$ . As an example

$$\begin{aligned} e_{124/3} &= x_1 x_2 x_1 x_3 + x_1 x_2 x_2 x_3 + x_1 x_2 x_3 x_3 + x_1 x_2 x_4 x_3 + \cdots \\ &= m_{13/2/4} + m_{1/23/4} + m_{1/2/34} + m_{1/2/3/4}. \end{aligned}$$

We now introduce a noncommutative version,  $Y_G$ , of Stanley's chromatic symmetric function,  $X_G$ . The latter is obtained from the former merely by letting the variables commute.

**DEFINITION 3.** For any multigraph  $G$  with vertices labeled  $v_1, v_2, \dots, v_d$  in a fixed order, define

$$Y_G = Y_G(\mathbf{x}) = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)},$$

where the sum is over all proper colorings  $\kappa: V \rightarrow \mathbb{P}$  of  $G$ .

As an example, if we let  $P_3$  be the path with edge set  $E = \{v_1 v_2, v_2 v_3\}$  then

$$\begin{aligned} Y_{P_3} &= x_1 x_2 x_1 + x_2 x_1 x_2 + x_1 x_3 x_1 + \cdots \\ &\quad + x_1 x_2 x_3 + x_1 x_3 x_2 + \cdots + x_3 x_2 x_1 + \cdots \\ &= m_{13/2} + m_{1/2/3}. \end{aligned}$$

Note that if we let  $1^n$  denote the substitution  $x_1 = x_2 = \cdots = x_n = 1$  and  $x_i = 0$  for  $i > n$  then

$$X_G(1^n) = Y_G(1^n) = \chi_G(n)$$

since the only terms surviving in the sum are those using the first  $n$  colors.

We will need two properties of  $Y_G$ ; proofs can be found in [7]. For the first one, consider  $\delta \in S_d$ , the symmetric group on  $[d]$ . We let  $\delta$  act on the vertices of  $G$  by  $\delta(v_i) = v_{\delta(i)}$ . This induces an action on graphs, denoted  $\delta(G) = H$ , where  $H$  is just a relabeling of  $G$ . We also have an action on noncommutative symmetric functions given by linearly extending

$$\delta \circ (x_{i_1} x_{i_2} \cdots x_{i_k}) \stackrel{\text{def}}{=} x_{i_{\delta^{-1}(1)}} x_{i_{\delta^{-1}(2)}} \cdots x_{i_{\delta^{-1}(k)}}.$$

These two actions are compatible.

**PROPOSITION 3.2 (Relabeling Proposition).** *For any finite multigraph  $G$ , we have*

$$\delta \circ Y_G = Y_{\delta(G)},$$

where the vertex order  $v_1, v_2, \dots, v_d$  is used in both  $Y_G$  and  $Y_{\delta(G)}$ .

In order to allow us to state the Deletion–Contraction Rule for  $Y_G$ , we make the following definition.

DEFINITION 3.3. Define an operation induction,  $\uparrow$ , on monomials in noncommuting variables by

$$(x_{i_1} x_{i_2} \cdots x_{i_{d-2}} x_{i_{d-1}}) \uparrow = x_{i_1} x_{i_2} \cdots x_{i_{d-2}} x_{i_{d-1}}^2$$

and extend linearly.

From Eq. (3) it is easy to see that if  $\pi \in \Pi_{d-1}$ , then  $m_\pi \uparrow = m_{\pi+(d)}$  where  $\pi+(d) \in \Pi_d$  is the partition obtained from  $\pi$  by inserting  $d$  into the block with  $d-1$ .

PROPOSITION 3.4 (Deletion–Contraction Rule). *If  $e = v_{d-1}v_d$  is in  $E(G)$  then*

$$Y_G = Y_{G \setminus e} - Y_{G/e} \uparrow,$$

where the contraction of  $e = v_{d-1}v_d$  is labeled  $v_{d-1}$ .

