Inversion polynomials for 321-avoiding permutations: addendum

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Abstract

This addendum contains results about the inversion number and major index polynomials for permutations avoiding 321 which did not fit well into the original paper. In particular, we consider symmetry, unimodality, behavior modulo 2, and signed enumeration.

1 Basic definitions

We recall the fundamental definitions from the original paper [2] for ease of reference.

Call two sequences of distinct integers $\pi = a_1 \ldots a_k$ and $\sigma = b_1 \ldots b_k$ *order isomorphic* whenever $a_i < a_j$ if and only if $b_i < b_j$ for all i, j. Let $\mathfrak{S}_n$ denote the symmetric group of permutations of $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$. Say that $\sigma \in \mathfrak{S}_n$ *contains* $\pi \in \mathfrak{S}_k$ *as a pattern* if there is a subsequence σ' of σ order isomorphic to π. If σ contains no such subsequence then we say σ *avoids* π and write $\mathrm{Av}_n(\pi)$ for the set of such $\sigma \in \mathfrak{S}_n$.

We will use a hash sign to denote cardinality. Our generating functions will keep track of four statistics for $\sigma = b_1 \ldots b_n \in \mathfrak{S}_n$:

- the *number of descents*
$$\operatorname{des}\sigma = \#\operatorname{Des}\sigma$$
where $\operatorname{Des}\sigma = \{i \mid b_i > b_{i+1}\}$,

- the *major index*
$$\operatorname{maj}\sigma = \sum_{i\in\operatorname{Des}\sigma} i,$$

- the *inversion number*
$$\operatorname{inv}\sigma = \#\{(i,j) \mid i < j \text{ and } b_i > b_j\},$$

- the number of *left-right maxima*
$$\operatorname{lrm}\sigma = \#\{i \mid b_i > b_j \text{ for all } j < i\}.$$

For Av(321) we will be studying the *major index polynomial*
$$M_n(q,t) = \sum_{\sigma\in\operatorname{Av}_n(321)} q^{\operatorname{maj}\sigma} t^{\operatorname{des}\sigma}$$
and the *inversion number polynomial*
$$I_n(q,t) = \sum_{\sigma\in\operatorname{Av}_n(321)} q^{\operatorname{inv}\sigma} t^{\operatorname{lrm}\sigma}.$$
Note that
$$M_n(1,1) = I_n(1,1) = \#\operatorname{Av}_n(321) = C_n$$
where C_n is the nth Catalan number.

2 Symmetry, unimodality, and mod 2 behavior of $M_n(q,t)$

The coefficients of the polynomials $M_n(q,t)$ have various nice properties which we now investigate. If $f(x) = \sum_k a_k x^k$ is a polynomial in x then we will use the notation
$$\begin{aligned} [x^k]f(x) &= \text{coefficient of } x^k \text{ in } f(x) \\ &= a_k. \end{aligned}$$
Our main object of study in this section will be the polynomial
$$A_{n,k}(q) = [t^k]M_n(q,t).$$
In other words, $A_{n,k}(q)$ is the generating function for the maj statistic over $\sigma \in \operatorname{Av}_n(321)$ having exactly k descents.

The first property which will concern us is symmetry. Consider a polynomial
$$f(x) = \sum_{i=r}^{s} a_i x^i$$
where $a_r, a_s \neq 0$. Call $f(x)$ *symmetric* if $a_i = a_j$ whenever $i + j = r + s$.

Theorem 2.1. *The polynomial $A_{n,k}(q)$ is symmetric for all n, k.*

Proof. If σ is counted by $A_{n,k}(q)$ then $\operatorname{des}\sigma = k$. Since σ avoids 321, it can not have two consecutive descents and so the minimum value of k is

