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On the 1/3–2/3 Conjecture

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Abstract Let $(P, \leq)$ be a finite poset (partially ordered set), where P has cardinality n. Consider linear extensions of P as permutations $x_1 x_2 \cdots x_n$ in one-line notation. For distinct elements $x, y \in P$, we define $\mathbb{P}(x \prec y)$ to be the proportion of linear extensions of P in which x comes before y. For $0 \leq \alpha \leq \frac{1}{2}$, we say (x, y) is an α-balanced pair if $\alpha \leq \mathbb{P}(x \prec y) \leq 1 - \alpha$. The 1/3–2/3 Conjecture states that every finite partially ordered set which is not a chain has a 1/3-balanced pair. We make progress on this conjecture by showing that it holds for certain families of posets. These include lattices such as the Boolean, set partition, and subspace lattices; partial orders that arise from a Young diagram; and some partial orders of dimension 2. We also consider various posets which satisfy the stronger condition of having a 1/2-balanced pair. For example, this happens when the poset has an automorphism with a cycle of length 2. Various questions for future research are posed.

Keywords 1/3–2/3 Conjecture · α-balanced · Automorphism · Dimension · Lattice · Linear extension · Pattern avoidance · Poset · Width · Young diagram

1 Introduction

Let $(P, \leq)$ be a poset, and let n be the cardinality of P. A *linear extension* is a total order $x_1 \prec x_2 \prec \cdots \prec x_n$ on the elements of P such that $x_i \prec x_j$ if $x_i <_P x_j$; more compactly, we can view a linear extension as a permutation $x_1 x_2 \cdots x_n$ in one-line notation. For distinct

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elements $x, y \in P$, we define $\mathbb{P}(x \prec y)$ to be the proportion of linear extensions of P in which x comes before y. For $0 \leq \alpha \leq \frac{1}{2}$, we say (x, y) is an α-*balanced pair* if

$$\alpha \leq \mathbb{P}(x \prec y) \leq 1 - \alpha,$$

and that P is α-*balanced* if it has some α-balanced pair. Notice that if (x, y) is α-balanced, then (y, x) is α-balanced as well.

Conjecture 1.1 (The 1/3–2/3 Conjecture) Every finite partially ordered set that is not a chain has a 1/3-balanced pair.

We can see, for instance, that the conjecture holds for the poset P depicted in Fig. 1. This poset has 15 linear extensions, which are

$$123456, \ 123465, \ 123645, \ 124356, \ 124365,$$
$$124536, \ 142356, \ 142365, \ 142536, \ 213456,$$
$$213465, \ 213645, \ 214356, \ 214365, \ 214536.$$

The matrix on the right in the figure has as its (i, j) entry the number of linear extensions of P where i comes before j. The entries in bold give the pairs (i, j) whose number of linear extensions with $i \prec j$ is between $1/3(15) = 5$ and $2/3(15) = 10$, thus satisfying the conjecture.

Conjecture 1.1 was first proposed by Kislitsyn [8] in 1968, although a number of resources attribute it to Fredman [5] and it was also independently discovered by Linial [9]. There are many types of posets for which the conjecture has already been proven. This includes posets of up to 11 elements [10], posets with height 2 [12], semiorders [2], posets with each element incomparable to at most 6 others [11], N-free posets [13], and posets whose Hasse diagram is a tree [14]. If the conjecture is true, the bounds are best possible, as seen by the poset T in Fig. 2. While the proof of the 1/3 bound for a general poset remains elusive, in 1984 Kahn and Saks [7] proved that for any poset P, there is some pair $x, y \in P$ such that $\frac{3}{11} < \mathbb{P}(x \prec y) < \frac{8}{11}$. In 1995, Brightwell, Felsner, and Trotter [4] improved the bound to be $\frac{5 - \sqrt{5}}{10} \leq \mathbb{P}(x \prec y) \leq \frac{5 + \sqrt{5}}{10}$. In [4], Conjecture 1.1 is described as "one of the most intriguing problems in combinatorial theory". The interested reader can refer to Brightwell's 1999 survey [3] for more information.

An alternative way of talking about this conjecture is as follows. We define the *balance constant* of P to be

$$\delta(P) = \max_{x,y \in P} \min\{\mathbb{P}(x \prec y), \mathbb{P}(y \prec x)\}.$$

For any poset P not a chain, it must be that $0 < \delta(P) \leq 1/2$. In the example in Fig. 1, P has a balance constant of $\frac{7}{15} \approx 0.4667$. So P has a 1/3-balanced pair if and only if P has a balance constant $\delta(P) \geq 1/3$. We will use these two phrases interchangeably, as does the literature.

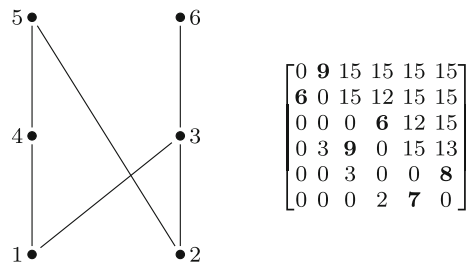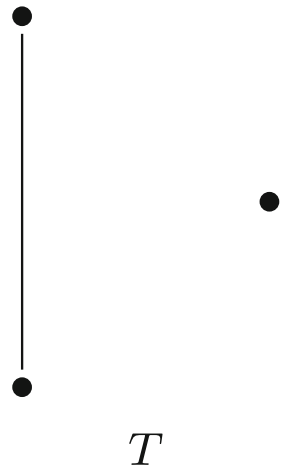**Fig. 1** A poset P with 6 elements and a matrix counting its linear extensions

$$\begin{bmatrix} 0 & \mathbf{9} & 15 & 15 & 15 & 15 \\ 6 & 0 & 15 & 12 & 15 & 15 \\ 0 & 0 & 0 & \mathbf{6} & 12 & 15 \\ 0 & 3 & \mathbf{9} & 0 & 15 & 13 \\ 0 & 0 & 3 & 0 & 0 & \mathbf{8} \\ 0 & 0 & 0 & 2 & \mathbf{7} & 0 \end{bmatrix}$$

Fig. 2 The poset T with three elements and one relation

$$T$$

The following notation will give us yet another way of discussing the conjecture. Let $E(P)$ be the set of linear extensions of P and $e(P)$ be the cardinality of $E(P)$. If $(P, \leq)$ is a poset and $x, y \in P$, let $P + xy$ denote the poset $(P, \leq')$, where $\leq'$ is the transitive closure of $\leq$ extended by the relation $x < y$. So $\mathbb{P}(x \prec y) = e(P + xy)/e(P)$. Note also that

$$e(P + xy) + e(P + yx) = e(P). \tag{1.1}$$

There are ideas of Zaguia which we will find useful in a number of our proofs. The following definitions were introduced in [14], although here we refer to them with different names which we find more descriptive. Given x in a poset P we let L_x and U_x denote the *strict lower* and *strict upper order ideals* generated by x, that is,

$$L_x = \{y \in P \mid y < x\}$$

and

$$U_x = \{y \in P \mid y > x\}.$$

Definition 1.2 Let P be a poset and x and y be distinct elements of P.