To illustrate, consider  $P_3$  again. So  $e = v_2v_3$  and

$$Y_{P_3} = Y_{P_2 \uplus \{v_3\}} - Y_{P_2} \uparrow.$$

We then compute

$$Y_{P_2 \uplus \{v_3\}} = m_{1/2/3} + m_{1/23} + m_{13/2}$$

$$Y_{P_2} = m_{1/2},$$

$$Y_{P_2} \uparrow = m_{1/2} \uparrow = m_{1/23}.$$

So

$$Y_{P_3} = m_{1/2/3} + m_{1/23} + m_{13/2} - m_{1/23}$$

$$= m_{1/2/3} + m_{13/2},$$

THEOREM 3.5. *Let  $Y_G = \sum_{\pi \in \Pi_d} c_\pi e_\pi$ . Then for any fixed vertex,  $v_0$ .*

$$|\mathcal{A}(G, v_0)| = (d-1)! c_{[d]}.$$

*Proof.* We induct on the number of non-loops in  $E$ . If all the edges of  $G$  are loops, then

$$Y_G = \begin{cases} e_{1/2/\dots/d} & \text{if } G \text{ has no edges} \\ 0 & \text{if } G \text{ has loops.} \end{cases}$$

So

$$c_{[d]} = \begin{cases} 1 & \text{if } G = K_1 \\ 0 & \text{if } d > 1 \text{ or } G \text{ has loops} \end{cases} = |\mathcal{A}(G, v_0)|.$$

Now suppose that  $G$  has non-loops. Then by the Relabeling Proposition, we may choose  $e = v_{d-1}v_d$  and  $Y_G = Y_{G \setminus e} - Y_{G/e} \uparrow$ . We are only interested in the leading coefficient, so let

$$Y_G = ae_{[d]} + \sum_{\sigma < [d]} a_\sigma e_\sigma,$$

$$Y_{G \setminus e} = be_{[d]} + \sum_{\sigma < [d]} b_\sigma e_\sigma,$$

and

$$Y_{G/e} = ce_{[d-1]} + \sum_{\sigma < [d-1]} c_\sigma e_\sigma,$$

where  $\leq$  is the partial order on set partitions. Using induction and Lemma 2.2, it suffices to prove that  $(d-1)! a = (d-1)! b + (d-2)! c$ .

From the change of basis formulae found in [4] one gets

$$e_\pi \uparrow = \sum_{\sigma \leq \pi} \frac{\mu(\hat{0}, \sigma)}{\mu(\hat{0}, \sigma + (d))} \sum_{\tau \leq \sigma + (d)} \mu(\tau, \sigma + (d)) e_\tau. \quad (5)$$

This permits us to compute the coefficient of  $e_{[d]}$  in  $Y_{G/e} \uparrow$ . The only term which contributes comes from  $ce_{[d-1]} \uparrow$ , and

$$\begin{aligned} ce_{[d-1]} \uparrow &= c \sum_{\sigma \in \Pi_{d-1}} \frac{\mu(\hat{0}, \sigma)}{\mu(\hat{0}, \sigma + (d))} \sum_{\tau \leq \sigma + (d)} \mu(\tau, \sigma + (d)) e_\tau \\ &= c \frac{\mu(\hat{0}, [d-1])}{\mu(\hat{0}, [d])} e_{[d]} + \sum_{\tau < [d]} d_\tau e_\tau \\ &= \frac{-c}{d-1} e_{[d]} + \sum_{\tau < [d]} d_\tau e_\tau. \end{aligned}$$

Now  $Y_G = Y_{G \setminus e} - Y_{G/e} \uparrow$  yields

$$\begin{aligned} (d-1)! a &= (d-1)! b + (d-1)! \frac{c}{d-1} \\ &= (d-1)! b + (d-2)! c, \end{aligned}$$

completing the proof.  $\blacksquare$



To see why this implies Greene–Zaslavsky, recall that  $Y_G(1^n) = X_G(1^n) = \chi_G(n)$ . Now, if  $\pi = B_1/B_2/\cdots/B_k$  then under this substitution

$$e_\pi(1^n) = \prod_{i=1}^k n(n-1)(n-2)\cdots(n-|B_i|+1).$$

For  $k \geq 2$ , this polynomial is divisible by  $n^2$ . So the only contribution to the linear term of  $\chi_G(n)$  occurs when  $\pi = [d]$ , which yields a coefficient with absolute value  $(d-1)! c_{[d]}$ .