$$1 + 3 + \cdots + (2k-1) = k^2$$

and the maximum value is

$$(n-1) + (n-3) + \cdots + (n-2k+1) = nk - k^2.$$

So it suffices to show that for $0 \le i \le nk$ we have $a_i = a_{nk-i}$ where

$$A_{n,k}(q) = \sum_i a_i q^i.$$

Let $\mathcal{A}_i$ be the permutations counted by a_i and let R_{180} denote rotation of the diagram of σ by 180 degrees. We claim R_{180} is a bijection between $\mathcal{A}_i$ and $\mathcal{A}_{nk-i}$ which will complete the proof. First of all, $R_{180}(321) = 321$ and so σ avoids 321 if and only if $R_{180}(\sigma)$ does so as well. If $\sigma \in \mathcal{A}_i$ then let $\operatorname{Des}\sigma = \{d_1, \ldots, d_k\}$ where $\sum_j d_j = i$. It is easy to see that $\operatorname{Des} R_{180}(\sigma) = \{n-d_1, \ldots, n-d_k\}$. It follows that $\operatorname{maj} R_{180}(\sigma) = nk - i$ and so $R_{180}(\sigma) \in \mathcal{A}_{nk-i}$. Thus R_{180} restricts to a well defined map from $\mathcal{A}_i$ to $\mathcal{A}_{nk-i}$. Since it is its own inverse, it is also a bijection. $\square$

Two other properties often studied for polynomials are unimodality and log concavity. The polynomial $f(x) = \sum_{i=0}^{s} a_i x^i$ is *unimodal* if there is an index r such that $a_0 \le \ldots \le a_r \ge \ldots \ge a_s$. It is *log concave* if $a_i^2 \ge a_{i-1}a_{i+1}$ for all $0 < i < s$. If all the a_i are positive, then log concavity implies unimodality.

Conjecture 2.2. *The polynomial $A_{n,k}(q)$ is unimodal for all n, k.*

This conjecture has been checked by computer for all $k < n \le 10$. The corresponding conjecture for log concavity is false, in particular, $A_{6,2}$ is not log concave.

The number-theoretic properties of the Catalan numbers have attracted some interest. Alter and Kubota [1] determined the highest power of a prime p dividing C_n using arithmetic means. Deutsch and Sagan [3] gave a proof of this result using group actions for the special case $p = 2$. Just considering parity, one gets the nice result that C_n is odd if and only if $n = 2^m - 1$ for some nonnegative integer m. Dokos et al. proved the following refinement of the "if" direction of this statement.

Theorem 2.3 (Dokos et al. [4])**.** *Suppose $n = 2^m - 1$ for some $m \ge 0$. Then*

$$[q^k]I_n(q,1) = \begin{cases} 1 & \text{if } k = 0 \\ \text{an even integer} & \text{if } k \ge 1. \end{cases} \qquad \square$$

In the same paper, the following statement was made as a conjecture which has now been proved by Killpatrick.

Theorem 2.4 (Killpatrick [6])**.** *Suppose $n = 2^m - 1$ for some $m \ge 0$. Then*

$$[q^k]M_n(q,1) = \begin{cases} 1 & \text{if } k = 0 \\ \text{an even integer} & \text{if } k \ge 1. \end{cases} \qquad \square$$

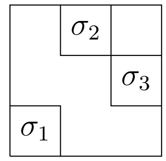

Figure 1: The diagram of $132[\sigma_1, \sigma_2, \sigma_3]$

We wish to prove a third theorem of this type. To do so, we will need the notion inflation for permutations. Given a permutation $\pi = a_1 \ldots a_n \in \mathfrak{S}_n$ and permutations $\sigma_1, \ldots, \sigma_n$, the *inflation* of π by the σ_i, written $\pi[\sigma_1, \ldots, \sigma_n]$, is the permutation whose diagram is obtained from the diagram of π by replacing the dot (i, a_i) by a copy of σ_i for $1 \le i \le n$. By way of example, Figure 1 shows a schematic diagram of an inflation of the form $132[\sigma_1, \sigma_2, \sigma_3]$. More specifically, $132[21, 1, 312] = 216534$.

Theorem 2.5. *Suppose $n = 2^m - 1$ for some $m \ge 0$. Then*

$$[t^k]M_n(1,t) = A_{n,k}(1) = \begin{cases} 1 & \text{if } k = 0, \\ \text{an even integer} & \text{if } k \ge 1. \end{cases}$$

Proof. We have $A_{n,0}(q) = 1$ since $\sigma = 12 \ldots n$ is the only permutation without descents and it avoids 321.