(a) We call the pair (x, y) *twin elements* if $L_x = L_y$ and $U_x = U_y$.
(b) We call the pair (x, y) *almost twin elements* if the following two conditions hold in P or in the dual of P:

    (i) $L_x = L_y$, and
    (ii) $U_x \setminus U_y$ and $U_y \setminus U_x$ are chains (possibly empty).

The result of Zaguia's which we will need is as follows.

Theorem 1.3 ([14]) *A finite poset that has an almost twin pair of elements is* $1/3$*-balanced.*

In fact, Zaguia proved a more general result by relaxing the definition of an almost twin pair. But we will not need that level of generality here. It is important to keep in mind that many of the known results for the $1/3$–$2/3$ Conjecture are existence proofs and do not compute $\mathbb{P}(x \prec y)$ exactly for any pair (x, y) in the given poset. This is particularly true in the case of almost twin pairs of elements. Further, an almost twin pair need not be $1/3$-balanced, even though its existence implies that a $1/3$-balanced pair exists.

The rest of this paper is structured as follows. In the next section, we will show that a poset with an automorphism containing a 2-cycle is 1/2-balanced. In particular, every poset with twin elements is 1/2-balanced. The automorphism result is applied in Section 3 to various types of lattices including the Boolean lattice, set partition lattice, subspace lattice, and certain distributive lattices. We also consider the lattice obtained by taking the product of two chains. This last example is just the poset of a rectangular Young diagram and in Section 4 we show that the poset of any Young diagram, including those which are skew or shifted, is 1/3-balanced. Section 5 is devoted to showing that certain posets of dimension 2 which satisfy a pattern avoidance condition have balance constant 1/2. We end with a section discussing posets which have balance constants near, but not equal to, 1/3. A number of questions concerning future research are scattered throughout.

2 Automorphisms of Posets

We first provide a proof of a simple observation about the linear extensions of a poset with an automorphism.

Proposition 2.1 *An automorphism ϕ of a poset P induces a bijection on $E(P)$. Further, $\mathbb{P}(x \prec y) = \mathbb{P}(\phi(x) \prec \phi(y))$ for all $x, y \in P$.*

Proof Let $\phi : P \to P$ be any automorphism. This means that for $x, y \in P$, $x \leq_P y$ if and only if $\phi(x) \leq_P \phi(y)$. Now, let $\pi = a_1 a_2 \cdots a_n$ be a linear extension of P, and by the definition of linear extension, we know that if $a_i \leq_P a_j$, then $i \leq j$. As ϕ is an automorphism, then we also have that if $\phi(a_i) \leq_P \phi(a_j)$, then $i \leq j$. This gives us, by definition, that $\phi(\pi) = \phi(a_1)\phi(a_2) \cdots \phi(a_n)$ is a linear extension of P. And the fact that ϕ is bijective implies that the induced map on $E(P)$ is as well.

We can also observe that the linear extensions with x before y map bijectively via ϕ to the linear extensions with $\phi(x)$ before $\phi(y)$. Hence, $\mathbb{P}(x \prec y) = \mathbb{P}(\phi(x) \prec \phi(y))$, as desired.   □

In [6], Ganter, Hafner, and Poguntke prove that posets with a nontrivial automorphism satisfy Conjecture 1.1.

Theorem 2.2 ([6]) *If a poset P has a non-trivial automorphism, then P is 1/3-balanced.*

We will give a more refined version of this result by giving a condition on the automorphism which will ensure a balance constant of 1/2.

Proposition 2.3 *If a poset P has an automorphism with a cycle of length 2, then P is 1/2-balanced. Further, if x and y are the elements in the cycle of length 2, then (x, y) is a 1/2-balanced pair.*

Proof Let $\phi : P \to P$ be an automorphism and $x, y \in P$ be distinct elements such that $\phi(x) = y$ and $\phi(y) = x$. Thus, using Proposition 2.1, we see that

$$e(P + xy) = e(P + \phi(x)\phi(y)) = e(P + yx).$$

Combining this with Eq. (1.1), we have

$$e(P) = e(P + xy) + e(P + yx) = 2e(P + xy),$$

and so $e(P + xy) = e(P)/2$. Hence, (x, y) is a 1/2-balanced pair, as desired. □

An example of a poset with an automorphism having a cycle of length 2 is given in Fig. 3. Poset P has a balance constant of 1/2. A counterexample to the converse of Proposition 2.3 is also provided in Fig. 3. Poset Q has a balance constant of 1/2, as it has 12 linear extensions and $e(P + 34) = 6$. However, we can see by inspection it has no nontrivial automorphisms.

The following is a corollary to Proposition 2.3.

Corollary 2.4 *A poset P with a twin pair of elements is 1/2-balanced.*

Proof Let P be a poset with x and y a twin pair of elements. We can see that P has a non-trivial automorphism which fixes all elements except for x and y and interchanges x and y. So, this poset has an automorphism with a cycle of length 2 and we are done by Proposition 2.3. □

While the above results depend on an automorphism of a poset, it is natural to ask if we can obtain results from other types of maps. Next, we consider anti-automorphisms σ. So if σ^2 is not the identity, then P has a non-trivial automorphism and we have the following immediate corollary to Theorem 2.2.

Corollary 2.5 *If σ is an anti-automorphism on P and σ^2 is non-trivial, then P is 1/3-balanced.*

We can also ask when an anti-automorphism guarantees a poset to be 1/2-balanced. One such case is as follows.