Immediately from the previous theorem we get

**COROLLARY 3.6.** *If  $Y_G = \sum_{\pi \in \Pi_d} c_\pi e_\pi$ , then the number of acyclic orientations of  $G$  with one sink is  $d! c_{[d]}$ .*

Before ending this section, we should state Stanley's theorem [13] relating  $X_G$  and sinks. In it,  $e_\lambda$  is the commutative elementary symmetric function corresponding to the integer partition  $\lambda$ , and  $l(\lambda)$  is the number of parts of  $\lambda$ .

**THEOREM 3.7 (Stanley).** *If  $X_G = \sum_\lambda c_\lambda e_\lambda$ , then the number of acyclic orientations of  $G$  with  $j$  sinks is  $\sum_{l(\lambda)=j} c_\lambda$ .*

We can prove an analogue of this theorem in the noncommutative setting by using his technique involving  $P$ -partitions. However, this only implies Corollary 3.6 (and Theorem 1.1) but not Theorem 3.5.

#### 4. THE MODIFIED BLASS–SAGAN ALGORITHM

We will now prove Theorem 1.2 a third time, using (2) to interpret the linear coefficient of  $\chi_G(n)$ . This demonstration will use a variant of an algorithmic bijection of Blass and Sagan to show that  $\mathcal{A}(G, v_0)$  and  $\{S \in \mathcal{B}_G : |S| = d-1\}$  have the same cardinality.

We first need some notation and definitions. For any arc  $a = \overrightarrow{wu}$ , the oppositely oriented arc is denoted  $a' = \overleftarrow{uw}$ . We also say that to *unorient* an arc,  $a$ , in a digraph we will just add the oppositely oriented arc  $a'$ . By the same token, an edge will also be considered as a pair of oppositely oriented arcs so that any graph is also a digraph. Since we are interested in acyclic digraphs, it is necessary to adopt the convention that a digraph is acyclic if it has no cycles of length  $\geq 3$ . With this convention, unorienting an arc will not necessarily produce a cyclic. Also for any acyclic digraph  $D$ , we will let  $c(D)$  be the *contraction* of  $D$ , which is the graph where all *unoriented* arcs of  $D$  have been contracted. We note that  $c(D)$  is still acyclic and has no unoriented arcs.

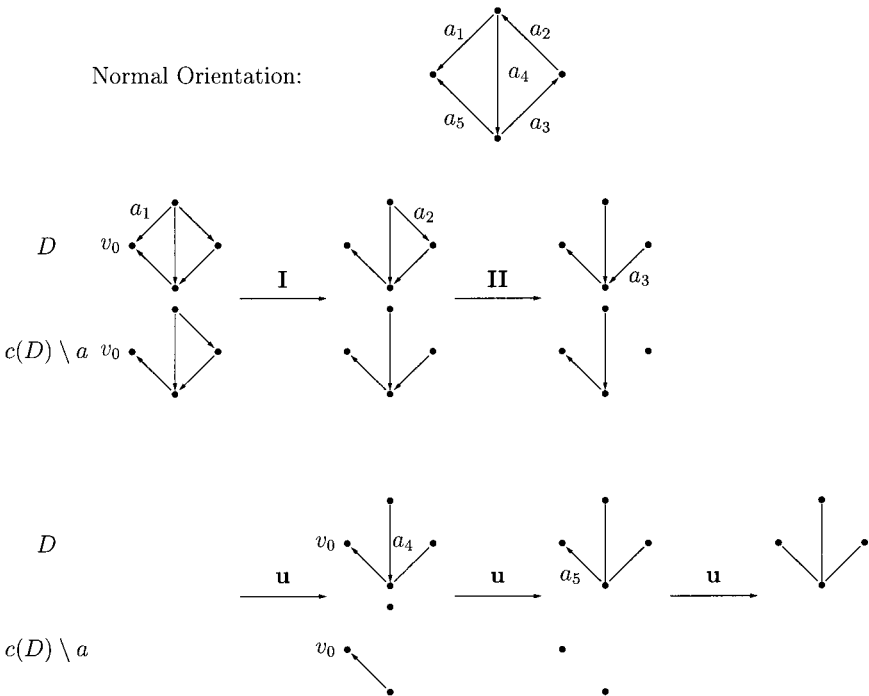
**THEOREM 4.1.** *For any fixed vertex  $v_0 \in V(G)$ , the number of acyclic orientations of  $G$  with a unique sink at  $v_0$  is the same as the number of sets,  $S \in B_G$  with  $|S| = d - 1$ .*