If $k \ge 1$, and $A_{n,k}(q)$ has an even number of terms then $A_{n,k}(1)$ must be even because it is a symmetric polynomial by Theorem 2.1. By the same token, if $A_{n,k}(q)$ has an odd number of terms, then $A_{n,k}(1)$ has the same parity as its middle term. Now consider R_{180} acting on the elements of $\mathcal{A}_{nk/2}$ as in the previous proof. Note that since n is odd, k must be even. Furthermore, this action partitions $\mathcal{A}_{nk/2}$ into orbits of size one and two. So it suffices to show that there are an even number of fixed points. If σ is fixed then its diagram must contain the center, c, of the square since this is a fixed point of R_{180}. Also, the NW and SE quadrants of σ with respect to c must be empty, since otherwise they both must contain dots (as one is taken to the other by R_{180}) and together with c this forms a 321. For the same reason, the SW quadrant of σ determines the NE one. Thus, the fixed points are exactly the inflations of the form $\sigma = 123[\tau, 1, R_{180}(\tau)]$ where $\tau \in \mathrm{Av}_{2^{m-1}-1}(321)$ has $k/2$ descents. By induction on m we have that the number of such τ, and hence the number of such σ, is even. $\square$

3 Refined sign-enumeration of 321-avoiding permutations

Simion and Schmidt [8] considered the signed enumeration of various permutation classes of the form $\sum_{\sigma \in \mathrm{Av}_n(\pi)} (-1)^{\mathrm{inv}\,\sigma}$. In this section we will rederive their theorem for $\mathrm{Av}_n(321)$ using a result from [2] about continued fractions. In addition, we will provide a more refined signed enumeration which also keeps track of the lrm statistic. We should note that Reifegerste [7] also has a refinement which takes into account the length of the longest increasing subsequence of σ.

We will use the following notation for continued fractions

$$F = \frac{a_1|}{|b_1} \pm \frac{a_2|}{|b_2} \pm \frac{a_3|}{|b_3} \pm \cdots \quad = \quad \cfrac{a_1}{b_1 \pm \cfrac{a_2}{b_2 \pm \cfrac{a_3}{b_3 \pm \cdots}}}. \tag{1}$$

Now consider the generating function $C(z) = \sum_{n\geq 0} C_n z^n$. It is well known that $C(z)$ satisfies the functional equation $C(z) = 1 + zC(z)^2$. Rewriting this as $C(z) = 1/(1 - zC(z))$ and iteratively substituting for $C(z)$, we obtain the also well-known continued fraction

$$C(z) = \frac{1|}{|1} - \frac{z|}{|1} - \frac{z|}{|1} - \frac{z|}{|1} - \frac{z|}{|1} - \frac{z|}{|1} - \cdots . \tag{2}$$

To return to our context, consider

$$\mathfrak{I}(q,t;z) = \sum_{n\geq 0} I_n(q,t) z^n.$$

We have a continued fraction expansion for this power series generalizing the one in equation (2).

Corollary 3.1 (Cheng et al. [2])**.** *The generating function $\mathfrak{I}(q,t;z)$ has continued fraction expansion*

$$\mathfrak{I}(q,t;z) = \frac{1|}{|1} - \frac{tz|}{|1} - \frac{qz|}{|1} - \frac{tqz|}{|1} - \frac{q^2z|}{|1} - \frac{tq^2z|}{|1} - \frac{q^3z|}{|1} - \frac{tq^3z|}{|1} - \frac{q^4z|}{|1} - \frac{tq^4z|}{|1} - \cdots \tag{3}$$

We will also need the following well-known result.

Theorem 3.2 (Jones and Thron [5])**.** *We have*

$$\begin{aligned} \frac{a_1|}{|1} + \frac{a_2|}{|1} + \frac{a_3|}{|1} + \cdots \quad &= \quad \frac{a_1|}{|1+a_2} - \frac{a_2a_3|}{|1+a_3+a_4} - \frac{a_4a_5|}{|1+a_5+a_6} - \frac{a_6a_7|}{|1+a_7+a_8} + \cdots \\ &= \quad a_1 - \frac{a_1a_2|}{|1+a_2+a_3} - \frac{a_3a_4|}{|1+a_4+a_5} - \frac{a_5a_6|}{|1+a_6+a_7} - \frac{a_7a_8|}{|1+a_8+a_9} - \cdots \end{aligned}$$

where the second and third continued fractions are called the even and odd parts, respectively, of the first continued fraction. □

Now plug $q = -1$ and $t = 1$ into the continued fraction (3) to obtain

$$\mathfrak{I}(-1,1;z) = \frac{1|}{|1} - \frac{z|}{|1} + \frac{z|}{|1} + \frac{z|}{|1} - \frac{z|}{|1} - \frac{z|}{|1} + \frac{z|}{|1} + \frac{z|}{|1} - \frac{z|}{|1} - \frac{z|}{|1} + \frac{z|}{|1} + \frac{z|}{|1} \cdots .$$