Proposition 2.6 *Let $\sigma : P \to P$ be an anti-automorphism. If σ has 2 fixed points, then P is 1/2-balanced.*

Proof Let P be a poset and $\sigma : P \to P$ be an anti-automorphism. Consider the bijection τ on linear extensions which takes a linear extension of P, applies σ to each of its elements, and then reads the resulting sequence backwards. Observe that for any $x, y \in P$, the linear extensions with x before y map via τ to the linear extensions with $\sigma(y)$ before $\sigma(x)$. Hence, $\mathbb{P}(x \prec y) = \mathbb{P}(\sigma(y) \prec \sigma(x))$. The proof is now completed in exactly the same way as the demonstration of Proposition 2.3. □

Fig. 3 The poset P has an automorphism with cycle length 2 and balance constant 1/2, while Q has no nontrivial automorphisms and balance constant 1/2
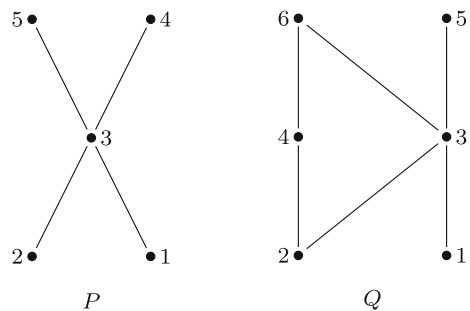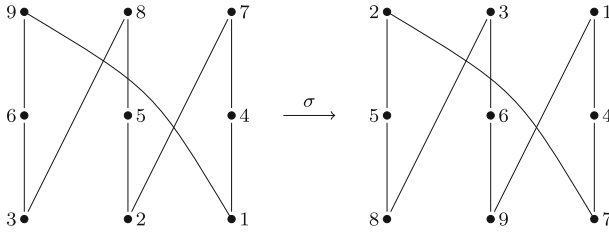
Fig. 4 P and anti-automorphism σ with 1 fixed point

We cannot weaken the assumption in Proposition 2.6, since a unique fixed point in an anti-automorphism is not enough to guarantee that the poset is $1/2$-balanced. For an example, consider the poset P and anti-automorphism σ in Fig. 4 where for $x \in P$ we place $\sigma(x)$ on the right in the same position as x on the left. Any anti-automorphism of P, including σ shown here, will have exactly 1 fixed point, and computer calculations give us that $\delta(P) = \frac{711}{1431} \neq \frac{1}{2}$.

We can also see that the converse of Proposition 2.6 is not true, as evidenced by the counterexample in Fig. 3. Any anti-automorphism of the poset P will have exactly 1 fixed point, and yet it is $1/2$-balanced.

3 Lattices

3.1 Boolean Lattices

The Boolean lattice, B_n, consists of all subsets of $[n] := \{1, 2, \ldots, n\}$ ordered by inclusion. The poset B_1 is a chain, and so we only need to consider $n \geq 2$. We present the following as a corollary to Proposition 2.3.

Corollary 3.1 *For all $n \geq 2$, the Boolean lattice B_n has an automorphism with a cycle of length 2. So the Boolean lattice is $1/2$-balanced.*

Proof We will first describe an automorphism of B_n using the symmetric difference operation Δ. For $S \subseteq [n]$, consider $\phi : B_n \to B_n$ defined by

$$\phi(S) = \begin{cases} S\Delta\{1, 2\}, & \text{if } S \cap \{1, 2\} = \{1\} \text{ or } S \cap \{1, 2\} = \{2\} \\ S, & \text{otherwise.} \end{cases}$$

One can easily check that ϕ is an automorphism. And if $A = \{1\}$ and $B = \{2\}$, then $\phi(A) = B$ and $\phi(B) = A$. Hence, by Proposition 2.3, B_n has a $1/2$-balanced pair. □

3.2 Set Partition Lattices

The lattice, Π_n, consists of all partitions of $[n]$ ordered by refinement. In writing set partitions, we separate subsets with slashes and dispense with set braces and commas. For $n = 1, 2$ we have that Π_n is a chain, and so will only consider $n \geq 3$.

Corollary 3.2 *For $n \geq 3$, the set partition lattice Π_n has an automorphism with a cycle of length 2. So the set partition lattice is $1/2$-balanced.*

Proof We consider the map that sends a partition π to the partition π', where π' has the same blocks as π with the elements 1 and 2 interchanged. This is an automorphism of the lattice. Indeed, it is a bijection because it is an involution and swapping 1 and 2 preserves ordering by refinement. To see that this automorphism has a 2-cycle, notice that the lattice contains partitions $\pi_1 = 13/2/4/\cdots/n$ and $\pi_2 = 1/23/4/\cdots/n$ since $n \geq 3$. Under the automorphism described above, π_1 and π_2 form a 2-cycle. Hence, by Proposition 2.3, the set partition lattice on n elements is $1/2$-balanced when $n \geq 3$. $\square$

3.3 Subspace Lattices

Consider the n-dimensional vector space $\mathbb{F}_q^n$ over the Galois field with q elements. Let $L_n(q)$ denote the lattice of subspaces of $\mathbb{F}_q^n$ ordered by inclusion. If $n \leq 1$ then $L_n(q)$ is a chain.

Corollary 3.3 *For $n \geq 2$, the subspace lattice $L_n(q)$ has an automorphism with a cycle of length 2. So the subspace lattice is $1/2$-balanced.*

Proof Let $B = \{e_1, \ldots, e_n\}$ be the standard basis of $L_n(q)$. Consider the linear transformation on $\mathbb{F}_q^n$ defined by the $n \times n$ matrix M that is all zero except for ones in the $(1, 2)$, $(2, 1)$, and (i, i) positions, $3 \leq i \leq n$. Clearly, multiplying by M sends e_1 to e_2, e_2 to e_1, and fixes all other basis elements of $\mathbb{F}_q^n$. If $U \in L_n(q)$ then let

$$\phi(U) = MU = \{Mu \mid u \in U\}.$$

It is now easy to check that ϕ is a well-defined automorphism of $L_n(q)$ which exchanges the subspaces spanned by e_1 and by e_2. So we are done by Proposition 2.3. $\square$

3.4 Distributive Lattices

By the Fundamental Theorem on Distributive Lattices, every distributive lattice is isomorphic to the lattice of lower order ideals of some poset P ordered by inclusion. So it would be interesting to determine results about $J(P)$, the distributive lattice corresponding to a poset P, depending on properties of P. Unfortunately, it is not true that if P is $1/2$-balanced, then $J(P)$ is $1/2$-balanced as well. An example can be seen in Fig. 5. While P is $1/2$-balanced by the pair $(1, 3)$, $J(P)$ is not $1/2$-balanced, as evidenced in the chart in Fig. 5 whose entries are $e(P + xy)$ for every x and y not comparable in $J(P)$. Since $J(P)$ has 14 linear extensions, we can see no pair is $1/2$-balanced. However, adding an extra condition allows us to prove that $J(P)$ is $1/2$-balanced.