*Proof.* We will construct a bijection using an algorithm that sequentially examines each arc of an element of  $\mathcal{A}(G, v_0)$  and either deletes the arc or unorients it.

Fix an orientation of  $G$  (not necessarily acyclic), which we will refer to as the *normal orientation*, and also choose a fixed vertex  $v_0$  of  $G$ . The algorithm will accept any acyclic orientation  $D$  of  $G$  which has a unique sink at  $v_0$ , and consider each arc in turn, using the total order on the edges which defines the broken circuits. At the stage when an arc  $a = \overrightarrow{wu}$  is being considered, the algorithm will delete  $a$  if either

- (I)  $D \cup a'$  has a cycle, or
- (II)  $c(D) \setminus a$  has only one sink, and  $a$  is not normally oriented.

Otherwise, the algorithm will unorient  $a$ . For an example of how this algorithm works, see Fig. 1. The steps of the algorithm are labeled either I, II,



**FIG. 1.** An example of the algorithm.

**II**, or **u**, indicating if the algorithm deleted the arc for reason I or II, or unoriented it.

To show that this algorithm actually does produce a bijection, we shall first introduce a sequence of sets,  $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_q$ , such that  $\mathcal{D}_0$  is the set of all acyclic orientations of  $G$  with a unique sink at  $v_0$ , and  $\mathcal{D}_q$  (where  $q = |E(G)|$ ) is the set of all  $S \in B_G$  with  $|S| = d - 1$ . Equivalently,  $\mathcal{D}_q$  is the set of all spanning trees,  $T$ , of  $G$  such that  $E(T)$  contains no broken circuits.

We will show that the  $k$ th step of the algorithm gives a bijection,  $f_k: \mathcal{D}_{k-1} \rightarrow \mathcal{D}_k$ , where  $\mathcal{D}_k$  is defined as the set of all spanning subdigraphs  $D$  of  $G$  satisfying the following conditions:

- (a) Each of the first  $k$  edges of  $G$  is either present in  $D$  (as an unoriented edge) or absent from  $D$ , but each of the remaining  $q - k$  edges is present in  $D$  in exactly one orientation.
- (b)  $D$  is acyclic.
- (c)  $D$  has an  $x \rightarrow v_0$  path for every  $x \in V(D)$ .
- (d) The unoriented part of  $D$  contains no broken circuit.

From these conditions, it should be clear that  $\mathcal{D}_0$  is indeed the set of acyclic orientations of  $G$  with a unique sink at  $v_0$  by Lemma 2.3. It is also clear that any element of  $\mathcal{D}_q$  will be an acyclic, connected graph, which implies that the elements of  $\mathcal{D}_q$  must be trees with exactly  $d - 1$  edges. So provided the algorithm gives a bijection at each step, we will have the desired bijection between acyclic orientations of  $G$  with a unique sink at  $v_0$ , and edge sets of size  $d - 1$  which contain no broken circuits.

We should also note here that conditions (b) and (c) together imply that  $c(D)$  must have a unique sink which occurs at the vertex identified with  $v_0$ . That this is the only possible sink of  $c(D)$  is clear from condition (c). We also know that  $v_0$  must be a sink of  $c(D)$ , since if it is not, then there is a vertex  $u$  and arc  $a = \overrightarrow{v_0 u}$  in  $c(D)$ . But from condition (c) there would have to be a  $u \rightarrow v_0$  path in  $D$ . This contradicts the acyclicity of  $D$ .