Using Theorem 3.2 to extract the odd part of this expansion gives

$$\mathfrak{I}(-1,1;z) = 1 + \frac{z|}{|1} - \frac{z^2|}{|1} - \frac{z^2|}{|1} - \frac{z^2|}{|1} - \frac{z^2|}{|1} - \frac{z^2|}{|1} - \cdots .$$

Comparing this to the continued fraction for $C(z)$ in (2), we see that

$$\mathfrak{I}(-1,1;z) = 1 + zC(z^2).$$

Taking the coefficient of z^n on both sides yields the following result.

Theorem 3.3 (Simion and Schmidt [8]). *For all $n \geq 1$, we have*

$$I_{2n}(-1,1) = \sum_{\sigma \in \mathrm{Av}_{2n}(321)} (-1)^{\mathrm{inv}\,\sigma} = 0 \quad \textit{and} \quad I_{2n+1}(-1,1) = \sum_{\sigma \in \mathrm{Av}_{2n+1}(321)} (-1)^{\mathrm{inv}\,\sigma} = C_n.$$

□

Since our refined sign-enumeration will involve the parameter lrm, we recall (but will not use) the folklore result that the enumerating polynomial of $\mathrm{Av}_n(321)$ according to the lrm statistic is the nth *Narayana polynomial*, i.e.,

$$I_n(1,t) = \sum_{\sigma \in \mathrm{Av}_n(321)} t^{\mathrm{lrm}\,\sigma} = \sum_{k=1}^{n} N_{n,k} t^k,$$

where the *Narayana number* $N_{n,k}$ is given by $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ for $n \geq k \geq 1$.

Theorem 3.4. *For all $n \geq 1$,*

$$I_n(-1,t) = \sum_{\sigma \in \mathrm{Av}_n(321)} (-1)^{\mathrm{inv}\,\sigma} t^{\mathrm{lrm}\,\sigma} = \sum_{k=1}^{n} (-1)^{n-k} s_{n,k} t^k \tag{4}$$

where $s_{n,k}$ is defined for $n \geq k \geq 1$ by

$$s_{n,k} = \binom{\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{k-1}{2} \rfloor}\binom{\lceil \frac{n-1}{2} \rceil}{\lceil \frac{k-1}{2} \rceil}.$$

Moreover,

$$\begin{aligned} I_{2n}(-1,t) &= (t-1) I_{2n-1}(-1,t), &(5)\\ (n+1) I_{2n+1}(-1,t) &= 2((1+t^2)n - t) I_{2n-1}(-1,t) - (1-t^2)^2 (n-1) I_{2n-3}(-1,t). &(6) \end{aligned}$$

Proof. Let $\mathfrak{I}(t;z)$ and $\mathfrak{I}_{odd}(t;z)$ be the power series defined as

$$\mathfrak{I}(t;z) = \sum_{n \geq 0} I_n(-1,t) z^n \quad \text{and} \quad \mathfrak{I}_{odd}(t;z) = \sum_{n \geq 0} I_{2n+1}(-1,t) z^n.$$

By equation (3), we have

$$\begin{aligned} \mathfrak{I}(t;z) &= \frac{1|}{|1} - \frac{tz|}{|1} + \frac{z|}{|1} + \frac{tz|}{|1} - \frac{z|}{|1} - \frac{tz|}{|1} + \frac{z|}{|1} + \frac{tz|}{|1} - \frac{z|}{|1} - \frac{tz|}{|1} + \frac{z|}{|1} + \frac{tz|}{|1} - \cdots \\ &= \cfrac{1}{1 - \cfrac{tz}{1 + \cfrac{z}{1 + \cfrac{tz}{1 - z\mathfrak{I}(t;z)}}}} \end{aligned}$$

which, after simplification, leads to the functional equation

$$(1+z-tz)z\,\mathfrak{I}(t;z)^2-(1+2z+z^2-t^2z^2)\,\mathfrak{I}(t;z)+(1+z+tz)=0.$$

Solving this quadratic equation, we obtain

$$\mathfrak{I}(t;z)=\frac{1+2z+(1-t^2)z^2-\sqrt{1-2(1+t^2)z^2+(1-t^2)^2z^4}}{2z(1+z-tz)}. \tag{7}$$