Proposition 3.4 *If P has an automorphism with cycle of length 2, then $J(P)$ is $1/2$-balanced.*

Proof Let $\phi : P \to P$ be an automorphism with $\phi(x) = y$ and $\phi(y) = x$ for some distinct $x, y \in P$. This induces an automorphism $\overline{\phi}$ of $J(P)$, given by $\overline{\phi}(I) = \{\phi(w) : w \in I\}$ for $I \in J(P)$. We claim that $\overline{\phi}$ has a cycle of length 2, namely that $\overline{\phi}(I_x) = I_y$ and $\overline{\phi}(I_y) = I_x$ where I_x, I_y are the lower order ideals generated by x, y respectively.

We will show $\overline{\phi}(I_x) = I_y$, as the proof of the other equality is similar. Let $z \in \overline{\phi}(I_x)$, so $z = \phi(w)$ for some $w \in I_x$. This means that $w \leq x$, and so $\phi(w) \leq \phi(x) = y$. Therefore, $z \leq y$ and we have $z \in I_y$. Hence, $\overline{\phi}(I_x) \subseteq I_y$. The proof of the other set containment is
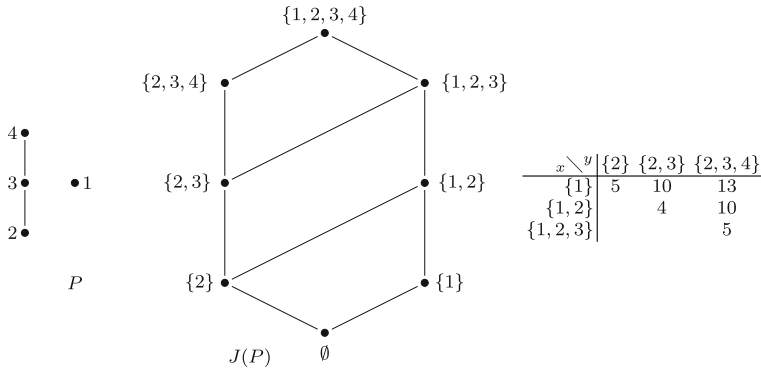
Fig. 5 A 4 element poset P, its corresponding $J(P)$, and a chart with values $e(P + xy)$

similar. Thus, we have proven the claim. Since $\overline{\phi}$ has a cycle of length 2, by Proposition 2.3, $J(P)$ is 1/2-balanced. □

This leads to another proof that Boolean lattices are 1/2-balanced.

Corollary 3.5 *For $n \geq 2$, the Boolean lattice B_n is 1/2-balanced.*

Proof The Boolean lattice B_n is the distributive lattice corresponding to the poset P with n elements and no relations. There is an automorphism on P that swaps elements 1 and 2 and is the identity on the remaining elements. Since P has an automorphism with a cycle of length 2, B_n is 1/2-balanced by Proposition 3.4. □

We note that if we use the construction in Proposition 3.4 on the ϕ from the proof of the previous corollary, then the resulting $\overline{\phi}$ is exactly the map used to prove Corollary 3.1. Also, the ideas in this subsection raise some interesting questions.

Question 3.6 Are all distributive lattices 1/3-balanced? What other characteristics of P would imply that $J(P)$ is 1/2-balanced?

3.5 Products of Two Chains

Let C_n be the chain with n elements. This section will be concerned with the product of two chains C_m and C_n, with $m, n \geq 2$. See Fig. 6 for the Hasse diagram of $C_3 \times C_4$. Such products can also be interpreted in terms of Young diagrams. An *integer partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_l)$ of positive integers. The corresponding *shape* is an array of l rows of left-justified boxes, also called cells, with λ_i boxes in row i. The Young diagram of shape $\lambda = (4, 4, 4)$ is displayed on the right in Fig. 6 and the diagram of $\lambda = (4, 4, 2)$ is given in Fig. 7 (ignoring the entries in the boxes for now). We often make no distinction between an integer partition and its Young diagram. Let (i, j) denote the box in row i and column j of λ. Then we turn this Young diagram into a poset by ordering the boxes component-wise: $(i, j) \leq (i', j')$ if and only if $i \leq i'$ and $j \leq j'$. It should now be clear that the posets $C_m \times C_n$ and $\lambda = (n^m)$ are isomorphic where n^m represents n repeated m times as in Fig. 6.

The linear extensions of λ can be thought of as a certain type of tableau. A *standard Young tableau* (SYT) of shape λ, Y, is a filling of the boxes of λ with the integers $1, \ldots, n =$
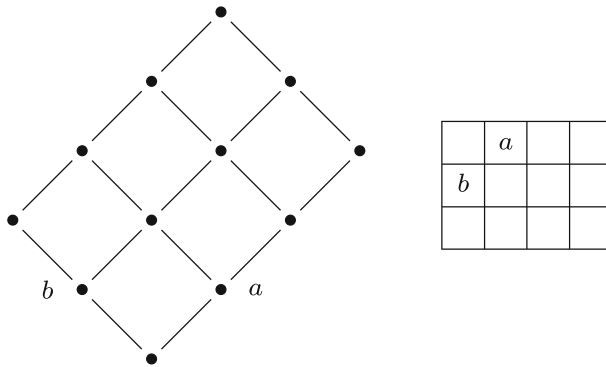
Fig. 6 The poset $C_3 \times C_4$ and its corresponding diagram

$\sum_i \lambda_i$ so that the rows and columns increase. An SYT of shape $\lambda = (4, 4, 2)$ is displayed on the left in Fig. 7. The SYT of shape λ are in bijection with the linear extensions of λ where k is the entry in box (i, j) if and only if (i, j) is the kth element of the linear extension. We will freely go back and forth between these two viewpoints.