To show that the algorithm does indeed produce a bijection at each step, we use the following three lemmas. We also use the notational convention that a digraph in  $\mathcal{D}_k$  will be denoted by  $D_k$ .

**LEMMA 4.2.**  $f_k$  maps  $\mathcal{D}_{k-1}$  into  $\mathcal{D}_k$ .

*Proof.* We need only prove that properties (a)–(d) listed previously are still satisfied after the algorithm is applied at the  $k$ th stage. We proceed to verify each one in turn.

(a) Since at the  $k$ th step the algorithm will either delete or unorient the  $k$ th arc, this is clear.

(b) Since any arc which would form a cycle if unoriented will be deleted by the algorithm, this also is clear.

(c) Since unorienting an arc can never destroy an  $x \rightarrow v_0$  path, we need only consider the case where the algorithm deletes an arc. In fact, if the arc  $a = \overrightarrow{wu}$  in  $D_{k-1}$  was deleted, we need only show that there is still a  $w \rightarrow v_0$  path.

Now, if the arc  $a = \overrightarrow{wu}$  in  $D_{k-1}$  was deleted for the first reason, then we must have had another (different)  $w \rightarrow u$  path in  $D_{k-1}$ . Since there was a  $u \rightarrow v_0$  path in  $D_{k-1}$ , (in fact, one which did not use the arc  $a$ ) we can then extend our other  $w \rightarrow u$  path into a walk containing a  $w \rightarrow v_0$  path in  $D_k$ .

If the arc  $a = \overrightarrow{wu}$  in  $D_{k-1}$  was deleted for the second reason, again we need only consider the possibility that for the vertex  $w$ , there is no  $w \rightarrow v_0$  path in  $D_k$ . But then there is no *oriented* arc  $\overrightarrow{wv}$  with  $u \neq v$ , since otherwise all  $v \rightarrow v_0$  paths must also use the arc  $a$ , as there are no  $w \rightarrow v_0$  paths in  $D_k$ . Thus  $D_{k-1}$  would have a cycle containing  $w$ . Contracting all unoriented arcs from  $w$  and repeating this argument as necessary, we see that  $w$  would then be a sink of  $c(D_{k-1}) \setminus a$ , which contradicts our reason for deleting  $a$ .

(d) Suppose for the sake of contradiction that the unoriented part of  $D_k$  contains a broken circuit,  $C \setminus x$ , where  $x$  is the greatest element of the cycle  $C$ . Since the unoriented part of  $D_{k-1}$  did not contain any broken circuits, and since the only difference between  $D_{k-1}$  and  $D_k$  is at the  $k$ th arc  $a$ , we see that  $a$  must be unoriented in  $D_k$  and that  $a \in C \setminus x$ . But then  $x$  is greater than  $a$ , and so  $x$  is present in  $D_k$  in one of its orientations. But all the other edges in  $C$  are also present and unoriented. Hence,  $C$  forms a cycle in  $D_k$ , contradicting the previously verified fact that  $D_k$  is acyclic. ■

LEMMA 4.3.  $f_k$  is one-to-one.

*Proof.* Suppose  $D_{k-1}$  and  $D'_{k-1}$  are two distinct elements of  $\mathcal{D}_{k-1}$  which are both mapped to  $D_k$  by the algorithm. Since the algorithm only affects the  $k$ th arc, we note that  $D_{k-1}$  and  $D'_{k-1}$  (and consequently  $c(D_{k-1})$  and  $c(D'_{k-1})$ ) must only differ in that arc. Without loss of generality, we may assume that this arc is  $a$  with normal orientation in  $D_{k-1}$  and  $a'$  with abnormal orientation in  $D'_{k-1}$ .

We note that  $D_k$  was not obtained from  $D_{k-1}$  and  $D'_{k-1}$  by deletion. For if  $a$  was deleted from  $D_{k-1}$  for the first reason then  $D'_{k-1}$  has a cycle and vice versa. And the second reason does not apply to  $a$ , which has normal orientation.