Noticing that $\mathfrak{I}_{odd}(t;z)=(\mathfrak{I}(t;\sqrt{z})-\mathfrak{I}(t;-\sqrt{z}))/2\sqrt{z}$ and using (7), we obtain after a routine computation

$$\mathfrak{I}_{odd}(t;z)=\frac{1-(1-t)^2z-\sqrt{1-2(1+t^2)z+(1-t^2)^2z^2}}{2z(1-(1-t)^2z)}. \tag{8}$$

It follows from (7) that

$$(1+z-tz)\mathfrak{I}(t;z)+(1-z+tz)\mathfrak{I}(t;-z)=2.$$

Extracting the coefficient of z^{2n} on both sides of the last equality, we obtain (5). Using (8), it is easily checked that $\mathfrak{I}_{odd}$ satisfies the differential equation

$$z\big(1-2(1+t^2)z+(1-t^2)^2z^2\big)\,\mathfrak{I}'_{odd}(t;z)+\big(1-2(1-t+t^2)z+(1-t^2)^2z^2\big)\big)\,\mathfrak{I}_{odd}(t;z)-t=0,$$

where $\mathfrak{I}'_{odd}(t;z)$ is the derivative with respect to z. Extracting the coefficient of z^n on both sides of the last equality, we obtain (6).

We now turn our attention to (4). Clearly, we have

$$[t^{2k+1}]I_{2n+1}(-1,t)=[t^kz^n]\frac{\mathfrak{I}_{odd}(\sqrt{t};z)-\mathfrak{I}_{odd}(-\sqrt{t};z)}{2\sqrt{t}}=[t^k][z^n]\frac{1}{\sqrt{1-2(1+t)z+(1-t)^2z^2}} \tag{9}$$

where the last equality follows from (8). Using the Lagrange inversion formula, one can show that

$$[z^n]\frac{1}{\sqrt{1-2(1+t)z+(1-t)^2z^2}}=[x^n](1+(1+t)x+tx^2)^n. \tag{10}$$

Combining (9) with the above relation, we obtain

$$[t^{2k+1}]I_{2n+1}(-1,t)=[x^n][t^k](1+(1+t)x+tx^2)^n=[x^n]\binom{n}{k}x^k(1+x)^n=\binom{n}{k}^2. \tag{11}$$

Similarly, we have

$$[t^{2k}]I_{2n+1}(-1,t)=[t^kz^n]\frac{\mathfrak{I}_{odd}(\sqrt{t};z)+\mathfrak{I}_{odd}(-\sqrt{t};z)}{2}=[t^k][z^n]\frac{1}{2z}-\frac{1-z-tz}{2z\sqrt{1-2(1+t)z+(1-t)^2z^2}}.$$

This, combined first with (9) and then (11), yields

$$[t^{2k}]I_{2n+1}(-1,t)=\frac{1}{2}\big([t^{2k+1}]I_{2n+1}(t,-1)+[t^{2k-1}]I_{2n+1}(t,-1)-[t^{2k+1}]I_{2n+3}(t,-1)\big)=-\binom{n}{k-1}\binom{n}{k}.$$

This proves that (4) is true when n is odd. Combining this with (5) shows that the formula also holds when n is even. $\square$

To see why the previous result implies the one of Simion and Schmidt, plug $t = 1$ into the equations for $I_{2n}(-1,t)$ and $I_{2n+1}(-1,t)$. In the former case we immediately get $I_{2n}(-1,1) = 0$ because of the factor of $t-1$ on the right. In the latter, we get the equation $(n+1)I_{2n+1}(-1,1) = 2(2n-1)I_{2n-1}(-1,1)$. The fact that $I_{2n+1}(-1,1) = C_n$ now follows easily by induction.

Finally, it is interesting to note that the numbers $s_{n,k}$ which arise in the signed enumeration of $\mathrm{Av}_n(321)$ have a nice combinatorial interpretation. Recall that *symmetric* Dyck paths are those $P = s_1 \ldots s_{2n}$ which are the same read forwards as read backwards. The following result appears in Sloane's Encyclopedia [9]: For $n \geq k \geq 1$, the number $s_{n,k}$ is equal to the number of symmetric Dyck paths of semilength n with k peaks.

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