Unlike many other demonstrations that a poset is 1/3-balanced, our proof for $C_m \times C_n$ finds the exact value of $\mathbb{P}(a \prec b)$ for a pair of elements (a, b). It also provides a nice application of the famous Hooklength Formula, Eq. (3.1) below. Consider the cells $a = (1, 2)$ and $b = (2, 1)$ as labeled in Fig. 6. In order to compute how many linear extensions of $C_m \times C_n$ have $a \prec b$, we will compute how many SYT have cell $(1, 2)$ filled with a smaller number than cell $(2, 1)$. Since the entry 2 must go in one of these two cells, this assumption forces the SYT to have the $(1, 1)$ cell filled with a 1 and the $(1, 2)$ cell filled with a 2. Flipping and rotating by 180 degrees, one sees that this is equivalent to counting the SYT of shape $(n^{m-1}, n - 2)$.

To prove the next lemma, we will need the hooklength formula for f^λ, the number of SYT of shape λ. For a given cell (i, j) in a diagram of shape λ, its *hook* is the set of all the cells weakly to its right and in the same row, together with all cells weakly below it and in the same column, and its hooklength $h_\lambda(i, j)$ is the number of cells in its hook. On the right in Fig. 7, each cell of $\lambda = (4, 4, 2)$ is labeled with its hooklength. The hooklength formula for a diagram with n cells is

$$f^\lambda = \frac{n!}{\prod h_\lambda(i, j)}, \tag{3.1}$$

where the product is over all cells (i, j) in λ. Returning to our example shape $(4, 4, 2)$, we see that the corresponding number of SYT is

$$f^{(4,4,2)} = \frac{10!}{6 \cdot 5^2 \cdot 4 \cdot 3 \cdot 2^3 \cdot 1^2} = 252.$$
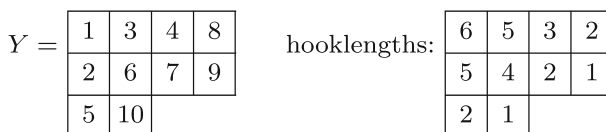
$$Y = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 8 \\ \hline 2 & 6 & 7 & 9 \\ \hline 5 & 10 \\ \cline{1-2} \end{array} \qquad \text{hooklengths:} \qquad \begin{array}{|c|c|c|c|} \hline 6 & 5 & 3 & 2 \\ \hline 5 & 4 & 2 & 1 \\ \hline 2 & 1 \\ \cline{1-2} \end{array}$$

Fig. 7 A SYT of shape $(4, 4, 2)$ and a diagram of its hooklengths

Lemma 3.7 *Let $m \geq 1$ and $n \geq 3$. We can relate the number of standard Young tableaux of shape (n^m) and of shape $(n^{m-1}, n-2)$ by the following equality:*

$$f^{(n^{m-1}, n-2)} = \frac{(n-1)(m+1)}{2(mn-1)} f^{(n^m)}.$$

Proof Let $\lambda = (n^m)$ and $\mu = (n^{m-1}, n-2)$. We will proceed by first describing which factors differ between f^λ and f^μ. We can observe that the hooklengths only disagree between λ and μ in those cells in the last two columns and those in the last row. The last two columns of λ have hooklengths of $m+1, m, \ldots, 2$ and $m, m-1, \ldots, 1$, while in μ the last two columns have hooklengths $m, m-1, \ldots, 2$ and $m-1, m-2, \ldots, 1$. Overall, f^μ is missing a factor of $(m+1)m$ which appears in the denominator of f^λ. Similarly, the hooklength values of the last row of λ, excluding the ones in the last two columns which have already been accounted for, are $n, n-1, \ldots, 3$, while those in μ are $n-2, n-3, \ldots, 1$. So our formula for f^μ is missing a factor of $n(n-1)$ from the denominator and a factor of 2 from the numerator. Finally f^λ has a numerator of $(mn)!$ while μ has a numerator of $(mn-2)!$, so there is a factor of $(mn)(mn-1)$ we need to remove from the numerator of f^λ. Overall, our hooklength formula for μ derived from f^λ is

$$f^\mu \quad = \quad \frac{n(n-1)(m+1)m}{2(mn)(mn-1)} f^\lambda \quad = \quad \frac{(n-1)(m+1)}{2(mn-1)} f^\lambda,$$

as desired.   □

Theorem 3.8 *Let C_m and C_n be chains of lengths $m \geq 2$ and $n \geq 2$, respectively. Then their product $C_m \times C_n$ has a 1/3-balanced pair.*

Proof Without loss of generality, we can let $m \leq n$. Let $P = C_m \times C_n$. If $m = 2$, then P has width 2, and so is 1/3-balanced by a result of Linial (see Theorem 6.1). If $m = n = 3$, then P has a non-trivial automorphism, and so by Theorem 2.2, P has a 1/3-balanced pair.

Next, let $m \geq 3$ and $n \geq 4$. Consider the cells $a = (1, 2)$ and $b = (2, 1)$. We claim that (a, b) are a 1/3-balanced pair. As discussed at the beginning of this subsection, $e(P) = f^\lambda$ and $e(P + ab) = f^\mu$, where $\lambda = (n^m)$ and $\mu = (n^{m-1}, n-2)$. Hence, by Lemma 3.7, we know that

$$e(P + ab) = \frac{(n-1)(m+1)}{2(mn-1)} e(P).$$

It remains to be shown that

$$\frac{1}{3} \leq \frac{(n-1)(m+1)}{2(mn-1)} \leq \frac{2}{3} \tag{3.2}$$

for all $m \geq 3$, $n \geq 4$. For the first inequality, cross multiply and bring everything to one side to get the equivalent inequality $(mn-1) + 3(n-m) \geq 0$. This inequality is true since $n \geq m$ and $mn \geq 1$.

For the second inequality, proceed in the same manner to get $mn + 3(m-n) - 1 \geq 0$. By the lower bounds for m, n we have $(m-3)(n-4) \geq 0$. So it suffices to prove $mn + 3m - 3n - 1 \geq (m-3)(n-4)$. Moving everything to one side yet again gives the equivalent inequality $7m - 13 \geq 0$ which is true since $m \geq 3$.

Therefore, we have shown that (3.2) holds, and so (a, b) is a 1/3-balanced pair in P. □

Question 3.9 We were motivated in part to study products of chains as they are isomorphic to divisor lattices. Can one show that a product of k chains is 1/3-balanced, for $k \geq 3$?