If the  $k$ th arc was unoriented then, by reason II,  $c(D'_{k-1}) \setminus a'$  must have an additional sink. So if  $a' = \overrightarrow{uw}$  then  $u$  must be the extra sink. But this means that  $u$  is also an additional sink in  $c(D_{k-1})$ , contradicting  $D_{k-1} \in \mathcal{D}_{k-1}$ . ■

LEMMA 4.4.  $f_k$  maps  $\mathcal{D}_{k-1}$  onto  $\mathcal{D}_k$ .

*Proof.* Given  $D_k \in \mathcal{D}_k$  we must construct  $D_{k-1} \in \mathcal{D}_{k-1}$  which maps onto it. Hence for any digraph,  $D_k \in \mathcal{D}_k$ , we must construct a digraph  $D_{k-1}$  and verify that the algorithm does indeed map  $D_{k-1}$  onto  $D_k$ , and that  $D_{k-1}$  satisfies properties (a)–(d). For all of the following cases, it will be immediate that the  $D_{k-1}$  we construct will satisfy properties (a), (b), and (d), so we will only do the verification of property (c). Let  $e$  be the  $k$ th edge of  $G$ . There are two cases.

In the first case  $e$  is not an edge of  $D_k$ . If there exists a unique orientation  $a$  of  $e$  in which  $D_k$  would remain acyclic, we give  $e$  that orientation in  $D_{k-1}$ . If both orientations of  $e$  would preserve the acyclicity of  $D_k$ , then we choose  $a$  to be the abnormal orientation for  $e$  in  $D_{k-1}$ . We note that at least one of the orientations of  $e$  must preserve acyclicity, since otherwise  $e$  completes two different cycles in  $D_{k-1}$ . These two cycles together would contain a cycle in  $D_k$ , which is a contradiction.

That the algorithm maps the digraph  $D_{k-1}$  obtained in the previous paragraph to  $D_k$  is obvious when only one orientation of  $a$  produces an acyclic orientation of  $D_{k-1}$ . However, if both produce acyclic orientations, we need to check that  $c(D_{k-1}) \setminus a$  has a unique sink at  $v_0$ . This is true, since it is easy to see that  $c(D_{k-1}) \setminus a = c(D_{k-1} \setminus a) = c(D_k)$ . To verify that  $c(D_{k-1})$  constructed above still satisfies property (c), we note that adding an arc cannot destroy any existing paths. So the first case is done.

In the second case we have  $e$  present in  $D_k$  and so neither orientation can produce a cycle in  $D_{k-1}$ . We note that there must be at least one orientation of  $e = wu$  such that there remains an  $x \rightarrow v_0$  path for every  $x \in D_{k-1}$ . If all  $x \rightarrow v_0$  paths  $P$  use the arc  $a = \overrightarrow{wu}$  for some  $x$ , and if all  $y \rightarrow v_0$  paths  $Q$  use  $a' = \overrightarrow{uw}$  for some  $y$ , then the  $x \rightarrow w$  portion of  $P$  together with the  $w \rightarrow v_0$  portion of  $Q$  contains an  $x \rightarrow v_0$  path avoiding  $a$ , which contradicts our assumption about  $x$ .

If there is a unique orientation of  $e = wu$  so that there remains an  $x \rightarrow v_0$  path for every  $x \in D_{k-1}$  we choose that one to maintain property (c) for  $D_{k-1}$ , say  $a = \overrightarrow{wu}$ . Using the same argument we used to prove the second case of (c) in Lemma 4.2, it is easy to verify that the algorithm will take the  $D_{k-1}$  so constructed and map it to  $D_k$  by unorienting  $a$  since  $c(D_{k-1}) \setminus a$  has an additional sink at  $w$ .