4 Other Diagrams

In Section 3.5, we considered the product of two chains as a rectangular Young diagram, and the linear extensions of the poset corresponded to the standard Young tableaux of that Young diagram. Given Theorem 3.8, it is natural to consider other posets that come from other diagrams. Suppose $\lambda_1 > \lambda_2 > \cdots > \lambda_k$, in which case $\lambda = (\lambda_1, \ldots, \lambda_k)$ is called a *strict partition*. The *shifted diagram* corresponding to a strict partition λ indents row i so that it begins at the diagonal cell (i, i). An example is given in Fig. 8b. A third type of diagram is a *skew diagram*, λ/μ, which is the set-theoretic difference between diagrams $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ such that $\mu \subseteq \lambda$, that is, $l \leq k$ and $\mu_i \leq \lambda_i$ for each $1 \leq i \leq l$. A skew diagram can be either *left-justified*, as seen in Fig. 8c, or *shifted*, as seen in Fig. 8d. Note that when μ is empty then $\lambda/\mu = \lambda$. Also, we will now use the term "Young diagram" to refer to any of the four possibilities we have described.

Any Young diagram can be turned into a poset using the same ordering on the cells as before. In addition, a standard Young tableau can be obtained from a diagram with n boxes by filling them with the numbers $1, \ldots, n$ so that rows and columns increase. Such tableaux correspond bijectively to linear extensions of the corresponding poset. We next present a generalized version of Theorem 3.8 for arbitrary shapes.

Theorem 4.1 *Let $P_{\lambda/\mu}$ be the poset corresponding to the Young diagram λ/μ. If $P_{\lambda/\mu}$ is not a chain, then it is $1/3$-balanced.*

Proof Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$. Assume first that μ is empty. We will show that P_λ has an almost twin pair of elements and so, by Theorem 1.3, is $1/3$-balanced.

When λ is left-justified, let x correspond to the $(1, 2)$ cell and y correspond to the $(2, 1)$ cell of λ. Both of these cells must exist in λ since P_λ is not a chain. It is now easy to verify that (x, y) is an almost twin pair. If λ is a shifted, then $\lambda_1 \geq 3$ and $\lambda_2 \geq 1$, as P_λ is not a chain. Let x correspond to the $(1, 3)$ cell and y correspond to the $(2, 2)$ cell. Again, (x, y) is an almost twin pair of elements in P_λ.

Next, we consider skew diagrams. If λ/μ is a disconnected diagram, observe that an almost twin pair in a connected component of $P_{\lambda/\mu}$ remains an almost twin pair in the entire poset. Therefore, we can assume λ/μ is a connected skew diagram that does not correspond to a poset that is a chain. First consider skew left-justified diagrams. By removing any empty columns on the left of the diagram, we can assume without loss of generality that $k \geq l + 1$. For ease in discussing the first and last rows of μ, define $\mu_0 = \lambda_1$ and $\mu_{l+1} = 0$. We have the following cases:
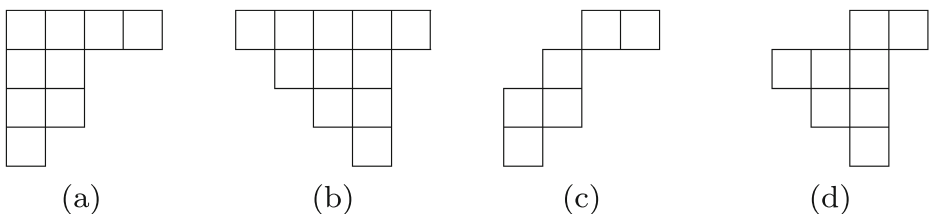

Fig. 8 **a** A left-justified Young diagram of shape $(4, 2^2, 1)$, **b** a shifted diagram of shape $(5, 3, 2, 1)$, **c** a skew left-justified diagram of shape $(4, 2^2, 1) / (2, 1)$, and **d** a shifted skew diagram of shape $(5, 3, 2, 1) / (3)$

(i) If there exists $i \in [l]$ such that
$$\mu_{i-1} - 1 \geq \mu_i = \mu_{i+1} + 1$$
then $(i, \mu_i + 1)$ and $(i + 1, \mu_{i+1} + 1)$ is an almost twin pair. For an example of this case, see the pair (a, b) in Fig. 9.

(ii) If there exists $i \in [l - 1]$ such that
$$\mu_{i-1} - 2 \geq \mu_i = \mu_{i+1}$$
then $(i, \mu_i + 2)$ and $(i + 1, \mu_{i+1} + 1)$ is an almost twin pair. Note that $(i, \mu_i + 2)$ exists in the diagram since λ/μ is connected. For an example of this, see the pair (c, d) in Fig. 9.

(iii) If $k = l + 1$ and $\mu_{l-1} - 1 \geq \mu_l$, then $(l, \mu_l + 1)$ and $(l + 1, 1)$ are an almost twin pair. For an example of this, see the pair (e, f) in Fig. 9.

(iv) If $k \geq l + 2$ and $\mu_l \geq 2$, then $(l + 1, 2)$ and $(l + 2, 1)$ are an almost twin pair. Notice this is similar to case (ii), only it occurs at the bottom of the skew diagram.

We can now decide what types of diagrams do not fall into cases (i)-(iv) above. We claim that any remaining diagram has μ of the form
$$(s^{m_1}, (s - 1)^{m_2}, \ldots, (s - p + 1)^{m_p})$$
where $1 \leq p \leq s$ and $m_i \geq 2$ for all $i \in [p]$. We call this case (v).

To verify the claim, note that all consecutive μ_i values differ by 1 or 0 since if there is some r with $\mu_{r-1} - 2 \geq \mu_r$, then to avoid cases (i) and (ii) above, it must be that $\mu_{s-1} - 2 \geq \mu_s$ for all $s \in [r, l + 1]$. In particular, this means $\mu_l \geq 2$, and this diagram will fall into case (iii) or (iv). So, any consecutive μ_i values differ by 1 or 0. Further, if $m_i = 1$ for any $i \in [p]$, the diagram would fall into case (i). Hence, μ must have the form above.

It also must be true that λ/μ has $\lambda_1 = \mu_1 + 1$, in order to avoid case (ii) above. An example of a diagram λ/μ that does not fall into cases (i)-(iv) is given in Fig. 10. In these remaining diagrams, $(1, \mu_1 + 1)$ and $(m_1 + 1, \mu_{(m_1+1)} + 1)$ is an almost twin pair. An example of such a pair is (a, b) in Fig. 10. Hence every skew left-justified diagram λ/μ satisfies one of these five cases, and so $P_{\lambda/\mu}$ has an almost twin pair.

Finally, we consider the skew shifted diagrams. Notice that that the first l rows of the diagram can be viewed as a skew left-justified diagram. Therefore, if any of the first $l - 1$ rows are of the forms found in cases (i) or (ii), or if the first rows correspond to case (v), then the almost twin pairs in those cases remain almost twin in this poset, and we are done.

If none of cases (i), (ii), or (v) apply, then consider μ_l. In particular, it must be the case that $\mu_l > 1$, else case (ii) or (v) applies. If $\mu_l > 3$, then the last $k - l$ rows of the diagram

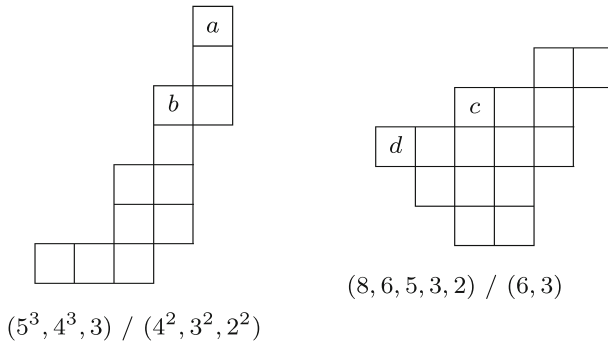**Fig. 9** The skew diagram $(9, 7^2, 5^4) / (6, 5, 3^3, 2)$
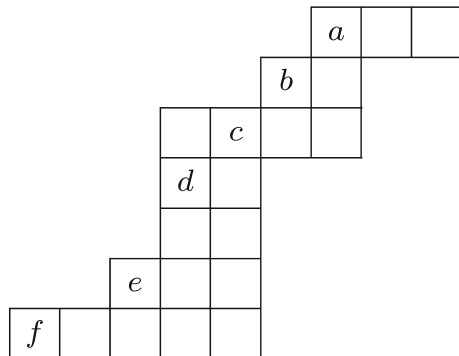
Fig. 10 A skew left-justified diagram and a skew shifted diagram

are a shifted diagram, and so we have the same almost twin pair as in the shifted case. If $\mu_l \in \{2, 3\}$, then $(l, \mu_l + l)$ and $(l + 1, l + 1)$ are an almost twin pair, as seen by the pair (c, d) in Fig. 10. Hence, for skew diagrams λ/μ, if $P_{\lambda/\mu}$ is not a chain, it has an almost twin pair of elements, as claimed. Hence, even when μ is not empty, we can always find an almost twin pair. □

5 Posets of Dimension 2

The set of linear extensions $E(P)$ of a labeled poset P with n elements can be considered as a subset of the symmetric group $\mathfrak{S}_n$, where permutations are written in one-line notation. The *dimension* of a poset is the least k such that there is some $U \subseteq E(P)$ of size k such that $\cap U = (P, \leq)$. An equivalent definition is that the dimension of P is the least k such that P can be embedded into the product $\mathbb{N}^k$ where $\mathbb{N} = \{0, 1, 2, \dots\}$.

We will be concentrating on posets of dimension 2. Every such poset can be realized using a permutation $\pi = \pi_1 \dots \pi_n$ since a poset of dimension 2 can be obtained by intersecting the linear order in π with the natural order $1 < \cdots < n$ on the integers. We will use P_π to denote this poset. Figure 11 displays the poset P_π when $\pi = 41325$.

To state our result, we will need some definitions from the theory of permutation patterns. If π and σ are permutations then we say that π *contains a copy of* σ if there is some

Fig. 11 The poset P_π for $\pi = 41325$

P_π
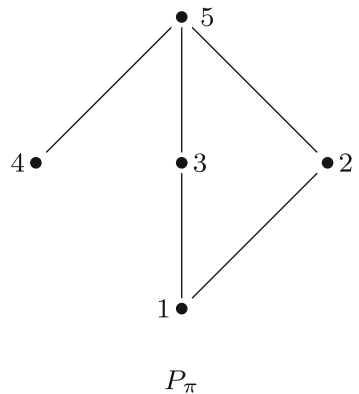
subsequence of π whose elements are in the same relative order as those of σ. Otherwise, π *avoids* σ. For example, if $\pi = 23154$ then π contains the pattern 132 because the subsequence 254, like the pattern, has its smallest element first, its largest element second, and its middle-sized element last. On the other hand, π avoids 321 since it does not contain a decreasing subsequence with three elements. Given a subsequence π' of π we say that π' is *contained in a copy of* σ if some copy of σ in π uses every element of π' (and perhaps others). Otherwise, we say π' *avoids* σ. Note that π' can avoid σ even if π contains it. Returning to our example, $\pi' = 14$ is contained in the pattern 132 because of the subsequence 154 of π. But π' avoids 123 since none of the copies of 123 in π use the 1.

Finally, define an *inversion* in $\pi = \pi_1 \ldots \pi_n$ to be a copy $\pi_i \pi_j$ of the pattern 21. Note that in this case some authors define the inversion to be the pair of corresponding indices (i, j).

Proposition 5.1 *Let* $\pi = \pi_1 \pi_2 \ldots \pi_n$ *be an element of* $\mathfrak{S}_n$, *and assume that* π *has an inversion* $\pi_i \pi_j$ *avoiding the patterns 312 and 231 in* π. *Then the pair* (π_i, π_j) *is 1/2-balanced in* P_π.

Before we proceed to the proof, we can observe an example of this in Fig. 11. Note that 32 is an inversion of $\pi = 41325$ which avoids 312 and 231 in π. So $(3, 2)$ is a 1/2-balanced pair in P_π.

Proof To simplify notation, let $\pi_i = y$ and $\pi_j = x$. Therefore, π has the form

$$\pi = \pi_1 \cdots y \cdots x \cdots \pi_n$$

where $y >_\mathbb{N} x$. Now, since yx avoids 312 and 231, there are no elements between x and y in π that are larger than y or smaller than x. Also, no elements to the right of x or left of y have values between x and y. To put this description another way, if yx avoids 312 and 231 in π, the elements between y and x in π are exactly those in the set $\{a \mid x <_\mathbb{N} a <_\mathbb{N} y\}$.