In the subcase, where  $e$  is present in  $D_k$  as an unoriented edge and we would still retain property (c) with either orientation of  $e$ , we will consider

the digraph  $D_{k-1}$  obtained from  $D$  by giving  $e$  the normal orientation, say  $a = \overrightarrow{wu}$ . It is clear that the algorithm maps  $D_{k-1}$  to  $D_k$ , since  $D_{k-1} \cup a' = D_k$  is acyclic and  $a$  has the normal orientation. ■

## 5. SOURCES AND SINKS

A *source* in a digraph is a vertex  $u_0$  with all incident arcs directed away from it. For this section only, we adopt the convention that an isolated vertex in a graph is neither a source nor a sink. Given an edge  $u_0v_0 \in E(G)$  we let  $\mathcal{A}(G, u_0v_0)$  be the set of all acyclic orientations of  $G$  having a unique source at  $u_0$  and a unique sink at  $v_0$ . Greene and Zaslavsky [9] also proved a theorem counting such orientations. If  $G$  has at least one edge, then Crapo's *beta invariant* [3] is

$$\beta(G) = (-1)^{d-k+1} \chi'_G(1),$$

where  $d = |V|$ ,  $k$  is the number of components, and the derivative is with respect to  $n$ . We should note that this is not the standard way to define  $\beta(G)$ , but agrees with that definition when  $|E| \geq 1$ . Also, the power of minus one (which is one more than the rank of the cycle matroid for  $G$ ) is present to make  $\beta(G)$  nonnegative, as is implied by the following theorem.

**THEOREM 5.1** (Greene and Zaslavsky). *Let  $u_0v_0$  be an edge of  $G$ . Then*

$$|\mathcal{A}(G, u_0v_0)| = \beta(G). \quad (6)$$

Again, the Greene–Zaslavsky proof used hyperplane arrangements and our objective is to give a simple bijective proof. First, we need to have a deletion–contraction recursion for  $\beta(G)$ . To describe it, we need the notion of a *bridge* which is an edge  $e \in E$  such that  $G \setminus e$  has more components than  $G$ .

**THEOREM 5.2** (Deletion–Contraction for  $\beta(G)$ ). *Suppose that  $|E| \geq 2$  and  $e$  is not a bridge. Then*

$$\beta(G) = \beta(G \setminus e) + \beta(G/e).$$

*Proof.* By the deletion–contraction rule for the characteristic polynomial,

$$\chi'_G(1) = \chi'_{G \setminus e}(1) - \chi'_{G/e}(1).$$

Since  $e$  is not a bridge,  $G$ ,  $G \setminus e$ , and  $G/e$  all have the same number of components. Since  $G/e$  has one fewer vertices than either  $G$  or  $G \setminus e$ , multiplying by the appropriate power of minus one finishes the proof. ■

The corresponding recursion for  $\mathcal{A}(G, u_0v_0)$  is as follows.

LEMMA 5.3. *Fix an edge  $u_0v_0 \in E$  and suppose  $uv_0 \in E$ ,  $u \neq u_0$ ,  $v_0$ , with corresponding arc  $a = \overrightarrow{uv_0}$ .*

$$D \rightarrow \begin{cases} D \setminus a \in \mathcal{A}(G \setminus e, u_0v_0) & \text{if } D \setminus a \in \mathcal{A}(G \setminus e, u_0v_0) \\ D/a \in \mathcal{A}(G/e, u_0v_0) & \text{if } D \setminus a \notin \mathcal{A}(G \setminus e, u_0v_0) \end{cases}$$

is a bijection between  $\mathcal{A}(G, u_0v_0)$  and  $\mathcal{A}(G \setminus e, u_0v_0) \uplus \mathcal{A}(G/e, u_0v_0)$ , where the vertex of  $G/e$  formed by contracting  $e$  is labeled  $v_0$ .