We claim that $U_x = U_y$ and $L_x = L_y$ in P_π. We will show that $U_x = U_y$ as the proof of $L_x = L_y$ is nearly identical. If $z \in U_x$, then z is to the right of x in π and thus also to the right of y in π. Since yx avoids 312 in π and $x <_\mathbb{N} z$, it must be that $y <_\mathbb{N} z$. Hence, $y <_P z$ and so $z \in U_y$.

If $z \in U_y$, then z is to the right of y in π and $y <_\mathbb{N} z$. Since yx avoids 231 in π, then z must also be to the right of x in π. Also, $x <_\mathbb{N} y <_\mathbb{N} z$. Thus $x <_P z$, which means $z \in U_x$. Hence, we have that $U_x = U_y$.

Now, because $U_x = U_y$ and $L_x = L_y$, (x, y) is a twin pair of elements. So, by Corollary 2.4, P_π is 1/2-balanced as desired. $\qquad \square$

Not every poset P_π is 1/2-balanced. For example, if $\pi = 13572468$ then P_π is isomorphic to the distributive lattice $J(P)$ in Fig. 5. But we have already noted that this lattice is not 1/2-balanced.

Question 5.2 Do posets of dimension 2 satisfy Conjecture 1.1?

6 Posets with Small Balance Constants

It would be of interest to characterize those posets whose balance constant is exactly 1/3, or to see if there are posets satisfying Conjecture 1.1 with $\delta(P)$ arbitrarily close to 1/3. For the second question, people have considered posets of width 2 (where width is the largest cardinality of an antichain) because Linial [9] proved that these posets satisfy the conjecture.
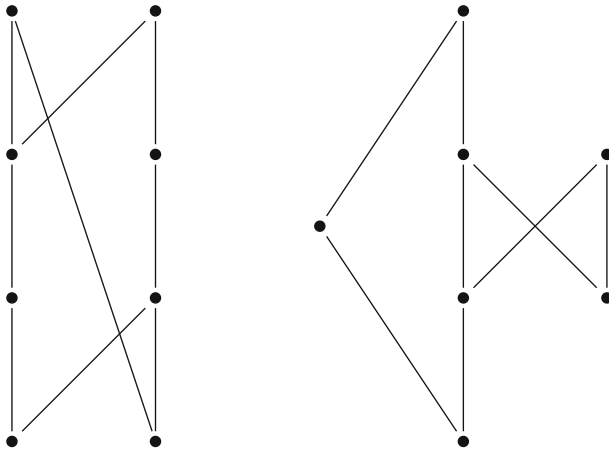
Fig. 12 Two posets with small balance constants

Theorem 6.1 ([9]) *Let* $(P, \leq)$ *be a poset of width 2. Then,* $\delta(P) \geq 1/3$.

Aigner [1] showed that posets of width 2 fit into one of two categories: either the poset is a linear sum of copies of the singleton poset and T (the poset from Fig. 2); or the poset has
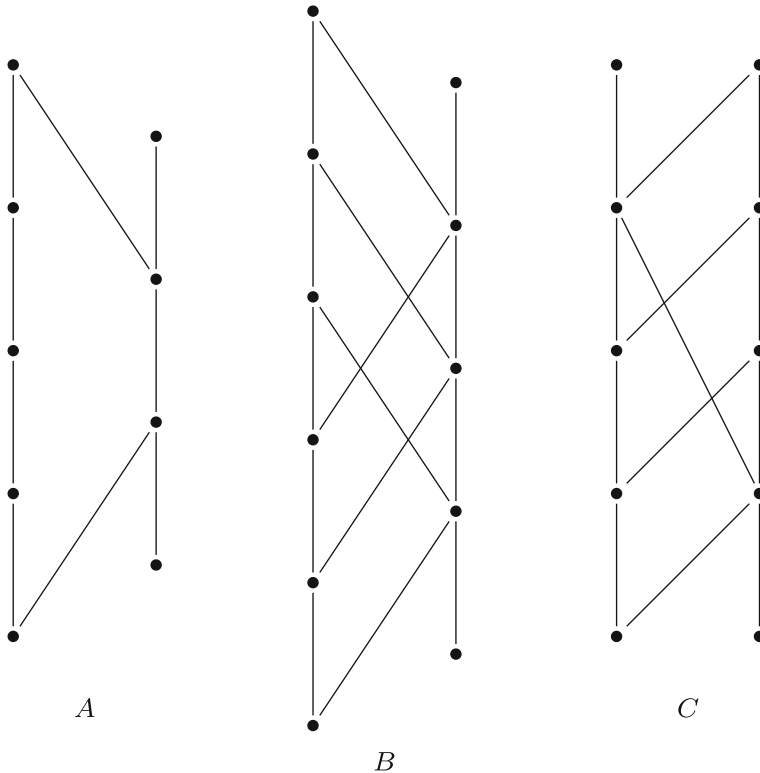

Fig. 13 Posets with the smallest balance constants greater than 1/3

an α-balanced pair with $1/3 < \alpha < 2/3$. In fact, the only known posets that have a balance constant of $1/3$ are the linear sums of singletons and T. The poset of width 2 in Fig. 12 has a balance constant of $\frac{16}{45} \approx 0.3556$, and until recently, it was the poset with the smallest known balance constant greater than $1/3$ [3]. Using computer search, we have found posets of width 2 that have balance constants closer to $1/3$. In Fig. 13, the three posets have balance constants $\delta(A) = \frac{6}{17} \approx 0.35294$, $\delta(B) = \frac{60}{171} \approx 0.350877$, and $\delta(C) = \frac{37}{106} \approx 0.349057$.

The smallest known balance constant for a poset with width strictly greater than 2 is $\frac{14}{39} \approx 0.3590$, as described in [3]. It belongs to the poset with 7 elements in Fig. 12. A computer search of all posets with up to 9 elements revealed no posets with balance constant smaller that $\frac{14}{39}$ and width greater than 2. These observations raise the following questions.

Question 6.2 Can one find a sequence of posets whose balance constants approach $1/3$? If P has width w, is there always a poset Q of smaller width such that $\delta(Q) < \delta(P)$?

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