*Proof.* We first worry about well-definedness, which is clear in the first case. As in Lemma 2.2, if  $D \setminus a \notin \mathcal{A}(G \setminus e, u_0v_0)$  then  $u$  becomes a source or a sink in  $D \setminus a$ . But if it becomes a source, then it must have also been a source in  $D$ , since the only extra arc is  $\overrightarrow{uv_0}$ , and this contradicts the fact that  $u_0$  was the unique source. So  $u$  must be a sink in  $D \setminus a$  which forces  $D/a \in \mathcal{A}(G/e, u_0v_0)$  as desired. Now the inverse is obtained as in Lemma 2.2. ■

To complete our induction, we will need a couple of lemmas about how  $\beta(G)$  and  $\mathcal{A}(G, u_0v_0)$  behave in graphs with more than one block. A *cutvertex* of  $G$  is  $v \in V$  such that  $G - v$  has more components than  $G$ . A *block* of  $G$  is a maximal connected subgraph having no cutvertex. A block or component is *trivial* if it is a single (isolated) vertex. It is well known that if  $G$  is connected and has at least two blocks, then  $G$  has at least two *endblocks* which are blocks containing only one cutvertex of  $G$ .

LEMMA 5.4. *If  $G$  has at least two nontrivial blocks then  $\beta(G) = 0$ .*

*Proof.* By definition of  $\beta(G)$ , it suffices to show that  $\chi_G(n)$  has  $n = 1$  as a double root. If  $G$  is any graph with at least one edge then  $\chi_G(1) = 0$ . So if  $G$  has two nontrivial components  $G_1, G_2$  then  $\chi_{G_1}(n)\chi_{G_2}(n)$  divides  $\chi_G(n)$  and we have the desired double root. Otherwise,  $G$  must be connected with at least two blocks. Let  $B$  be an endblock and let  $H$  be the union of all the other blocks of  $G$  so that  $V(B) \cap V(H) = \{v\}$  for some vertex  $v$ . Coloring first  $B$  and then  $H$  we see, directly from the definition of  $\chi$ , that  $\chi_G(n) = \chi_B(n)\chi_H(n)/n$ . So again we have a double root. ■

LEMMA 5.5. *If  $G$  has at least two nontrivial blocks then for any  $u_0v_0 \in E$  we have  $|\mathcal{A}(G, u_0v_0)| = 0$ .*

*Proof.* It suffices to show that  $G$  must have two vertices not connected by an edge which are sources or sinks. If  $G$  has at least two nontrivial components, then this is taken care of by Lemma 2.3. So we can assume  $G$  is

connected and let  $B, B'$  be two endblocks. By Lemma 2.3 again, the orientation of  $G$  restricted to  $B$  has at least one source and one sink in  $B$ . Since  $B$  is only connected to  $G$  by a single vertex, at least one of these survives as a source or a sink in  $G$ . Similar considerations apply to  $B'$ , so we have found the pair we seek. ■

We can now finish our inductive proof of Theorem 5.1. If  $G$  has a loop or two nontrivial blocks, then both sides of (6) are zero. So we need only consider the case when  $G$  is a nontrivial loopless block. If  $E(G) = \{u_0 v_0\}$  then  $\beta(G) = 1 = |\mathcal{A}(G, u_0 v_0)|$ . Otherwise  $u_0 v_0$  is not a bridge and we can apply induction using our two recursions. ■

## 6. OPEN PROBLEMS

In the hope that the reader will be inspired to work on them, we present some questions raised by this work.

(a) It would be interesting to find proofs of Theorem 5.1 using either chromatic symmetric functions or a direct bijection. Since  $\beta(G)$  is an alternating sum, the latter would probably involve the Involution Principle of Garsia and Milne [6].

(b) There are a number of open problems involving Stanley's symmetric function,  $X_G$ , and the noncommutative analog,  $Y_G$ . We will only mention one of them here. The reader can consult [7, 13, 14] for other questions. If  $T$  is a tree on  $d$  vertices, then it is easy to see that  $\chi_n(G) = n(n-1)^{d-1}$  so all trees with the same  $|V|$  have the same chromatic polynomial. However, it has been verified up to  $d=9$  that  $X_{T_1} \neq X_{T_2}$  if  $T_1, T_2$  are nonisomorphic trees [2], which leads to the following.

QUESTION 6.1 [13]. Does  $X_T$  distinguish among nonisomorphic trees?

It is easy to see that  $Y_G$  distinguishes between any two nonisomorphic graphs [7]. (In fact, one can reconstruct  $G$  from  $Y_G$ .) But it is still hoped that noncommutativity could be of some help in investigating this question.

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